2. Kinematic Analysis

Representation of Plane Vectors by Complex Numbers

Review of Complex Numbers

- -1 is an operator that rotates a vector 180°.
- The unit imaginary number i ($i^2 \equiv -1$) is an operator that rotates a vector 90° counter clockwise.
  - *Please note that twice 90° counter clockwise rotation which is 180° is $i^2 = -1$!*
  - $-i = \frac{1}{i}$ *rotates a vector 90° clockwise.*
- The complex plane (also called Gauss-Argand plane) is analogous to the two dimensional Cartesian coordinates (x → Re, y → Im)
  - $\vec{r} = x\hat{\imath} + y\hat{\jmath} \rightarrow z = x + iy$
- Complex numbers can be represented in polar form as well:
  - $z = x + iy = r(cos\theta + isin\theta), r = \sqrt{x^2+y^2}, \theta = Pol(x,y) \neq tan^{-1}\left(\frac{y}{x}\right), \begin{cases} Pol(x,y) = r,\theta \\ Rec(r,\theta) = x,y \end{cases}$
  - Euler's identity $e^{i\theta} = cos\theta + isin\theta$ so $z = re^{i\theta}$
  - Multiplication of a real number, r, with $e^{i\theta}$ rotates the real number θ counter clockwise.
- Vector addition and addition of two complex numbers are analogous.
- In complex numbers we **do not** need cross product or out of plane angular vectors like $\vec{\omega}$ and $\vec{\alpha}$!

2. Kinematic Analysis

Kinematics of Rigid Body in Plane Motion

- Motion of a rigid body in plane can be described fully by the motion of two points on the plane.
- Rigidity condition ensures that the velocity components of the two selected points along the line connecting the two points should be equal.
- It is sufficient to represent a rigid body (which may be considered as an infinite plane) by the two representative points and the line connecting them.

2. Kinematic Analysis

Coincident Points

Permanently Coincident Points: Two points on two different rigid bodies are coincident for all possible positions of the mechanism.

Typically the points on the axis of a revolute joint which connects two rigid bodies are permanently coincident.

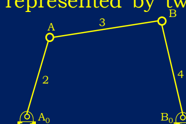
Instantly Coincident Points: Two points on two different rigid bodies are coincident only for the current position of the mechanism.

Typically the instant center of zero velocity of a link does not have a fixed location with respect to another link including the fixed link (and also it does not have a fixed location relative to its own body).

2. Kinematic Analysis

Vector Loops of Mechanisms (*Constraint Equations*!)

The three moving bodies are each represented by two points on them:



Link 2: A_0A

Link 3: AB

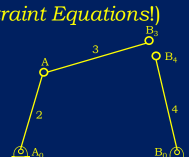
Link 4: B_0B

Rather than searching for tedious geometric relations as we did in ME 208 Dynamics we will assume one of the permanently coincident points to be *non-coincident* and the constraint equation we will write is going to *force* these two points to be coincident. This will be a vector loop equation and the constraint equation for the mechanism.

2. Kinematic Analysis

Vector Loops of Mechanisms (*Constraint Equations*!)

Suppose we select point B:



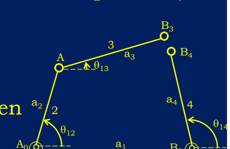
$\overrightarrow{A_0A} + \overrightarrow{AB_3} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B_4}$

This vector equation forces B_3 and B_4 be a permanently coincident point, the revolute joint between links 3 and 4, B. It is called *loop closure equation* and is the constraint equation of the four-bar mechanism.

2. Kinematic Analysis

Vector Loops of Mechanisms (*Constraint Equations*!)

$\overrightarrow{A_0A} + \overrightarrow{AB_3} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B_4}$



This vector equation can be written using complex numbers as:

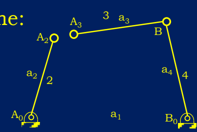
$a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} = a_1 + a_4e^{i\theta_{14}}$

This is a *complex* equation in three ***real*** unknowns, θ_{12}, θ_{13} and θ_{14}. If one of those variables (recall F =1 for a four-bar) is known the other two can be determined (*as we will see later!*).

