Chapter 2

Discrete Random Variables

2.1 Preliminaries

**Definition 4** A random variable is a mapping (a function) from the sample space into real numbers.

- We can define an arbitrary number of different random variables on the same sample space.

**Ex:** Toss a fair 6-sided die. Let the random variable $X$ take on the value 1 if the outcome is 6, and 0 otherwise. Let the random variable $Y$ be equal to the outcome of the die. Illustrate the mappings from the sample space associated with $X$ and $Y$. (Note that $\{X = 1\} = \{\text{outcome is 6}\} = A$, and $\{X = 0\} = A^c$.)

**Definition 5** A discrete random variable takes a discrete set of values. The Probability Mass Function (PMF) of a discrete random variable is defined as

$$p_X(x) = P(X = x)$$

**Ex:** Find and plot the PMFs of $X$ and $Y$ defined in the previous example.

- A discrete random variable is completely characterized by its PMF.

**Ex:** Let $M$ be the maximum of the two rolls of a fair die. Find $p_M(m)$ for all $m$. (Think of the sample space description and the sets of outcomes where $M$ takes on the value $m$.)
2.2 Some Discrete Random Variables

2.2.1 The Bernoulli Random Variable

In the rest of this course, we shall define the Bernoulli random variable with parameter $p$ as the following:

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

In shorthand we say $X \sim \text{Ber}(p)$.

**Ex:** Express and sketch the PMF of a Bernoulli($p$) random variable.

Despite its simplicity, the Bernoulli r.v. is very important since it can model generic probabilistic situations with just two outcomes (often referred to as binary r.v.).

Examples:

- Indicator function: Consider the random variable $X$ defined previously. $X(w) = 1$ if outcome $w \in A$, and $X(w) = 0$ otherwise. So, $X$ indicates whether the outcome is in set $A$ or $A'$. $X$, a Bernoulli random variable, is sometimes called the “indicator function” of the event $A$. This is sometimes denoted as $X(w) = I_A(w)$.

- Consider $n$ tosses of a coin. Let $X_i = 1$ if the $i^{th}$ roll comes up H, and $X_i = 0$ if it comes up T. Each of the $X_i$’s are independent Bernoulli random variables. The $X_i$’s, $i = 1, 2, \ldots$ are a sequence of independent “Bernoulli Trials”.

- Let $Z$ be the total number of successes in $n$ independent Bernoulli trials. Express $Z$ in terms of $n$ independent Bernoulli random variables.

2.2.2 The Geometric Random Variable

Consider a sequence of independent Bernoulli trials where the probability of success in each trial is $p$ (We will later call this a “Bernoulli Process”). Let $Y$ be the number of trials up to and including the first success. $Y$ is a Geometric random variable with parameter $p$.

$$P(Y = k) = \text{for } k = 1, 2, 3, \ldots$$
Sketch $p_Y(k)$ for all $k$.

Check that this is a legitimate PMF.

**Ex:** Let $Z$ be the number of trials up to (but not including) the first success. Find and sketch $p_Z(z)$.

2.2.3 The Binomial Random Variable

Consider $n$ independent Bernoulli Trials each with probability of success $p$, and let $B$ be the number of successes in the $n$ trials. $B$ is Binomial with parameters $(n, p)$.

$$P(B = k) = \text{for } k =$$

**Ex:** Let $R$ be the number of Heads in $n$ independent tosses of a coin with bias $p$. 

![Histograms for Binomial distributions]
2.2.4 The Poisson Random Variable

A Poisson random variable $X$ with parameter $\lambda$ has the PMF

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

**Ex:** Show that $\sum_k p_X(k) = 1$ (Hint: use the Taylor series expansion of $e^\lambda$).

- The Binomial is a good approximation for the Poisson with $\lambda = np$ when $n$ is very large and $p$ is small, for small values of $k$. That is, if $k \ll n$

  $$\frac{\lambda^k e^{-\lambda}}{k!} \approx \frac{n!}{(n-k)!} p^k (1-p)^{(n-k)}$$

2.2.5 The Discrete Uniform R.V.

The discrete uniform random variable takes consecutive integer values within a finite range with equal probability. That is, $X$ is Discrete Uniform in $[a, b]$, $b > a$ if and only if

$$p_X(k) = 1/(b - a + 1) \quad \text{for } k = a, a + 1, a + 2, \ldots, b$$

**Ex:** A four-sided die is rolled. Let $X$ be equal to the outcome, $Y$ be equal to the outcome divided by three, and $Z$ be equal to the square of the outcome.

(Note that $Y$ and $Z$ both take four equally likely values, however they do not have the discrete uniform distribution.)

2.3 Functions of Random Variables

**Y = f(X)**

**Ex:** Let $X$ be the temperature in Celsius, and $Y$ be the temperature in Fahrenheit. Clearly, $Y$ can be obtained if you know $X$.

$$Y = 1.8X + 32$$

**Ex:** $P(Y \geq 14) = P(X \geq ?)$

**Ex:** A uniform r.v. $X$ whose range is the integers in $[-2, 2]$. It is passed through a transformation $Y = |X|$.

To obtain $p_Y(y)$ for any $y$, we add the probabilities of the values $x$ that results in $g(x) = y$:

$$p_Y(y) = \sum_{x : g(x) = y} p_X(x).$$

**Ex:** A uniform r.v. whose range is the integers in $[-3, 3]$. It is passed
through a transformation $Y = u(X)$ where $u(\cdot)$ is the discrete unit step function.

### 2.4 Expectation, Mean, and Variance

We are sometimes interested in a summary of certain properties of a random variable.

**Ex:** Instead of comparing your grade with each of the other grades in class, as a first approximation you could compare it with the class average.

**Ex:** A fair die is thrown in a casino. If 1 or 2 shows, the casino will pay you a net amount of 30,000 TL (so they will give you your money back plus 30,000), if 3, 4, 5 or 6 shows you they will take the money you put down. Up to how much would you pay to play this game?

**Ex:** Alternatively, suppose they give you a total of 30,000 if you win (regardless of how much you put down), and nothing if you lose. How much would you pay to play this game?

(answer: the value of the first game (the break-even point) is 15,000, and for the second game, it is 10,000. In the second game, you expect to get 30,000 with probability 1/3, so you expect to get 10,000 on average.)