ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

ME 301
Theory of Machines I

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Kinematics of Mechanisms

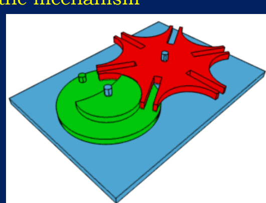
a. Functional Synthesis: Determination of candidate mechanisms that can realize a set of given (or implied) functional requirements.
b. Type Determination: Investigation of known mechanisms for their topological characteristics.
c. Kinematic Analysis: Determination of kinematic characteristics (position, velocity and acceleration) of a known mechanism.
d. Kinematic Synthesis: Determination of mechanism parameters (mostly link lengths) to realize a given motion (position, velocity and/or acceleration) for a mechanism whose topological characteristics are known.

Mechanisms

Kinematic element is the part of a body used to connect it to another body *permitting relative motion of two bodies*.

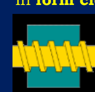
Kinematic pair is the combination of two kinematic elements and mostly termed as joint.

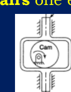
- **Open kinematic pair** pairs and unpairs during the operation of the mechanism



Mechanisms

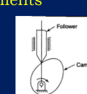
- **Closed kinematic pair** maintains contact for all possible positions of the mechanism
  - In **form closed kinematic pairs** one element envelopes the other





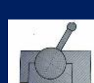
  - In **force closed kinematic pairs** an external force maintains contact of two kinematic elements



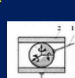


Mechanisms

- **Lower kinematic pair** has contact of two kinematic elements along a surface therefore contact stresses are *lower*

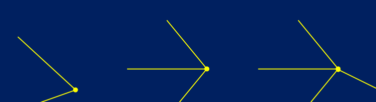




- **Higher kinematic pair** has contact of two kinematic elements along a line or at a point therefore contact stresses are *higher*

Mechanisms

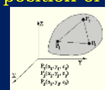
Degree of a kinematic pair is number of links that come together at the kinematic pair *minus one*.



Binary Joint (Degree: 1) — *Ternary Joint* (Degree: 2) — *Quaternary Joint* (Degree: 3) — …

Degree of Freedom

Degree of Freedom of a Rigid Body in Space: It is the number of *independent* parameters to define the position of a rigid body in that space.



3-D Space: Three *non-collinear* points

9 parameters: $x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3$

3 equations: $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$$|P_2P_3| = \sqrt{(x_3-x_2)^2 + (y_3-y_2)^2 + (z_3-z_2)^2}$$

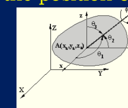
$$|P_3P_1| = \sqrt{(x_3-x_1)^2 + (y_3-y_1)^2 + (z_3-z_1)^2}$$

We may select six of the nine parameters as we like to locate the body as we like, however the distances $|P_1P_2|$, $|P_2P_3|$ and $|P_3P_1|$ are fixed once the points P_1, P_2 and P_3 are selected on the rigid body.

Figure from electronic lecture notes of Eres Söylemez https://ocw.metu.edu.tr/course/view.php?id=132

Degree of Freedom

Degree of Freedom of a Rigid Body in Space: It is the number of *independent* parameters to define the position of a rigid body in that space.



3-D Space: One point, four angles

7 parameters: $x_A, y_A, z_A; \theta_1, \theta_2, \theta_3; \phi$

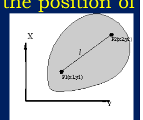
1 equation: $\sqrt{cos^2\theta_1 + cos^2\theta_2 + cos^2\theta_3{}^2} = 1$

We may select six of the seven parameters as we like to locate the body as we like, however the angles θ_1, θ_2 and θ_3 cannot be *all* arbitrary.

Figure from electronic lecture notes of Eres Söylemez https://ocw.metu.edu.tr/course/view.php?id=132

Degree of Freedom

Degree of Freedom of a Rigid Body in Space: It is the number of *independent* parameters to define the position of a rigid body in that space.



2-D Space: Two points

4 parameters: $x_1, y_1; x_2, y_2$

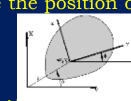
1 equation: $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

We may select three of the four parameters as we like to locate the body as we like, however the distance $|P_1P_2|$ is fixed once the points P_1 and P_2 are selected on the rigid body.