2. Kinematic Analysis

Vector Loops of Mechanisms (*Constraint Equations*!)

Suppose we select point A this time:



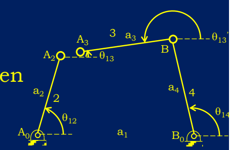
$\overrightarrow{A_0A_2} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B} + \overrightarrow{BA_3}$

This vector equation forces A_2 and A_3 be a permanently coincident point, the revolute joint between links 2 and 3, A. It looks different from the previous equation where we disconnected revolute joint B and force it to be permanently coincident.

$\overrightarrow{A_0A} + \overrightarrow{AB_3} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B_4}$

2. Kinematic Analysis

Vector Loops of Mechanisms (*Constraint Equations*!)



$\overrightarrow{A_0A_2} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B} + \overrightarrow{BA_3}$

This vector equation can be written using complex numbers as:

$a_2e^{i\theta_{12}} = a_1 + a_4e^{i\theta_{14}} + a_3e^{i\theta_{13}'}$

$\theta_{13}' = \theta_{13} + \pi$

$a_2e^{i\theta_{12}} = a_1 + a_4e^{i\theta_{14}} + a_3e^{i(\theta_{13}+\pi)}$

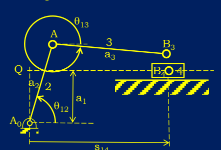
$e^{i(\theta_{13}+\pi)} = e^{i\theta_{13}}e^{i\pi} = -e^{i\theta_{13}}$

$a_2e^{i\theta_{12}} = a_1 + a_4e^{i\theta_{14}} - a_3e^{i\theta_{13}}$

Identical equation with when B is disconnected!

2. Kinematic Analysis

Slider-Crank



$\overrightarrow{A_0A} + \overrightarrow{AB_3} = \overrightarrow{A_0B_4}$

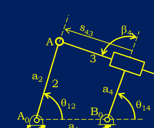
This vector equation forces B_3 and B_4 be a permanently coincident point, the revolute joint between links 3 and 4, B. This vector equation can be written using complex numbers as:

$a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} = ia_1 + s_{14}$

This is a *complex* equation in three ***real*** unknowns, θ_{12}, θ_{13} and s_{14}. If one of those variables (recall F =1 for a slider-crank) is known the other two can be determined.

2. Kinematic Analysis

Inverted Slider-Crank



$\overrightarrow{A_0A_2} = \overrightarrow{A_0B_0} + \overrightarrow{B_0A_3}$

This vector equation forces A_2 and A_3 be a permanently coincident point, the revolute joint between links 2 and 3, A. This vector equation can be written using complex numbers as:

$a_2e^{i\theta_{12}} = a_1 + a_4e^{i\theta_{14}} + s_{43}e^{i(\theta_{14}+\beta_4)}$

This is a *complex* equation in three ***real*** unknowns, θ_{12}, θ_{14} and s_{43}. If one of those variables (recall F =1 for an inverted slider-crank) is known the other two can be determined.

2. Kinematic Analysis

Vectors Allowed in a Loop Closure Equation

1. Body Vectors: They are the vectors that connect the two points on the *same* link. These vectors have a constant magnitude but the orientation may change.
2. Translational Joint Variable Vectors: They are the vectors between two links which are connected by a prismatic or cylinder in slot joint. These vectors are parallel to the relative sliding axis and have a variable magnitude. Direction may be variable as well.
3. Zero Vectors: They are the vectors connecting two permanently coincident points on two different links. *They are not written!*

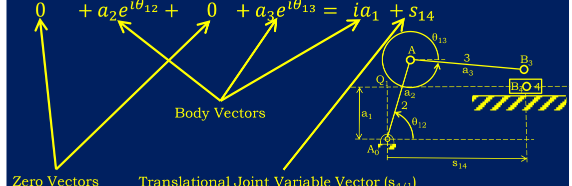
This definition is due to Reşit Soylu, Department of Mechanical Engineering, Middle East Technical University.

