2. Kinematic Analysis

Solution of Loop Closure Equations

1. Graphical Solution
2. Stepwise Solution
3. Analytic – Closed Form Solution
4. Numerical Solution

2. Kinematic Analysis

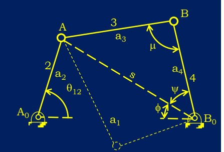
2. Stepwise Solution of Loop Closure Equations

Law of cosines:

$$s^2 = a_1{}^2 + a_2{}^2 - 2a_1a_1cos\theta_{12}$$

$$s^2 = a_3{}^2 + a_4{}^2 - 2a_3a_4cos\mu \rightarrow$$

$$cos\mu = \frac{a_3{}^2 + a_4{}^2 - s^2}{2a_3a_4}$$



similarly

$$cos\phi = \frac{a_1{}^2 + s^2 - a_2{}^2}{2a_1s}, cos\psi = \frac{s^2 + a_4{}^2 - a_3{}^2}{2a_4s}$$

$$\psi = \pm cos^{-1}\left(\frac{s^2 + a_4{}^2 - a_3{}^2}{2a_4s}\right)$$

Therefore angular orientations of all links can be determined.

2. Kinematic Analysis

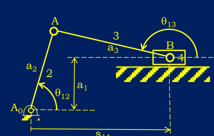
3. Analytic–Closed Form Solution of Loop Closure Equations

Slider-Crank Mechanism

Disconnect and reconnect A

$$\overrightarrow{A_0A_2} = \overrightarrow{A_0B} + \overrightarrow{BA_3}$$

$$a_2e^{i\theta_{12}} = ia_1 + s_{14} + a_3e^{i\theta_{13}}$$



Re: $a_2cos\theta_{12} = s_{14} + a_3cos\theta_{13}$

Im: $a_2sin\theta_{12} = a_1 + a_3sin\theta_{13}$

These derivations are due to M. Kemal Özgören, Department of Mechanical Engineering, Middle East Technical University.

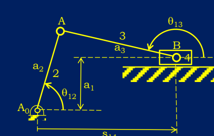
2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Slider-Crank Mechanism

Re: $a_2cos\theta_{12} = s_{14} + a_3cos\theta_{13}$ (1)

Im: $a_2sin\theta_{12} = a_1 + a_3sin\theta_{13}$ (2)



1. Let θ_{12} be the input
   a. First solve s_{14} then θ_{13}

$(1) \rightarrow a_2cos\theta_{12} - s_{14} = a_3cos\theta_{13}$

$(2) \rightarrow a_2sin\theta_{12} - a_1 = a_3sin\theta_{13}$

$(1)^2 + (2)^2$: $(a_2\,cos\theta_{12} - s_{14})^2 + (a_2\,sin\theta_{12} - a_1)^2 = a_3{}^2$

$$s_{14}{}^2 - 2a_2cos\theta_{12}s_{14} + a_1{}^2 + a_2{}^2 - a_3{}^2 - 2a_1a_2sin\theta_{12} = 0$$

$$s_{14} = a_2cos\theta_{12} + \sigma\sqrt{(a_2cos\theta_{12})^2 - (a_1{}^2 + a_2{}^2 - a_3{}^2 - 2a_1a_2sin\theta_{12})}$$

$\sigma = \pm 1$

2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Slider-Crank Mechanism

1. Let θ_{12} be the input
   a. First solve s_{14} then θ_{13} (cont'ed)

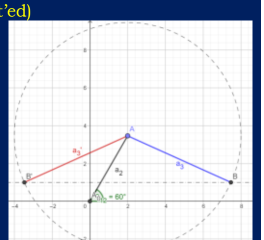
$(1) \rightarrow a_2cos\theta_{12} - s_{14} = a_3cos\theta_{13}$

$(2) \rightarrow a_2sin\theta_{12} - a_1 = a_3sin\theta_{13}$

$$cos\theta_{13} = \frac{a_2cos\theta_{12} - s_{14}}{a_3} = x_{13}$$

$$sin\theta_{13} = \frac{a_2sin\theta_{12} - a_1}{a_3} = y_{13}$$

$$\theta_{13} = Pol(x_{13}, y_{13})$$

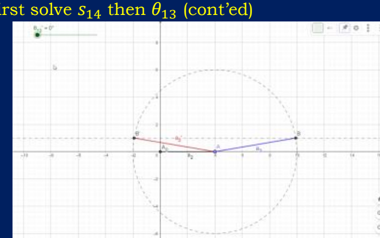


2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Slider-Crank Mechanism

1. Let θ_{12} be the input
   a. First solve s_{14} then θ_{13} (cont'ed)



2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Slider-Crank Mechanism

Re: $a_2 cos\theta_{12} = s_{14} + a_3 cos\theta_{13}$ (1)