Figure from electronic lecture notes of Eres Söylemez https://ocw.metu.edu.tr/course/view.php?id=132

Degree of Freedom

Degree of Freedom of a Rigid Body in Space: It is the number of *independent* parameters to define the position of a rigid body in that space.



2-D Space: One point, one angle

3 parameters:

- $x_A, y_A; \phi$ (Cartesian representation of point A)
- $r, \theta; \phi$ (polar representation of point A)

In any case we may select three parameters as we like to locate the body.

Figure from electronic lecture notes of Eres Söylemez https://ocw.metu.edu.tr/course/view.php?id=132

Degree of Freedom

Degree of Freedom of a Kinematic Pair (Joint): It is the number of *independent* parameters to define the position of one link (rigid body) relative to the other link connected by the kinematic pair.

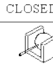
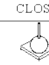
Since degree of freedom of a rigid body in 3-D space is 6, the body is fully free without any joint for degree of freedom 6.

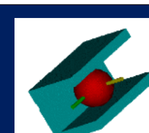
Similarly the body is fully free without any joint for degree of freedom 3 for planar motion.

Degree of Freedom of Joints (1/6)

TABLE I

KINEMATIC PAIRS WITH INDEPENDENT ROTATIONAL AND TRANSLATIONAL MOTION

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	FORM CLOSED	FORCE CLOSED
5	3	2	Sphere between parallel planes		



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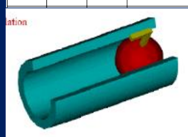
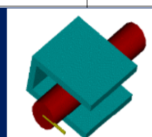
Degree of Freedom of Joints (2/6)

TABLE I

KINEMATIC PAIRS WITH INDEPENDENT ROTATIONAL AND TRANSLATIONAL MOTION

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	FORM CLOSED	FORCE CLOSED
4	3	1	Sphere in a cylinder		
	2	2	Cylinder between parallel planes		



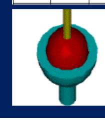


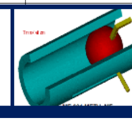
Degree of Freedom of Joints (3/6)

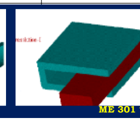
TABLE I

KINEMATIC PAIRS WITH INDEPENDENT ROTATIONAL AND TRANSLATIONAL MOTION

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	FORM CLOSED	FORCE CLOSED
3	3	0	Spherical pair (Ball joint)		
	2	1	Slotted sphere in a cylinder		
	1	2	Plane joint		







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Degree of Freedom of Joints (4/6)

TABLE I

KINEMATIC PAIRS WITH INDEPENDENT ROTATIONAL AND TRANSLATIONAL MOTION

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	FORM CLOSED	FORCE CLOSED
2	2	0	Slotted sphere		
	2	0	Torus		

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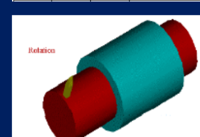
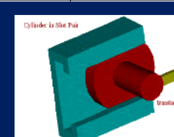
Degree of Freedom of Joints (5/6)

TABLE I

KINEMATIC PAIRS WITH INDEPENDENT ROTATIONAL AND TRANSLATIONAL MOTION

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	FORM CLOSED	FORCE CLOSED
2	1	1	Cylindrical joint		
	1	1	Slotted cylinder		





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Degree of Freedom of Joints (6/6)

TABLE I

KINEMATIC PAIRS WITH INDEPENDENT ROTATIONAL AND TRANSLATIONAL MOTION

DEGREE OF FREEDOM	ROTATIONAL FREEDOM	TRANSLATIONAL FREEDOM	NAME	FORM CLOSED	FORCE CLOSED
1	1	0	Revolute pair (turning joint)		
	0	1	Prismatic pair (sliding joint)		

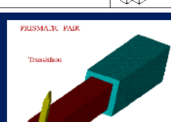

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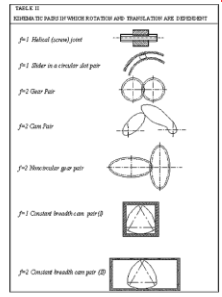
TABLE II

KINEMATIC PAIRS IN WHICH ROTATION AND TRANSLATION ARE DEPENDENT



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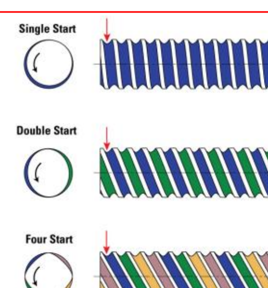