2. Kinematic Analysis

Slider-Crank

$\overrightarrow{A_{0_1}A_{0_2}} + \overrightarrow{A_0A_2} + \overrightarrow{A_2A_3} + \overrightarrow{A_3B_3} = \overrightarrow{A_0Q_1} + \overrightarrow{Q_1B_4}$

$0 + a_2e^{i\theta_{12}} + 0 + a_3e^{i\theta_{13}} = ia_1 + s_{14}$

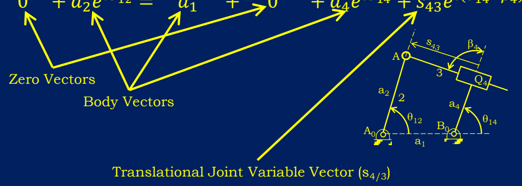


2. Kinematic Analysis

Inverted Slider-Crank

$$\overrightarrow{A_{0_1}A_{0_2}} + \overrightarrow{A_{0_2}A_2} = \overrightarrow{A_{0_1}B_{0_1}} + \overrightarrow{B_{0_1}B_{0_4}} + \overrightarrow{B_{0_4}Q_4} + \overrightarrow{Q_4A_3}$$

$$0 \quad + a_2e^{i\theta_{12}} = \quad a_1 \quad + \quad 0 \quad + a_4e^{i\theta_{14}} + s_{43}e^{i(\theta_{14}+\beta_4)}$$



2. Kinematic Analysis

Multiloop Mechanisms

There are many loops like:

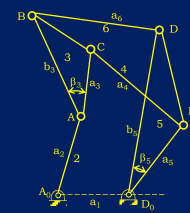
$\overrightarrow{A_0A} + \overrightarrow{AB} + \overrightarrow{BD_6} = \overrightarrow{A_0D_0} + \overrightarrow{D_0D_5}$

$\overrightarrow{A_0A} + \overrightarrow{AC} + \overrightarrow{CE_4} = \overrightarrow{A_0D_0} + \overrightarrow{D_0E_5}$

$\overrightarrow{CE} + \overrightarrow{ED_5} = \overrightarrow{CB} + \overrightarrow{BD_6}$

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

$\overrightarrow{D_0D} + \overrightarrow{DE} + \overrightarrow{ED_0} = \vec{0}$



*The question is which of these equations are **independent** (i.e. contain new information)?*

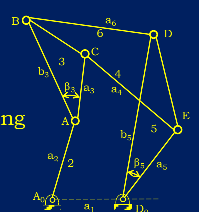
2. Kinematic Analysis

Multiloop Mechanisms

Euler's polyhedron formula tells the number of independent loops as:

$L = j - \ell + 1 = 7 - 6 + 1 = 2$

Next question is which of the following five equations are independent?



$\overrightarrow{A_0A} + \overrightarrow{AB} + \overrightarrow{BD_6} = \overrightarrow{A_0D_0} + \overrightarrow{D_0D_5}$ (1)

$\overrightarrow{A_0A} + \overrightarrow{AC} + \overrightarrow{CE_4} = \overrightarrow{A_0D_0} + \overrightarrow{D_0E_5}$ (2)

$\overrightarrow{CE} + \overrightarrow{ED_5} = \overrightarrow{CB} + \overrightarrow{BD_6}$ (3)

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ (4)

$\overrightarrow{D_0D} + \overrightarrow{DE} + \overrightarrow{ED_0} = \vec{0}$ (5)

2. Kinematic Analysis

Multiloop Mechanisms

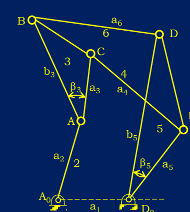
$\overrightarrow{A_0A} + \overrightarrow{AB} + \overrightarrow{BD_6} = \overrightarrow{A_0D_0} + \overrightarrow{D_0D_5}$ (1)

