3.5.2 Computing Conditional Expectation with PDFs

The conditional expectation of a continuous r.v. X is defined similar to its expectation. The only difference is that the conditional PDFs are used.

$$E[X|A] = \int \qquad E[X|Y=y] = \int$$

$$E[g(X)|A] = \int \qquad E[g(X)|Y=y] = \int$$

Ex: Consider a r.v. U which is uniform in $[0,100]$. Find $E[U|B]$ where $B = \{U > 60\}$. Compare it to $E[U]$.

Total expectation theorem: the divide-and-conquer principle

$E[X] =$

$E[g(X)] =$

Ex: A coin is tossed 5 times. Knowing that the probability of heads is a r.v. P uniformly distributed in $[0.4, 0.7]$, find the expected value of the number of heads to be observed.

Independence for Continuous Random Variables

Definition 11 *In general, two random variables X and Y are called independent if for any events A and B,*

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B), \tag{3.1}$$

where A and B are two arbitrary subsets real numbers.

- Suppose X and Y are continuous random variables, with $f_{XY}(x,y) = f_X(x)f_Y(y)$. Show that this condition implies the independence of X and Y.

Sometimes, observing that random variables are NOT independent is obvious from the region in which the joint PDF exists (recall the earlier joint PMF example.) However, proving that they ARE independent usually requires analytical operation (again, recall the discrete examples.)

Example:

$$f_{XY}(x,y) = \begin{cases} c & , 0 < x \leq 1, 0 < y \leq x \\ 0 & , o.w. \end{cases} \tag{3.2}$$

Ex: Show that each of the following conditions are equivalent to the definition of independence for continuous random variables X and Y:

$$f_{X|Y}(x|y) = f_X(x)$$
$$f_{Y|X}(y|x) = f_Y(y)$$

Ex: If two random variables X and Y are independent, then

$E(g(X)h(Y)) = E(g(X))E(h(Y))$ for any two functions $g(\cdot)$ and $h(\cdot)$.

Ex: As a consequence of the property above, var$(X+Y)$= var(X)+ var(Y) when X and Y are independent. (Proof is identical to the proof of the discrete version, which we did before.)

Inference and Continuous Bayes's Rule

$$f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)}$$

Ex: Consider the lightbulb with exponential lifetime Y, with parameter λ (failures per day). However, this time, λ is also random, and known to be uniformly distributed in $[1/4, 1]$. Having recorded the actual lifetime of a particular bulb as 4 days, what can we say about the distribution of λ?

Now, consider inference of the probability of an event based on the observation of the value of a random variable:

$$\begin{aligned} \mathrm{P}(A|Y=y) &= \\ &= \ldots \\ &= \frac{\mathrm{P}(A) f_{Y|A}(y)}{f_Y(y)} \end{aligned}$$

where the denominator can be expressed as

$$f_Y(y) = \mathrm{P}(A) f_{Y|A}(y) + \mathrm{P}(A^c) f_{Y|A^c}(y)$$

Ex: *Antipodal signaling under additive zero-mean unit-variance Gaussian noise.* Suppose that probabilities of sending a "1" or "−1" are p and $1-p$, respectively. What is the posterior probability that a "1" was sent, given that the noisy signal is measured as y. (Also consider the usual case where $p = 1/2$).

Chapter 4

Further Topics on Random Variables

4.1 Derived Distributions

Let $Y = g(X)$ be a function of a continuous random variable X. The general procedure for deriving the distribution of Y is as follows:

1. Calculate the CDF of Y:

$$F_Y(y) = \mathrm{P}(g(x) \le y) = \int_{\{x: g(x) \le y\}} f_X(x) dx$$

2. Differentiate to obtain the PDF of Y:

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

Ex: Find the distribution of $g(X) = \frac{180}{X}$ when $X \sim U[30, 60]$.