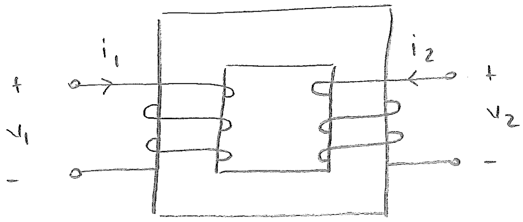
Coupled Inductors

When two or more inductors are brought close or wound on the same core they start affecting each other. They become coupled.



uncoupled	coupled
$v_1 = L_1 \frac{di_1}{dt}$	$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$
$v_2 = L_2 \frac{di_2}{dt}$	$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$

In matrix form:

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\substack{v \\ \text{voltage vector}}} = \underbrace{\begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}}_{\substack{L \\ \text{inductance matrix}}} \cdot \frac{d}{dt} \underbrace{\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}_{\substack{i \\ \text{current vector}}} \Rightarrow v = L \frac{di}{dt}$$

→ L_1, L_2 are called "self inductance"

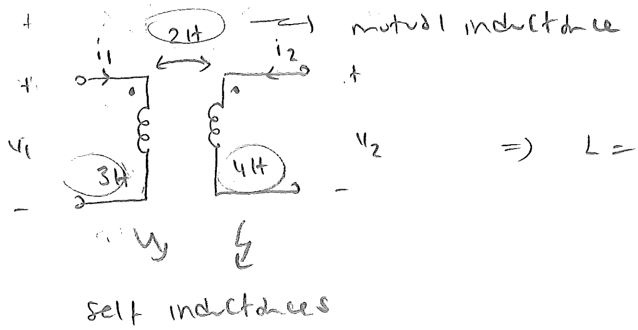
→ $|M|$ is called "mutual inductance"

Note that the inductance matrix is symmetric. For 3 coupled inductors:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{12} & L_2 & M_{23} \\ M_{13} & M_{23} & L_3 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

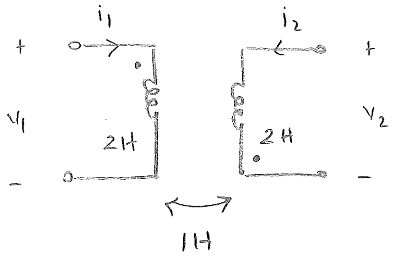
<u>Dot convention</u> Dots indicate the relative orientation of the coupled inductors. Considering a pair of inductors, if both currents are leaving / entering from the dotted terminal then the generated fluxes contribute to each other (the associated off diagonal terms of the inductance matrix are positive). Otherwise, they are opposing each other (the off diagonal terms are negative).

Ex



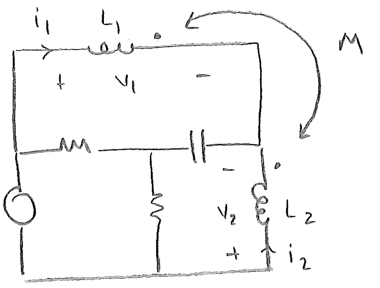
$$\Rightarrow L = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{cases} v_1 = 3 \frac{di_1}{dt} + 2 \frac{di_2}{dt} \\ v_2 = 2 \frac{di_1}{dt} + 4 \frac{di_2}{dt} \end{cases}$$

Ex



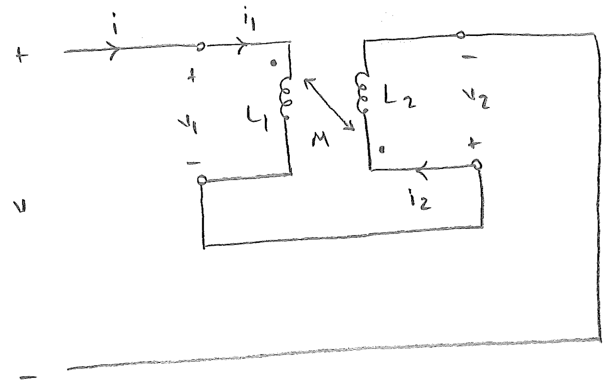
$$\Rightarrow \quad L = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Ex



$$\Rightarrow \quad L = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$$

Ex Find the equivalent inductance L_{eq} where $v = L_{eq} \frac{di}{dt}$



$$v = v_1 + v_2$$

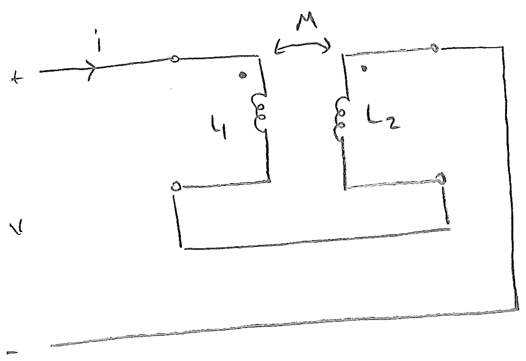
$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$i = i_1 = i_2$$