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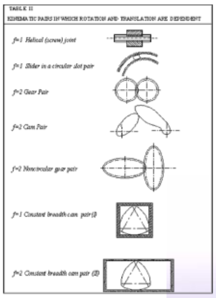
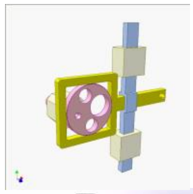
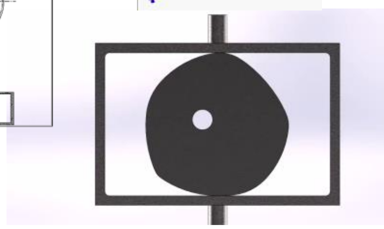
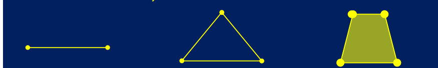






Definitions

Link: A rigid body with at least two (*in robotics there are open kinematic chains therefore one is also allowed*) kinematic elements



Binary Link *Ternary Link* *Quaternary Link*

Kinematic Chain: Formed by links connected by kinematic elements

Dimension of a Link: The linear and angular measurements of relative positions of kinematic elements

Mechanism: When one fixes one of the links of a kinematic chain a mechanism is obtained

Degree of Freedom of Mechanisms

It is the total number of *independent* parameters that is required to define the position of every link (hence every point on any link) in a mechanism.

Can be determined by inspection for simple mechanisms but there is a systematic approach (*the general degree of freedom equation*) that works for simple and complicated systems.

Degree of Freedom of Mechanisms

Inspection: Four-bar mechanism (*British call it three-bar mechanism because there are three moving bars but Germans and Americans call it four-bar because it is obtained from a four-bar kinematic chain by fixing one of the bars.* We will use German-American nomenclature).

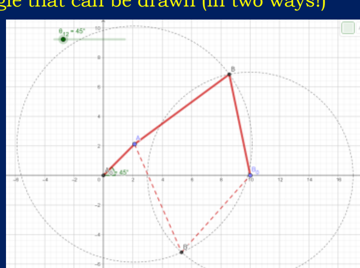
The lengths of links are known:

$|A_0A| = a_2 = 3\ cm$
$|AB| = a_3 = 8\ cm$
$|B_0B| = a_4 = 7\ cm$
$|A_0B_0| = a_1 = 10\ cm$

For a given θ_{12} location of A is fixed therefore $|AB_0|$ can be determined. ΔABB_0 is an SSS triangle that can be drawn (in two ways!). Therefore once we define θ_{12} as we like everything else is fixed. Degree of freedom of the four-bar is one.

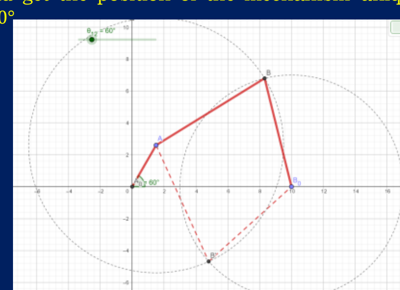
Degree of Freedom of Mechanisms

Inspection: Four-bar mechanism, for a given θ_{12} location of A is fixed therefore $|AB_0|$ can be determined. ΔABB_0 is an SSS triangle that can be drawn (in two ways!)



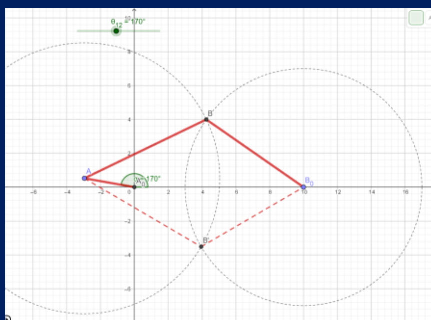
Degree of Freedom of Mechanisms

Inspection: Four-bar mechanism, select any θ_{12} value you like and get the position of the mechanism uniquely. Say $\theta_{12} = 60°$