$\overrightarrow{A_0A} + \overrightarrow{AC} + \overrightarrow{CE_4} = \overrightarrow{A_0D_0} + \overrightarrow{D_0E_5}$ (2)

$\overrightarrow{CE} + \overrightarrow{ED_5} = \overrightarrow{CB} + \overrightarrow{BD_6}$ (3)

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ (4)

$\overrightarrow{D_0D} + \overrightarrow{DE} + \overrightarrow{ED_0} = \vec{0}$ (5)



Equations 4 & 5 are identities.

Equations 1, 2 and 3 are valid loop closure equations. However they are not **all** independent.

Is there a way to find out independent loop closure equations?

2. Kinematic Analysis

Multiloop Mechanisms

Is there a way to find out independent loop closure equations?

Yes, there is!

1. Disconnect gear pair(s) (if any) and write the gear relation(s).
2. Disconnect as many **revolute joints** as necessary to eliminate **all** loops. (*However no link should be totally disconnected*!)
3. By connecting **only one joint at a time** (*all others should be disconnected during this process*) write the loop formed by connecting this joint.

2. Kinematic Analysis

Multiloop Mechanisms

1. Disconnect gear pairs (if any) and write the gear relations.

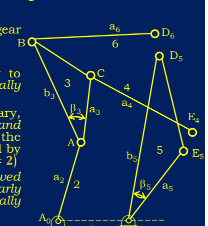
   No gears!

2. Disconnect as many **revolute joints** as necessary to eliminate **all** loops. (*However no link should be totally disconnected*!)

   Let's disconnect D and E (selection is totally arbitrary, *you could as well select (B and C) or (B and E) or (D and C) or (A and C) or (A and E) etc.* however in all cases the number of joints to be disconnected is **2** as predicted by Euler's polyhedron formula: $L = j - \ell + 1 = 7 - 6 + 1 = 2$)

   Please note that disconnecting B and D is not allowed since link 6 becomes totally disconnected. Similarly disconnecting C and E will make link 4 totally disconnected therefore not allowed!

3. By *re*connecting **only one joint at a time** (*all others should be disconnected during this process*) write the loop formed by connecting this joint.



Reconnect D (*E is disconnected!*)

$\overrightarrow{A_0A} + \overrightarrow{AB} + \overrightarrow{BD_6} = \overrightarrow{A_0D_0} + \overrightarrow{D_0D_5}$

Reconnect E (*D is disconnected!*)

$\overrightarrow{A_0A} + \overrightarrow{AC} + \overrightarrow{CE_4} = \overrightarrow{A_0D_0} + \overrightarrow{D_0E_5}$

Two *possible* independent loop closure equations.

2. Kinematic Analysis

Multiloop Mechanisms

$$\overrightarrow{A_0A} + \overrightarrow{AB} + \overrightarrow{BD_6} = \overrightarrow{A_0D_0} + \overrightarrow{D_0D_5}$$

$$a_2e^{i\theta_{12}} + b_3e^{i(\theta_{13}+\beta_3)} + a_6e^{i\theta_{16}} = a_1 + b_5e^{i(\theta_{15}+\beta_5)}$$

$$\overrightarrow{A_0A} + \overrightarrow{AC} + \overrightarrow{CE_4} = \overrightarrow{A_0D_0} + \overrightarrow{D_0E_5}$$

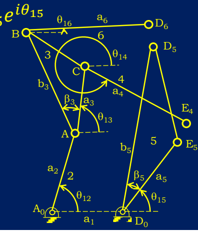
$$a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} + a_4e^{i\theta_{14}} = a_1 + a_5e^{i\theta_{15}}$$

Variables: $\theta_{12}, \theta_{13}, \theta_{14}, \theta_{15}$ and θ_{16} (5)