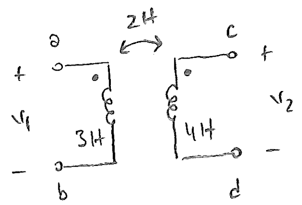
$$\Rightarrow \quad v = (L_1 + L_2 + 2M) \frac{di}{dt} \quad \Rightarrow \quad \boxed{L_{eq} = L_1 + L_2 + 2M}$$

How about ?

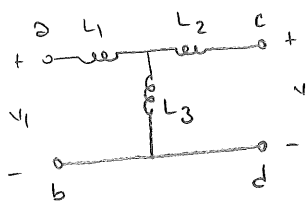


$$\boxed{L_{eq} = L_1 + L_2 - 2M}$$

Exercise Suppose



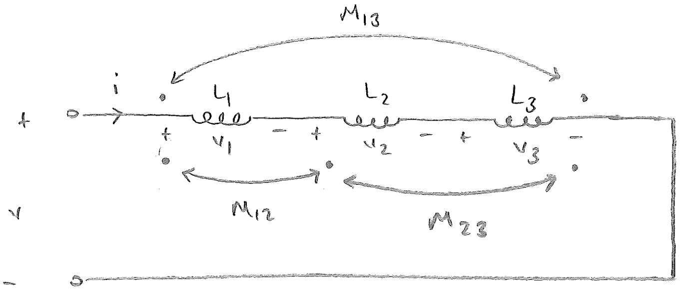
$\equiv$



(inductors uncoupled)

Find L_1, L_2, L_3.

Example



$L_{eq} = ?$

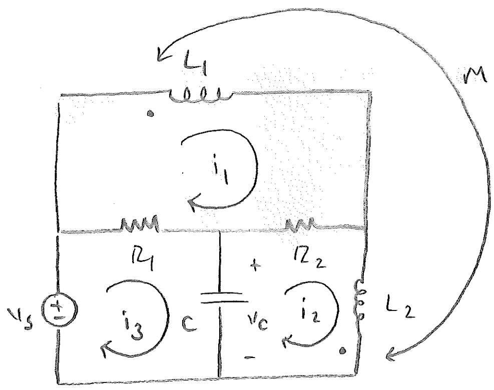
$$\left(Di = \frac{di}{dt} \right)$$

$v = v_1 + v_2 + v_3$

$$\left.\begin{aligned} v_1 &= L_1 Di + M_{12} Di - M_{13} Di \\ v_2 &= L_2 Di + M_{12} Di - M_{23} Di \\ v_3 &= L_3 Di - M_{13} Di - M_{23} Di \end{aligned}\right\} \quad v = \{ L_1 + L_2 + L_3 + 2M_{12} - 2M_{23} - 2M_{13} \} Di$$

$$\Rightarrow \boxed{L_{eq} = L_1 + L_2 + L_3 + 2(M_{12} - M_{23} - M_{13})}$$

Ex



Write the mesh equations.

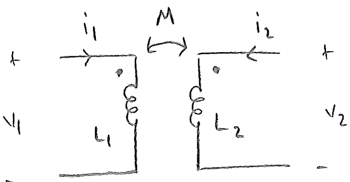
mesh ①: $L_1 Di_1 - M Di_2 + R_2 (i_1 - i_2) + R_1 (i_1 - i_3) = 0$

mesh ②: $L_2 Di_2 - M Di_1 - \left\{ v_c(t_0) + \frac{1}{C} \int_{t_0}^{t} (i_3(\tau) - i_2(\tau)) d\tau \right\}$

$+ R_2 (i_2 - i_1) = 0$

mesh ③: $-v_s + R_1 (i_3 - i_1) + v_c(t_0) + \frac{1}{C} \int_{t_0}^{t} (i_3(\tau) - i_2(\tau)) d\tau = 0$