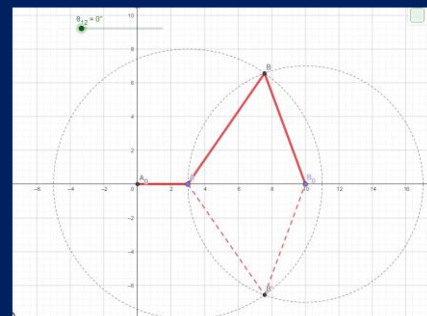
Degree of Freedom of Mechanisms

Inspection: Four-bar mechanism, say $\theta_{12} = 170°$



Degree of Freedom of Mechanisms

Inspection: Four-bar mechanism, say $0° \leq \theta_{12} \leq 360°$



Software: Geogebra Classic 6, https://www.geogebra.org/

Degree of Freedom of Mechanisms

Inspection: Five-bar mechanism

The lengths of links are known:

$|A_0A| = a_2 = 4\ cm$

$|AB| = a_3 = 3\ cm$

$|BC| = a_3 = 8\ cm$

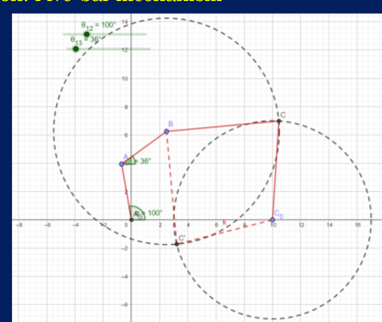
$|C_0C| = a_4 = 7\ cm$

$|A_0C_0| = a_1 = 10\ cm$

For a given θ_{12} location of A is fixed. $ABCC_0$ forms a four-bar that requires one more angle (say θ_{13}) to be known. Therefore there are *two independent* parameters. The degree of freedom of five-bar is two.

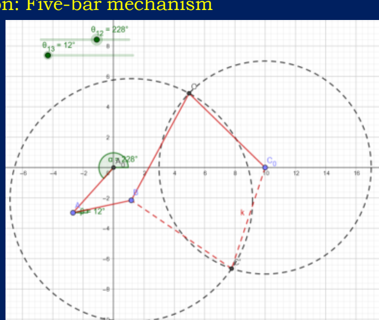
Degree of Freedom of Mechanisms

Inspection: Five-bar mechanism



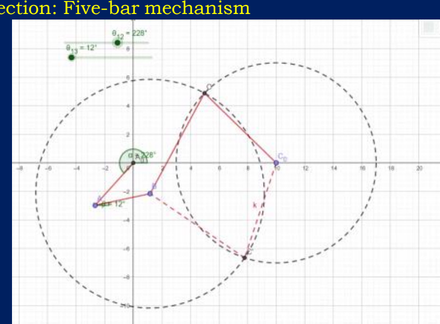
Degree of Freedom of Mechanisms

Inspection: Five-bar mechanism



Degree of Freedom of Mechanisms

Inspection: Five-bar mechanism



Degree of Freedom of Mechanisms

Systematic determination of degree of freedom:

In planar motion ℓ links with no joints has $F = 3(\ell - 1)$

k_1 joints (revolute & prismatic) constrain 2 freedoms

k_2 joints (cylinder in slot) constrain 1 freedom

$F = 3(\ell - 1) - 2k_1 - k_2$

$F = 6(\ell - 1) - 5k_1 - 4k_2 - 3k_3 - 2k_4 - k_5$

Kutzbach formula!

Replace 3 and 6 in the above equation with λ, degree of freedom of a rigid body in the space,

Constraints imposed by i^{th} joint is $\lambda - f_i$

Constraints imposed by all joints $\sum_{i=1}^{j}(\lambda - f_i) = \lambda j - \sum_{i=1}^{j} f_i$

Then $F = \lambda(\ell - 1) - \left[\lambda j - \sum_{i=1}^{j} f_i\right]$

Simplification yields $F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$

Degree of Freedom of Mechanisms

λ: Degree of freedom of the unconstrained bodies in the mechanism space

ℓ: Number of links of the mechanism (including fixed link)

j: Number of joints of the mechanism (ternary joints!)

f_i: Degree of freedom of i^{th} joint

F: Degree of freedom of the mechanism

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

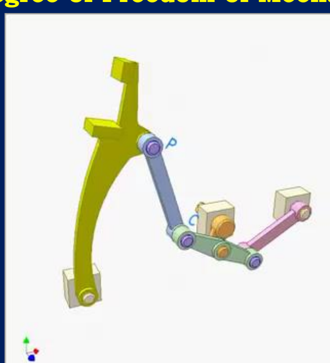
Remember exceptions!