Constraints: Two complex equations (i.e. four scalar equations)

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i = 3(6-7-1) + 7 = 1$$

$$V = 2L + F$$



2. Kinematic Analysis

Multiloop Mechanisms

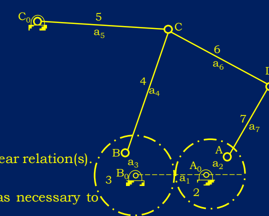
$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

$$\ell = 7$$

$$j = 9\ (8R + GP^*), \sum_{i=1}^{9} f_i = 10$$

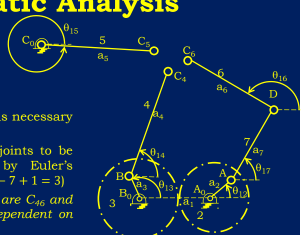
$$F = 3(7-9-1) + 10 = 1$$

1. Disconnect gear pair(s) and write the gear relation(s).
   $r_3\theta_{13} = -r_2(\theta_{12} - \beta_2)$
2. Disconnect as many **revolute joints** as necessary to eliminate **all** loops.
   Let's disconnect C. The number of joints to be disconnected is 3 as predicted by Euler's polyhedron formula: $L = j - \ell + 1 = 9 - 7 + 1 = 3$
   GP is one of the joints the other two are C_{46} and C_{56} (please note that joint C_{45} is dependent on other two!)*
3. By connecting **only one joint at a time** (*all others should be disconnected during this process*) write the loop formed by connecting this joint.



2. Kinematic Analysis

Multiloop Mechanisms



1. Disconnect gear pair(s) and write the gear relation(s).
   $r_3\theta_{13} = -r_2(\theta_{12} - \beta_2)$
2. Disconnect as many **revolute joints** as necessary to eliminate **all** loops.
   Let's disconnect C. The number of joints to be disconnected is 3 as predicted by Euler's polyhedron formula: $L = j - \ell + 1 = 9 - 7 + 1 = 3$)
   GP is one of the joints the other two are C_{46} and C_{56} (please note that joint C_{45} is dependent on other two!)*
3. By connecting **only one joint at a time** (*all others should be disconnected during this process*) write the loop formed by connecting this joint.

Reconnect C_{46} (*C_{56} and GP^*_{23} are disconnected*)

$$\overrightarrow{A_0A} + \overrightarrow{AD} + \overrightarrow{DC_6} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B} + \overrightarrow{BC_4}$$

Reconnect C_{56} (*C_{46} and GP^*_{23} are disconnected*)

$$\overrightarrow{A_0A} + \overrightarrow{AD} + \overrightarrow{DC_6} = \overrightarrow{A_0C_0} + \overrightarrow{C_0C_5}$$

2. Kinematic Analysis

Multiloop Mechanisms

$$r_3\theta_{13} = -r_2(\theta_{12} - \beta_2)$$

$$\overrightarrow{A_0A} + \overrightarrow{AD} + \overrightarrow{DC_6} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B} + \overrightarrow{BC_4}$$

$$a_2e^{i\theta_{12}} + a_7e^{i\theta_{17}} + a_6e^{i\theta_{16}} = -a_1 + a_3e^{i\theta_{13}} + a_4e^{i\theta_{14}}$$

$$\overrightarrow{A_0A} + \overrightarrow{AD} + \overrightarrow{DC_6} = \overrightarrow{A_0C_0} + \overrightarrow{C_0C_5}$$

$$a_2e^{i\theta_{12}} + a_7e^{i\theta_{17}} + a_6e^{i\theta_{16}} = -(a_1 + b_1) + ic_1 + a_5e^{i\theta_{15}}$$

V: $\theta_{12}, \theta_{13}, \theta_{14}, \theta_{15}, \theta_{16}$ and θ_{17} (6)

Equations: 5 (2 Complex LCE + Gear Relation)

F = 1

$V \neq 2L + F$