Chapter 6

The Bernoulli and Poisson Processes

6.1 The Bernoulli Process

By this time in the course, we have actually covered all the background for the Bernoulli Process under the disguise of "Bernoulli trials". All that remains is to introduce it as a *random process* (in other words, a stochastic process).

A discrete stochastic process is a sequence of random variables, $\{X_i\}$, indexed by i. Typically, $i = 1, 2, \ldots$.

The Bernoulli process with rate p is a sequence of IID Bernoulli random variables with parameter p:

$$X_i = \begin{cases} 1 & \text{, with probability p} \\ 0 & \text{, with probability } 1 - \text{p} \end{cases}$$

You can think of this as the result of a sequence of independent tosses of a coin with bias p. Thus it is easy to see that it has the following properties.

6.1.1 Properties

1. Binomial sums: Let S be the number of 1's among $X_1, X_2, \ldots, X_n$ is Binomial(n, p) (number of successes in n trials). In other words, $S = \sum_{i=1}^{n} X_i$. Then, $E(S) = np$, $\text{var}(S) = np(1-p)$ and $p_S(k) =$, $k = 0, 1, \ldots, n$.

2. Geometric first arrival time: Let T be the time of the first success, $T \geq 1$. Then, $E(T) = \frac{1}{p}$, $\text{var}(T) = \frac{1-p}{p^2}$ and $p_T(t) = (1-p)^{t-1}p$, $t = 1, 2, \ldots$.

3. Fresh-start: For any given time i, the sequence of random variables $X_{i+1}, X_{i+2}, \ldots$ (the future of the process) is also a Bernoulli process, and is independent of $X_1, X_2, \ldots, X_n$ (the past.)

4. Memorylessness and Geometric Inter-arrivals: Let $i + T$ be the time of the first success *after* time i, $T \geq 1$. Then, $E(T) = \frac{1}{p}$, $\text{var}(T) = \frac{1-p}{p^2}$ and $p_T(t) = (1-p)^{t-1}p$, $t = 1, 2, \ldots$.

Ex: Ayse and Burak are playing a computer game. In each round of the game, Ayse wins with probability p, and otherwise, Burak wins. Find the PMF of the number of games won by Ayse between any two games won by Burak.

Ex: Let Y_k be the total number of games played up to and including the k^{th} game won by Ayse. Find the mean, variance and the PMF of Y_k. (This is called the Pascal PMF of order k with parameter p.)

6.1.2 Splitting and Merging the Bernoulli Process

Ex: Show that when we split each arrival of a Bernoulli(p) process independently with probability r, the resulting subprocesses are Bernoulli with rates rp and $(1-r)p$. Are the resulting processes independent?

Ex: Show that when we merge two INDEPENDENT Bernoulli processes of rates p_1 and p_2, the resulting process is Bernoulli with rate $p_1 + p_2 - p_1 p_2$.