$F > 0$ mechanism requires F actuations for kinematically deterministic motion

$F = 0$ structure (immobile) unless has special dimensions

$F < 0$ over-constraint (number of redundant supports is $|F|$) and immobile unless has special dimensions (*also forces cannot be determined unless equations of equilibrium/motion are complemented with* $|F|$ *number of equations relating deformations*).

Degree of Freedom of Mechanisms



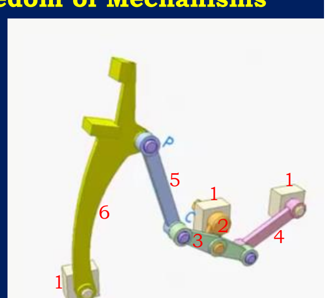
Degree of Freedom of Mechanisms

λ = 3 (co-planar)

Degree of Freedom of Mechanisms

λ = 3 (co-planar)

ℓ = 6

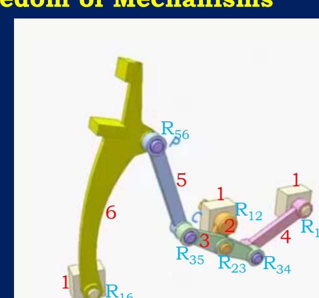


Degree of Freedom of Mechanisms

λ = 3 (co-planar)

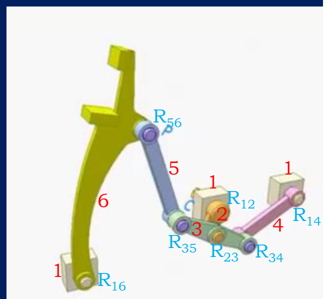
ℓ = 6

j = 7



Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 6
j = 7
f_i = 1 (all joints revolute)



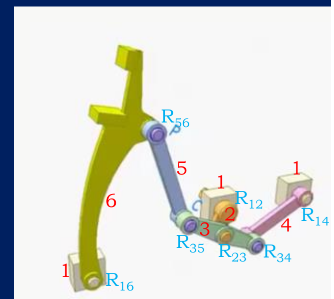
Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 6
j = 7
f_i = 1 (all joints revolute)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

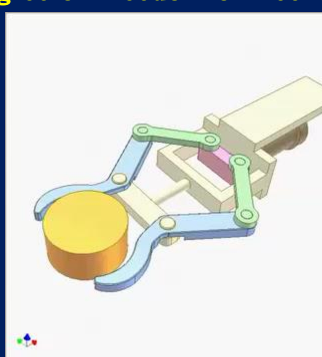
$$F = 3(6 - 7 - 1) + \sum_{i=1}^{7} f_i$$

$$F = 1$$



Therefore just rotating the crank 2 through R_{12} is sufficient to control the mechanism.

Degree of Freedom of Mechanisms

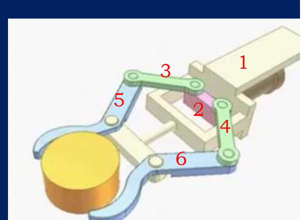


Degree of Freedom of Mechanisms

λ = 3 (co-planar)

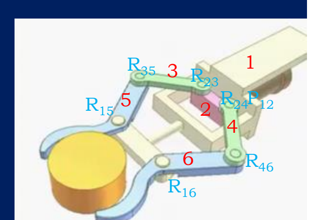
Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 6



Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 6
j = 7

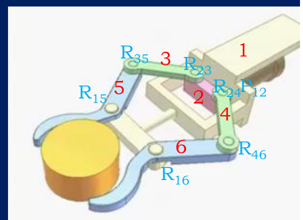


Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 6
j = 7
$\sum_{i=1}^{7} f_i$ = 7 (6R+P)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

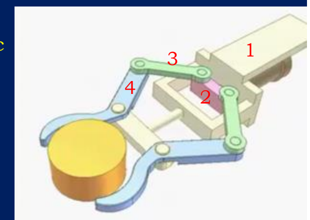
$F = 3(6 - 7 - 1) + 7$
$F = 1$



Therefore just the piston 2 in fixed cylinder through P_{12} is sufficient to control the mechanism.

