Terminal Velocity

- In liquids and gases, friction is not constant, but velocity dependent.
- For small velocities $\vec{F}_D = -b\vec{v}$
- Consider a mass $m$ left from rest at some height. (1D motion)
- Newton’s second law:

$$mg - bv = ma \equiv m\frac{dv}{dt}$$  \hspace{1cm} (71)

$$ (72) $$

- If $v = mg/b$, $a = 0$. The terminal velocity is $v_t = mg/b$
Terminal Velocity

Newton’s second law:

\[ mg - bv = ma \equiv m \frac{dv}{dt} \]  \hspace{2cm} (71)

\[ \frac{dv}{dt} = \frac{mg}{b} \]  \hspace{2cm} (72)

If \( v = \frac{mg}{b} \), \( a = 0 \). The terminal velocity is \( v_t = \frac{mg}{b} \)

For any other velocity

\[ m \frac{dv}{mg - bv} = dt \]  \hspace{2cm} (73)
Terminal Velocity

- For any other velocity

\[ m \frac{dv}{mg - bv} = dt \]  

(71)

- Integration both sides from \( t_i \) to \( t \) (\( v_i \) to \( v(t) \))

\[ \int_{t_i}^{t} dt = \int_{v_i}^{v(t)} \frac{dv}{mg - bv} \]  

(72)

\[ \implies (t - t_i) = \frac{m}{b} \log \frac{mg - bv_i}{mg - bv(t)} \]  

(73)
Terminal Velocity

- Integration both sides from $t_i$ to $t$ ($v_i$ to $v(t)$)

\[
\int_{t_i}^{t} dt = \int_{v_i}^{v(t)} \frac{dv}{mg - bv} \tag{71}
\]

\[
\implies (t - t_i) = \frac{m}{b} \log \frac{mg - bv_i}{mg - bv(t)} \tag{72}
\]

- Solving for $v(t)$:

\[
v(t) = v_t - e^{-\frac{b}{m}(t-t_i)}(v_t - v_i) \tag{73}
\]

\[
= v_i e^{-\frac{b}{m}(t-t_i)} + v_t \left(1 - e^{-\frac{b}{m}(t-t_i)}\right) \tag{74}
\]
Kepler’s Laws

Kepler’s laws are based on observation only:

1. The orbit of planets around the sun are ellipses with the Sun positioned at one of the centers

2. The vector from the sun to the planet, sweeps equal areas at equal times

3. Let $s_i$ and $T_i$, $i = 1, 2$ be the semi major axis and the period of rotation respectively, of two planets. Then

\[
\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{s_1}{s_2}\right)^3
\]  \hspace{1cm} (75)

or

\[
\frac{T^2}{s^3}
\]  \hspace{1cm} (76)

is the same for every planet.
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Kepler’s First Law

- $a$ ($b$) is the semi minor (major) axis
- Definition of an ellipse:
  
  $$|F_1 P_1| + |P_1 F_2| = |F_1 P_2| + |P_2 F_2| \equiv 2b$$
Kepler’s Second Law

- $t_{12}$ ($t_{34}$) time it takes for the planet to go from $P_1$ ($P_3$) to $P_2$ ($P_4$)
- If $t_{12} = t_{34}$ then $A_1 = A_2$. 
Kepler’s Second Law

- The area covered in time interval $\delta t$ is
  \[ \delta A = \frac{1}{2} r \delta s \sin(\pi - \theta) = \frac{1}{2} rv \sin \theta \delta t \]
  - $\delta A$ is the same independent of where the planet is on its orbit
  - As the planet moves, $rv \sin \theta$ is constant.
- $rv \sin \theta = |\vec{r} \times \vec{v}|$
Kepler’s Second Law

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Kepler’s Second Law

- \( \frac{T^2}{s^3} \) is constant
- Consider a circular orbit \( s = R \)
- \( T = \frac{2\pi R}{v} \)
- Kepler’s Law:

\[
\left( \frac{2\pi R}{v} \right)^2 \left( \frac{1}{R^3} \right) = \frac{2\pi}{Rv^2} = \frac{2\pi}{R^2v^2} \Rightarrow \frac{v^2}{R^2} = \text{constant} \quad (77)
\]

\[
\Rightarrow |\vec{F}|R^2 = \text{constant} \quad (78)
\]

Kepler’s second law implies that the central force decreases with the square of the distance.
Newton’s Law of Gravitation

- Kepler’s third Law $\implies F \propto \frac{1}{r^2}$
- Law of uniform gravitational acceleration $\implies F = mg \propto m$
- Symmetry of forces (action reaction pairs) $\implies F \propto m_E$

$$|\vec{F}| = G_N \frac{mm_E}{r^2}$$  \hspace{1cm} (79)

where $m$ and $m_E$ are the masses of two gravitating objects, $r$ is the distance between their centres.

- $G_N = 6.67384 \times 10^{-11} N(m/kg)^2$
Newton’s Law of Gravitation

- \( \vec{F}_{12} \): Force acting on \( m_1 \) due to \( m_2 \)

\[
\vec{F}_{12} = G_N \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12}
\]  

- \( \vec{F}_{21} \): Force acting on \( m_2 \) due to \( m_1 \)

\[
\vec{F}_{21} = G_N \frac{m_2 m_1}{r_{21}^2} \hat{r}_{21} \equiv \vec{F}_{12}
\]

- On the surface of the Earth, the force acting on a mass \( m \) is:

\[
|\vec{F}| = mg = G_N \frac{mm_E^2}{R_E^2} \implies g = G_N \frac{m_E}{R_E^2}
\]
\[ \vec{F}_{12} = G_N \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12} \] is valid for point masses

If various masses \( m_i \) exert gravitational attraction on a mass \( M \), the total force acting on \( M \) is:

\[ \vec{F} = G_N \sum_{i} \frac{m_i M}{r_i^2} \hat{r}_i \quad (83) \]

where \( r_i \) is the distance of mass \( m_i \) from \( M \), and \( \hat{r}_i \) is the unit vector pointing from \( M \) towards \( m_i \).