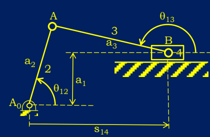
Im: $a_2 sin\theta_{12} = a_1 + a_3 sin\theta_{13}$ (2)

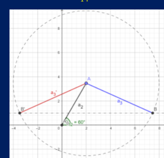
1. Let θ_{12} be the input
   b. First solve θ_{13} then s_{14}

$(2) \rightarrow sin\theta_{13} = \frac{a_2 sin\theta_{12} - a_1}{a_3} = y_{13}$

$cos\theta_{13} = \sigma\sqrt{1 - sin^2\theta_{13}} = \sigma\sqrt{1 - y_{13}{}^2}, \sigma = \pm 1$

$(1) \rightarrow s_{14} = a_2 cos\theta_{12} - a_3 cos\theta_{13}$





2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Slider-Crank Mechanism

Re: $a_2 cos\theta_{12} = s_{14} + a_3 cos\theta_{13}$ (1)

Im: $a_2 sin\theta_{12} = a_1 + a_3 sin\theta_{13}$ (2)

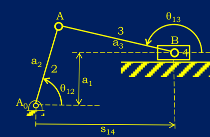
2. Let s_{14} be the input
   a. First solve θ_{12} then θ_{13}

$(1) \rightarrow a_2 cos\theta_{12} - s_{14} = a_3 cos\theta_{13}$

$(2) \rightarrow a_2 sin\theta_{12} - a_1 = a_3 sin\theta_{13}$

$(1)^2 + (2)^2 : (a_2\, cos\theta_{12} - s_{14})^2 + (a_2\, sin\theta_{12} - a_1)^2 = a_3{}^2$

$s_{14}{}^2 - 2a_2 cos\theta_{12} s_{14} + a_1{}^2 + a_2{}^2 - a_3{}^2 - 2a_1 a_2 sin\theta_{12} = 0$



2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

2. Let s_{14} be the input
   a. First solve θ_{12} then θ_{13}

$s_{14}{}^2 - 2a_2 cos\theta_{12} s_{14} + a_1{}^2 + a_2{}^2 - a_3{}^2 - 2a_1 a_2 sin\theta_{12} = 0$ (3)

i. Half-tangent method:

$t_{12} = tan\left(\frac{\theta_{12}}{2}\right), sin\theta_{12} = \frac{2t_{12}}{1 + t_{12}{}^2}$ and $cos\theta_{12} = \frac{1 - t_{12}{}^2}{1 + t_{12}{}^2}$

$(3) \rightarrow s_{14}{}^2 - 2a_2 s_{14}\frac{1 - t_{12}{}^2}{1 + t_{12}{}^2} + a_1{}^2 + a_2{}^2 - a_3{}^2 - 2a_1 a_2 \frac{2t_{12}}{1 + t_{12}{}^2} = 0$

$(a_1{}^2 + a_2{}^2 - a_3{}^2 + s_{14}{}^2 + 2a_2 s_{14})t_{12}{}^2 - 4a_1 a_2 t_{12} + (a_1{}^2 + a_2{}^2 - a_3{}^2 + s_{14}{}^2 - 2a_2 s_{14}) = 0$

$At_{12}{}^2 + Bt_{12} + C = 0$

$t_{12} = \frac{-B + \sigma\sqrt{B^2 - 4AC}}{2A}, \sigma = \pm 1$

$\theta_{12} = 2tan^{-1} t_{12}$

2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

2. Let s_{14} be the input
   b. First solve θ_{12} then θ_{13}

$s_{14}{}^2 - 2a_2 cos\theta_{12} s_{14} + a_1{}^2 + a_2{}^2 - a_3{}^2 - 2a_1 a_2 sin\theta_{12} = 0$ (3)

ii. Phase angle method:

$2a_2 s_{14} cos\theta_{12} + 2a_1 a_2 sin\theta_{12} = s_{14}{}^2 + a_1{}^2 + a_2{}^2 - a_3{}^2$

$acos\theta_{12} + bsin\theta_{12} = c$

$r > 0, a = rcos\phi, b = rsin\phi$

$r, \phi = Pol(a, b)$

$rcos\phi cos\theta_{12} + rsin\phi sin\theta_{12} = c$

$rcos(\theta_{12} - \phi) = c, \theta_{12} = \phi + \sigma cos^{-1}\left(\frac{c}{r}\right), \sigma = \pm 1$

2. Kinematic Analysis

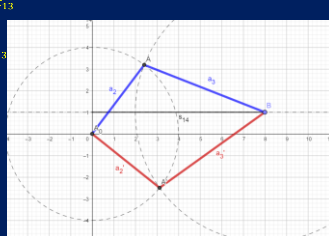
3. Analytic–Closed Form Solution of Loop Closure Equations