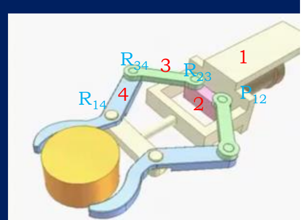
Degree of Freedom of Mechanisms

Alternative method:
Two grippers are symmetric
Just consider one of them
λ = 3 (co-planar)
ℓ = 4



Degree of Freedom of Mechanisms

Alternative method:
Two grippers are symmetric
Just consider one of them
λ = 3 (co-planar)
ℓ = 4
j = 4

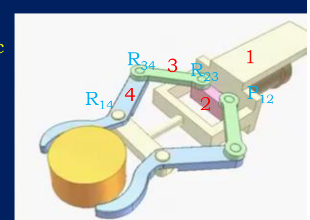


Degree of Freedom of Mechanisms

Alternative method:
Two grippers are symmetric
Just consider one of them
λ = 3 (co-planar)
ℓ = 4
j = 4
$\sum_{i=1}^{4} f_i$ = 4 (3R+P)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

$F = 3(4 - 4 - 1) + 4$
$F = 1$

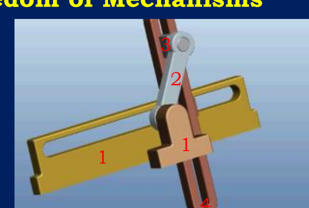


Therefore **still** just the piston 2 in fixed cylinder through P_{12} is sufficient to control the mechanism.

Degree of Freedom of Mechanisms

Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 4

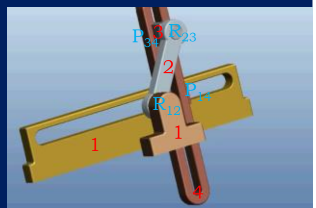


Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 4
j = 4
$\sum_{i=1}^{4} f_i$ = 4 (2R+2P)

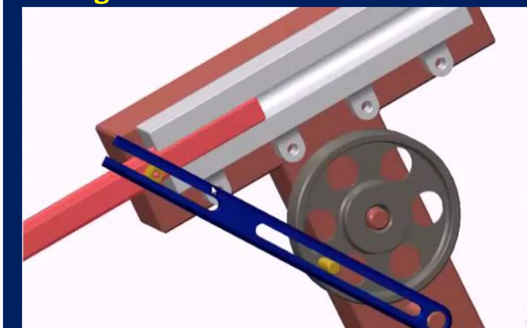
$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

$F = 3(4 - 4 - 1) + 4$
$F = 1$



Therefore controlling crank 2 through R_{12} is sufficient to control the Scotch yoke mechanism.

Degree of Freedom of Mechanisms

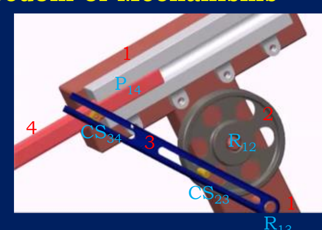


Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 4

Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 4
j = 5
$\sum_{i=1}^{5} f_i$ = 7 (2CS+2R+P)

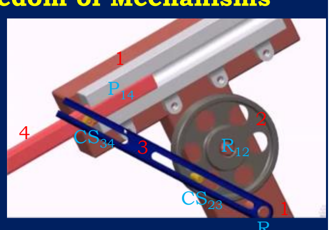


Degree of Freedom of Mechanisms

λ = 3 (co-planar)
ℓ = 4
j = 5
$\sum_{i=1}^{5} f_i$ = 7 (2CS+2R+P)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

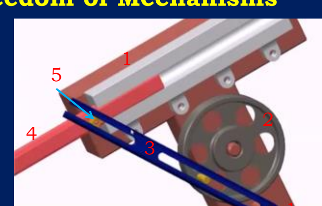
$F = 3(4 - 5 - 1) + 7$
$F = 1$



Therefore controlling crank 2 through R_{12} is sufficient to control the quick return mechanism.

