Example-Calculating Moments of Inertia

Moment of Inertia Of a Rigid Rod Rotating Around CM

\[
 dm = \frac{M}{L} \, dx
\]

\[
 I = \sum_i m_i R_i^2 = \sum_i \frac{M}{L} \, dx \left( \frac{L}{2} - x \right)^2
\]

\[
 = \frac{M}{L} \int_0^L \left( \frac{L}{2} - x \right)^2 \, dx = \frac{M}{3L} \left( \frac{L}{2} - x \right)^3 \bigg|_{x=0}^{x=L}
\]

\[
 = \frac{M}{3L} \left( \frac{L}{2} \right)^3 - \frac{M}{3L} \left( -\frac{L}{2} \right)^3 = \frac{M}{12} L^2
\]
Parallel Axis Theorem

Proof

- Let $I_{CM}$ be the moment of inertia around an axes going through the CM.
- Choose $z$ axis to be along the this axes.
- Choose a second axes that goes through the point $(x_0, y_0)$.
- The distance of point with coordinates $(x_i, y_i, z_i)$ from the second axis is $R^2 = (x_i - x_0)^2 + (y_i - y_0)^2$
- The moment of inertial with respect to the second axis is

$$I = \sum_i m_i \left[ (x_i - x_0)^2 + (y_i - y_0)^2 \right]$$  \hspace{1cm} (125)

$$= \sum_i m_i (x_i^2 + y_i^2) + \sum_i m_i (x_0^2 + y_0^2) - 2 \sum_i m_i (x_i x_0 + y_i y_0)$$  \hspace{1cm} (126)

$$= I_{CM} + Md^2$$  \hspace{1cm} (127)

where $d^2 = x_0^2 + y_0^2$ and $\sum_i m_i x_i = \sum_i m_i y_i = 0$
Parallel Axis Theorem

Example

\[ dm = \frac{M}{L} \, dx, \quad I_{CM} = \frac{1}{2} ML^2 \]

Moment of inertia of a thin rod around one end:

\[ I = I_{CM} + M \left( \frac{L}{2} \right)^2 = \frac{1}{12} ML^2 + \frac{1}{4} ML^2 = \frac{1}{3} ML^2 \]
Example: Thin Rod Rotating Around One End

The net torque on the rod around the fixed axes:

\[ \vec{\tau} = \sum_i \vec{r}_i \times (m_i \vec{g}) = \left( \sum_i m_i \vec{r}_i \right) \times \vec{g} = (M\vec{r}_{CM}) \times \vec{g} = \vec{r}_{CM} \times \vec{\omega} \]

Hence the center of gravity for an object in uniform gravitational field is its CM.

\[ \tau = \frac{MgL}{2} \]

\[ \alpha = \frac{\tau}{I} = \frac{\frac{MgL}{2}}{\frac{1}{3}ML^2} = \frac{3g}{2L} \]

If the rod is initially at rest: \( a^{CM} = a^{CM}_{tan} = \alpha \frac{L}{2} = \frac{3}{4}g < g \)

\( a_{CM} = \frac{F_{ext}}{M} \). Which other force is acting on the rod?
Perpendicular Axis Theorem

Proof

Consider a very thin object in the $xy$ plane.

For any point in the object $z_i \approx 0$

$l_x = \sum_i m_i(y_i^2 + z_i^2) \approx \sum_i m_i y_i^2$

Similarly $l_y = \sum_i m_i(x_i^2 + z_i^2) \approx \sum_i m_i x_i^2$

Then $l_z = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 = l_x + l_y$
Perpendicular Axis Theorem

Example

- The moment of inertia of a loop of mass $M$ and radius $R$ for rotations around an axis that is perpendicular to its plane and going through its CM: $I_z = \sum_i m_i R^2 = MR^2$

- Its moment of inertia around any axis that goes through its CM and is in its plane:
  - If $x$ and $y$ axes are the two axes in the plane of the loop, due to symmetry $I_x = I_y$
  - By perpendicular axis theorem: $I_z = I_x + I_y = 2I_x$
  - Hence $I_x = \frac{1}{2} MR^2$.

- Moment of inertia for rotation around an axis that goes through the edge and is in the plane of the loop:
  $$ I' = I_{CM} +Md^2 = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2 $$
Heavy Pulley

Since the pulley has a mass, the tensions at each end of the pulley will be different.

Let $x$ axis be out of the screen
Heavy Pulley

The net forces acting on mass $m_1$ and $m_2$ are:

$$\vec{F}_{1T} = (T_1 - m_1g)\hat{z} \quad (126)$$
$$\vec{F}_{2T} = (T_2 - m_2g)\hat{z} \quad (127)$$
The torque acting on the pulley is
\[ \vec{\tau} = R(T_1 - T_2)\hat{x} \]
Let \( \ddot{a}_1 = a_i \hat{z} \) and \( \ddot{\alpha} = \alpha \hat{x} \). Then

\[
T_1 - m_1 g = m_1 a_1 \quad (126)
\]
\[
T_2 - m_2 g = m_2 a_2 \quad (127)
\]
\[
R(T_1 - T_2) = l\alpha \quad (128)
\]

**Unknowns:** \( T_1, T_2, a_1, a_2, \) and \( \alpha \): 5 unknowns

**The remaining two eqns are:**

\[
a_1 = -a_2 \quad (129)
\]
\[
\alpha R = -a_1 \quad (130)
\]
Heavy Pulley

The solutions of these equations are:

\[ a_1 = -a_2 = \frac{(m_2 - m_1)R^2}{l + (m_1 + m_2)R^2} g \]  \hspace{1cm} (126)

\[ \alpha = \frac{(m_1 - m_2)R}{l + (m_1 + m_2)R^2} g \]  \hspace{1cm} (127)

\[ T_1 = \frac{m_1 g(l + 2m_2 R^2)}{l + (m_1 + m_2)R^2} \]  \hspace{1cm} (128)

\[ T_2 = \frac{m_2 g(l + 2m_1 R^2)}{l + (m_1 + m_2)R^2} \]  \hspace{1cm} (129)