2. Let s_{14} be the input
   a. First solve θ_{12} then θ_{13} (cont'ed)

$cos\theta_{13} = \frac{a_2 cos\theta_{12} - s_{14}}{a_3} = x_{13}$

$sin\theta_{13} = \frac{a_2 sin\theta_{12} - a_1}{a_3} = y_{13}$

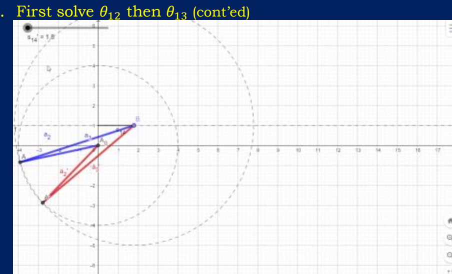
$\theta_{13} = Pol(x_{13}, y_{13})$



2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

2. Let s_{14} be the input
   a. First solve θ_{12} then θ_{13} (cont'ed)



2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

2. Let s_{14} be the input

   b. First solve θ_{13} then θ_{12}

   $a_2 cos\theta_{12} = s_{14} + a_3 cos\theta_{13}$ (1)

   $a_2 sin\theta_{12} = a_1 + a_3 sin\theta_{13}$ (2)

   $(1)^2 + (2)^2: 2a_3 s_{14} cos\theta_{13} + 2a_1 a_3 sin\theta_{13} = -({a_1}^2 - {a_2}^2 + {a_3}^2 + {s_{14}}^2)$

   Either use half-tangent or phase angle to find θ_{13}

   $cos\theta_{12} = \frac{s_{14} + a_3 cos\theta_{13}}{a_2} = x_{12}$

   $sin\theta_{12} = \frac{a_1 + a_3 sin\theta_{13}}{a_2} = y_{12}$

   $\theta_{12} = Pol(x_{12}, y_{12})$

2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Inverted Slider-Crank Mechanism

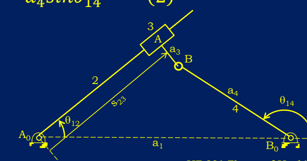
Disconnect and reconnect B

$\overrightarrow{A_0A} \;+\; \overrightarrow{AB_3} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B_4}$

$s_{23}e^{i\theta_{12}} + a_3 e^{i\left(\theta_{12}-\frac{\pi}{2}\right)} = a_1 + a_4 e^{i\theta_{14}}$

Re: $s_{23}cos\theta_{12} + a_3 sin\theta_{12} = a_1 + a_4 cos\theta_{14}$(1)

Im: $s_{23}sin\theta_{12} - a_3 cos\theta_{12} = a_4 sin\theta_{14}$ (2)



2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Inverted Slider-Crank Mechanism

$s_{23}e^{i\theta_{12}} + a_3 e^{i\left(\theta_{12}-\frac{\pi}{2}\right)} = a_1 + a_4 e^{i\theta_{14}}$

Re: $s_{23}cos\theta_{12} + a_3 sin\theta_{12} = a_1 + a_4 cos\theta_{14}$(1)

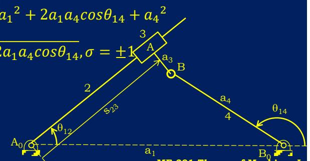
Im: $s_{23}sin\theta_{12} - a_3 cos\theta_{12} = a_4 sin\theta_{14}$ (2)

1. Let θ_{14} be the input

   a. First solve s_{23} then θ_{12}

   $(1)^2 + (2)^2 \rightarrow {s_{23}}^2 + {a_3}^2 = {a_1}^2 + 2a_1 a_4 cos\theta_{14} + {a_4}^2$

   $s_{23} = \sigma\sqrt{{a_1}^2 - {a_3}^2 + {a_4}^2 + 2a_1 a_4 cos\theta_{14}}, \sigma = \pm 1$



2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Inverted Slider-Crank Mechanism

1. Let θ_{14} be the input

   a. First solve s_{23} then θ_{12} (cont'ed)

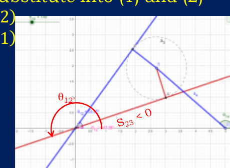
   Re: $s_{23}cos\theta_{12} + a_3 sin\theta_{12} = a_1 + a_4 cos\theta_{14}$(1)

   Im: $s_{23}sin\theta_{12} - a_3 cos\theta_{12} = a_4 sin\theta_{14}$ (2)

   Let $cos\theta_{12} = x_{12}$ and $sin\theta_{12} = y_{12}$, substitute into (1) and (2)

   $\begin{bmatrix} s_{23} & -a_3 \\ a_3 & s_{23} \end{bmatrix}\begin{bmatrix} x_{12} \\ y_{12} \end{bmatrix} = \begin{bmatrix} a_4 sin\theta_{14} \\ a_1 + a_4 cos\theta_{14} \end{bmatrix}\begin{matrix}(2)\\(1)\end{matrix}$

   $\theta_{12} = Pol(x_{12}, y_{12})$



2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Inverted Slider-Crank Mechanism

1. Let θ_{14} be the input