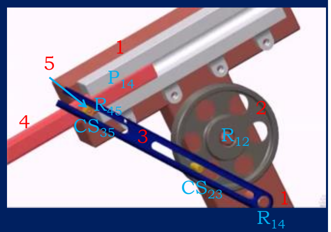
Degree of Freedom of Mechanisms

Alternative approach:
λ = 3 (co-planar)
ℓ = 5



Degree of Freedom of Mechanisms

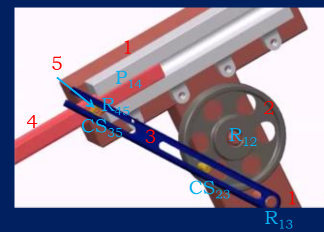
Alternative approach:
λ = 3 (co-planar)
ℓ = 5
j = 6
$\sum_{i=1}^{6} f_i$ = 8 (2CS+3R+P)



Degree of Freedom of Mechanisms

Alternative approach:
λ = 3 (co-planar)
ℓ = 5
j = 6
$\sum_{i=1}^{6} f_i$ = 8 (2CS+3R+P)

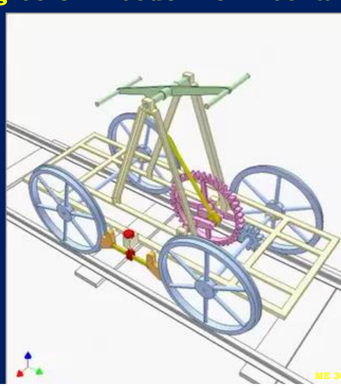
$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$
$$F = 3(5 - 6 - 1) + 8$$
$$F = 2$$



Therefore by controlling crank 2 through R_{12} although we can still control the quick return mechanism the angular position of roller 5 is another free parameter that cannot be controlled. This is called a *redundant freedom* that has no effect on overall motion of the mechanism therefore the actual degree of freedom is:

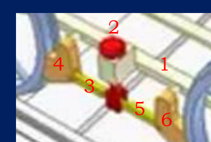
$$F_{actual} = 1$$

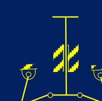
Degree of Freedom of Mechanisms



Degree of Freedom of Mechanisms

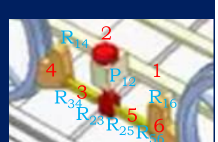
The Brakes (Approach 1):
λ = 3 (co-planar)
ℓ = 6

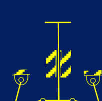




Degree of Freedom of Mechanisms

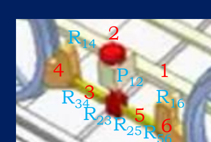
The Brakes (Approach 1):
λ = 3 (co-planar)
ℓ = 6
j = 7
$\sum_{i=1}^{7} f_i$ = 7 (6R+P)





Degree of Freedom of Mechanisms

The Brakes (Approach 1):
λ = 3 (co-planar)
ℓ = 6
j = 7
$\sum_{i=1}^{7} f_i$ = 7 (6R+P)

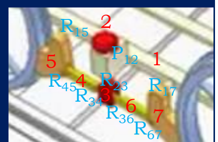


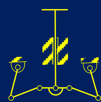
$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$
$$F = 3(6 - 7 - 1) + 7$$
$$F = 1$$

Therefore by controlling the brake "button" 2 through P_{12} we can apply the brakes to both wheels.

Degree of Freedom of Mechanisms

The Brakes (Approach 2):
λ = 3 (co-planar)
ℓ = 7
j = 8
$\sum_{i=1}^{8} f_i$ = 8 (7R+P)



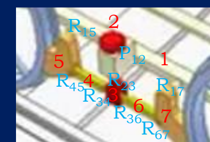


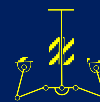
Degree of Freedom of Mechanisms

The Brakes (Approach 2):
λ = 3 (co-planar)
ℓ = 7
j = 8
$\sum_{i=1}^{8} f_i$ = 8 (7R+P)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

$F = 3(7 - 8 - 1) + 8$
$F = 2$!

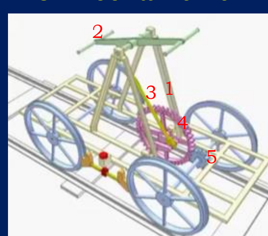




Therefore by controlling the brake "button" 2 through P_{12} we can still apply the brakes to both wheels. However if the wear in shoes are uneven (typical in real life) say just by applying brake to rear wheel by shoe 5 would not stop the mechanism but also shoe 7 will apply the brake. That sort of mechanisms where F > number of actuators is called underactuated mechanisms. A very typical example is car differential where F = 2 but the only actuation is by the engine.

