Example: A mass Inside a Spherical Mass

The shown sphere has a radius $R$ and a mass $M$ uniformly distributed over its surface. Another mass $m$ is placed at a distance $r < R$ from the center. What will be the gravitational force that the object will feel?
Example: A mass Inside a Spherical Mass

- Divide the sphere into an inner sphere with radius $r$ and outer shell.
- Outer shell will not exert any force.
- Inner shell will have a mass:

$$M(r) = \frac{M}{\frac{4}{3} \pi R^3} \frac{4}{3} \pi r^3 = M \left( \frac{r}{R} \right)^3$$

- The force exerted by the inner shell is

$$\vec{F} = -G_N M \left( \frac{r}{R} \right)^3 \frac{m}{r^2} \hat{r} = -G_N \frac{M m}{R^3} \hat{r}$$
Example: A sphere with another sphere carved out

A sphere of radius $R_2$ is carved out of another sphere of radius $R_1$. The position of the center of the carved sphere is denoted by $\vec{d}$. The mass density of the system is $\rho$. If a mass $m$ is placed inside the cavity, what will be the force that this mass $m$ will feel?
Example: A sphere with another sphere carved out

Let \( \vec{r}_1 (\vec{r}_2) \) be the position of the mass \( m \) relative to the center of the large (small) sphere.
Example: A sphere with another sphere carved out

Let $\vec{r}_1$ ($\vec{r}_2$) be the position of the mass $m$ relative to the center of the large (small) sphere.

$\vec{F}_{\text{full sphere}} = \vec{F}_T + \vec{F}_{\text{carved out mass}} \quad \rightarrow \quad \vec{F}_T = \vec{F}_{\text{full sphere}} - \vec{F}_{\text{carved out mass}}$

The cavity can be modeled as a mass with mass density $-\rho$. 
Example: A sphere with another sphere carved out

\[ \vec{F}_{\text{full sphere}} = \vec{F}_T + \vec{F}_{\text{carved out mass}} \]
\[ \vec{F}_T = \vec{F}_{\text{full sphere}} - \vec{F}_{\text{carved out mass}} \]

- The cavity can be modeled as a mass with mass density \(-\rho\).
- Large sphere:
  \[ \vec{F}_L = -G_N \frac{\rho \frac{4}{3} \pi r_1^3}{r_1^2} \hat{r}_1 = -\frac{4\pi}{3} G_N \rho \hat{r}_1 \]
- Small sphere:
  \[ \vec{F}_s = -\frac{4\pi}{3} G_N (-\rho) \hat{r}_2 \]

Gravitational attraction is uniform inside the cavity.

CHALLENGE: Can you prove this without using vectors? (not recommended)
Example: Circular Orbits

Let $M_E$ ($M_S$) be the mass of Earth (satellite)

What is the speed of the satellite?

$$m \frac{v^2}{R} = G_N \frac{Mm}{R^2} \implies v = \sqrt{\frac{G_N M}{R}}$$

The closer the satellite is to the Earth, the faster it should be.

The period of the satellite is

$$T = \frac{2\pi R}{v} = \frac{2\pi}{\sqrt{G_N M}} R^\frac{3}{2} \implies \frac{T^2}{R^3} = \frac{2\pi}{\sqrt{G_N M}}$$

Measuring the ratio $T^2/R^3$, it is possible to determine the mass of the sun.
Example: Circular Orbits

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Geocentric orbits are those for which the relative position of the satellite is fixed with respect to the surface of the planet.