1.6.2 Independent Trials and Binomial Probabilities

Consider n tosses of a coin with bias p. P(k H's in an n-toss sequence) $=$ $\binom{n}{k}p^k(1-p)^{n-k}$

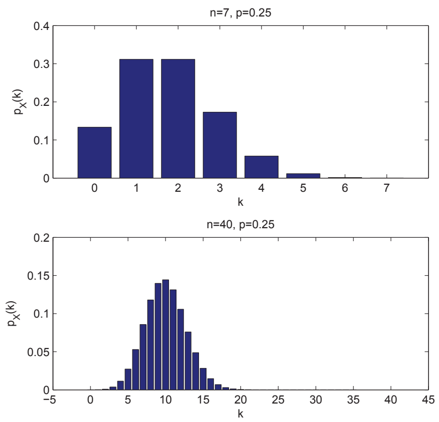

Figure 1.2: Binomial probability law.

Ex: Binary symmetric channel: Fig. 1.3 depicts a binary symmetric channel, where each symbol ("0" or "1") sent is inverted (turned to "1" or "0", respectively)with probability p_o, independently of all other symbols. (First consider the case where 1s and 0s are equiprobable, then the case when they are not.)

(a) What is the probability that a string of length n is received correctly?

(b) Given that a "110" is received, what is the probability that actually a "100" was sent?

(c) In an effort to improve reliability, each symbol is repeated 3 times and the received string is decoded by majority rule. What is the probability that a transmitted "1" is correctly decoded?

(d) Can you think about a better coding scheme than the one in (c)?

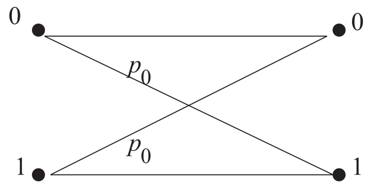

Figure 1.3: Binary symmetric channel.

1.7 Counting

A special case: all outcomes are equally likely.

$$\begin{aligned}\Omega &= \{s_1, s_2, \ldots, s_n\}\\ \mathrm{P}(\{s_j\}) &= \frac{1}{n}, \text{for all } j\\ A &= \{s_{j_1}, s_{j_2}, \ldots, s_{j_k}\}, j_k \in \{1, 2, \ldots, n\}\\ \mathrm{P}(A) &= \end{aligned}$$

The problem of finding $\mathrm{P}(A)$ reduces to counting its elements.

Ex: 6 balls in an urn, $\Omega = \{1, 2, \ldots, 6\}$.
$A = \{$the number on the ball drawn is divisible by 3$\}$.

Ex: A. (Permutations) The number of different ways of picking an ordered set of k out of n distinct objects

Ex: B. (Combinations) The number of different ways of choosing a group of k out of n distinct objects

Ex: C. (Partitions) How many different ways can a set of size n be partitioned into r disjoint subsets, with the i^{th} subset having size n_i? For example, pick a captain, a goalie and five players from among 7 friends.

Ex: D. (Distributing n identical objects into to r boxes) Consider n identical balls, to be colored red, black or white. How many possible configurations for the numbers of red, black and white balls? (Think about putting dividers between objects and shuffling objects and dividers.)

Ex: Categorize the following examples with respect to the following two criteria: Is the sampling with or without replacement? Does ordering matter or not?

(a) How many distinct words can you form by shuffling the letters of PROBABILITY?

(b) As a result of a race with 100 entrants, how many possibilities for the gold, silver and bronze medalists?

(c) Choose a captain, goalie and 5 players from a group of 9 friends.

(d) Choose a team of 7 from among 9 friends.

(e) How many possible car plate numbers are there in Ankara (assume two or three letters, and two or three digits are used on a car plate, chosen out of 23 letters and 9 numerals)?

(f) I can use the numbers 0, 1, and 9 arbitrarily many times to form a sequence of length 8. How many possibilities are there for the total weight of my sequence (sum of all numbers in the sequence)?

(g) Find the number of solutions of the equation $x_1 + x_2 + \ldots + x_r = n$, where $n \geq 1$ and $x_i \geq 0$'s are integers. (Hint: note that this is an example of "type-D" as well. Also think of the case where $x_i > 0$. In that case, there has to be at least one ball in each bin.)