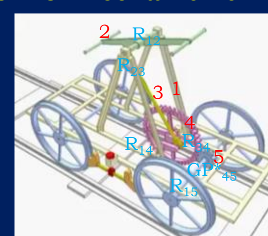
Degree of Freedom of Mechanisms

The Drive:
λ = 3 (co-planar)
ℓ = 5



Degree of Freedom of Mechanisms

The Drive:
λ = 3 (co-planar)
ℓ = 5
j = 6
$\sum_{i=1}^{6} f_i$ = 7 (5R+GP*)

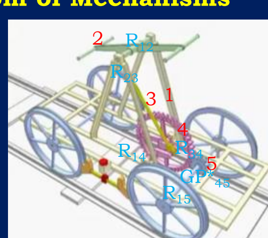


Degree of Freedom of Mechanisms

The Drive:
λ = 3 (co-planar)
ℓ = 5
j = 6
$\sum_{i=1}^{6} f_i$ = 7 (5R+GP*)

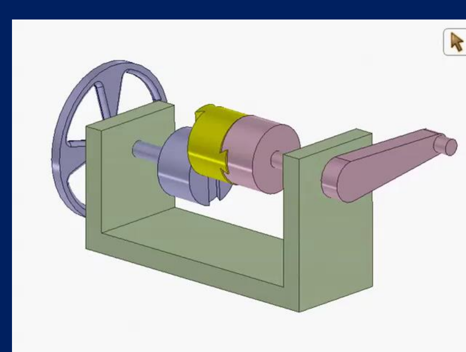
$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

$F = 3(5 - 6 - 1) + 7$
$F = 1$



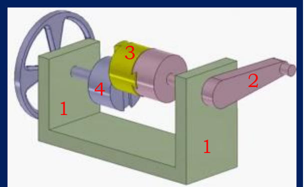
Therefore by moving the crank 2 up and down the axle of the vehicle can be powered to inspect the railroad.

Degree of Freedom of Mechanisms



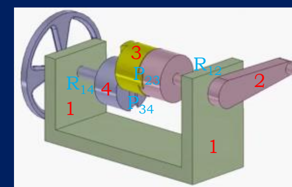
Degree of Freedom of Mechanisms

Oldham Coupling:
λ = 3 (co-planar)
ℓ = 4



Degree of Freedom of Mechanisms

Oldham Coupling:
λ = 3 (co-planar)
ℓ = 4
j = 4
$\sum_{i=1}^{4} f_i$ = 4 (2R+2P)

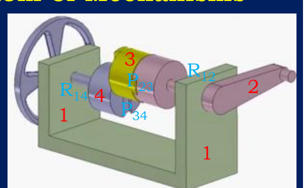


Degree of Freedom of Mechanisms

Oldham Coupling:
λ = 3 (co-planar)
ℓ = 4
j = 4
$\sum_{i=1}^{4} f_i$ = 4 (2R+2P)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

$F = 3(4 - 4 - 1) + 4$
$F = 1$
Therefore by driving the crank 2 the pulley 4 can be driven.



Degree of Freedom of Mechanisms

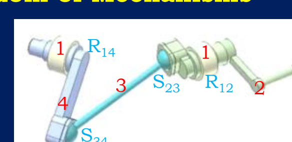
Degree of Freedom of Mechanisms

Spatial Four Bar:
λ = 6 (3-D)
ℓ = 4

Degree of Freedom of Mechanisms

Spatial Four Bar:
λ = 6 (3-D)
ℓ = 4
j = 4
$\sum_{i=1}^{4} f_i$ = 8 (2R+2S)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$



Degree of Freedom of Mechanisms

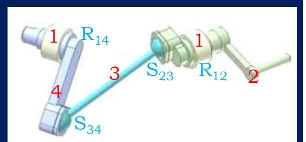
Spatial Four Bar:

$\lambda = 6$ (3-D)

$\ell = 4$

j = 4

$\sum_{i=1}^{4} f_i = 8$ (2R+2S)



$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

$F = 6(4 - 4 - 1) + 8$

$F = 2$

There is one uncontrolled motion which is the rotation of link 3 (the coupler) about its own axis due to the spherical joints S23 and S34. This is a redundant freedom and if ignored the position of follower 4 can be controlled by the position of crank 2 so $F_{actual} = 1$.