Power & Energy



(instanteneous) power of 1st inductor:

$$P_1(t) = i_1(t) v_1(t) = i_1(t) \left\{ L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \right\}$$

power of 2nd inductor

$$P_2(t) = i_2(t) v_2(t) = i_2(t) \left\{ L_2 \frac{di_2(t)}{dt} + M \frac{di_1(t)}{dt} \right\}$$

$\Rightarrow$ total power $p(t) = P_1(t) + P_2(t) = L_1 i_1 Di_1 + M i_1 Di_2 + M i_2 Di_1 + L_2 i_2 Di_2$

$$= \frac{d}{dt} \left\{ \frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2 \right\}$$

Hence, the <u>energy stored</u> at time t :

$$w(t) = \int_{-\infty}^{t} p(\tau)d\tau = \frac{1}{2}L_1 i_1(t)^2 + M i_1(t) i_2(t) + \frac{1}{2}L_2 i_2(t)^2$$

$$= \boxed{\frac{1}{2} i(t)^T L\, i(t)} \quad \text{where} \quad i(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} \quad \& \quad L = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$$

For <u>passive</u> inductors the stored energy cannot be negative. This implies

$$\frac{1}{2} i^T L i \geq 0 \quad \text{for all } i \qquad (1)$$

Inductance matrices L satisfying (1) are said to be <u>positive semi definite</u>

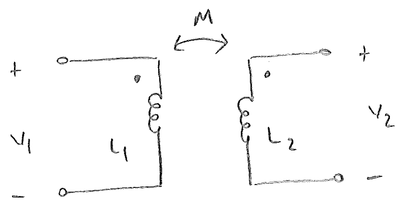
<u>Question</u> How to check positive semidefiniteness (PSD) ?

<u>Answer</u> $L = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$ is PSD if and only if $L_1 \geq 0$, $L_2 \geq 0$, & $L_1 L_2 - M^2 \geq 0$.

(is PSD if and only if $\lambda_i(L) \geq 0$ $i = 1,2$.)

↳ eigenvalues of L

<u>Terminal Equations in Integral Form</u>



$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} di_1(t)/dt \\ di_2(t)/dt \end{bmatrix}$$

$i_1(0^-) = I_{10}$, $i_2(0^-) = I_{20}$

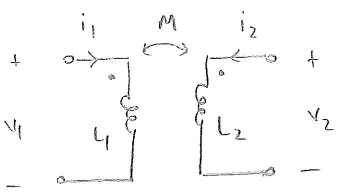
$$\Rightarrow \begin{bmatrix} \int_{0^-}^{t} v_1(\tau)\,d\tau \\ \int_{0^-}^{t} v_2(\tau)\,d\tau \end{bmatrix} = \underbrace{\begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}}_{L} \begin{bmatrix} i_1(t) - i_1(0^-) \\ i_2(t) - i_2(0^-) \end{bmatrix}$$

$$\Rightarrow L \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = L \begin{bmatrix} I_{10} \\ I_{20} \end{bmatrix} + \begin{bmatrix} \int_{0^-}^{t} v_1(\tau)\,d\tau \\ \int_{0^-}^{t} v_2(\tau)\,d\tau \end{bmatrix}$$

When L^{-1} exists we can write

$$\underbrace{\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}}_{i(t)} = \underbrace{\begin{bmatrix} I_{10} \\ I_{20} \end{bmatrix}}_{I_0} + L^{-1} \int_{0^-}^{t} \underbrace{\begin{bmatrix} v_1(\tau) \\ v_2(\tau) \end{bmatrix}}_{v(\tau)} d\tau \quad \Rightarrow \quad i(t) = I_0 + L^{-1} \int_{0^-}^{t} v(\tau)\,d\tau$$

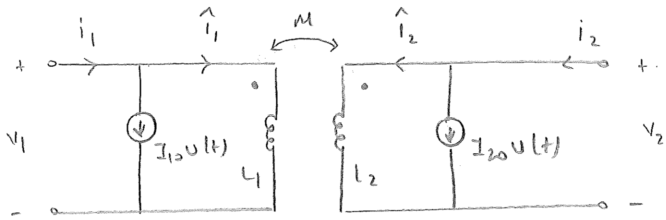
Initial Condition Models



$i_1(0^-) = I_{10}$, $i_2(0^-) = I_{20}$

(I)

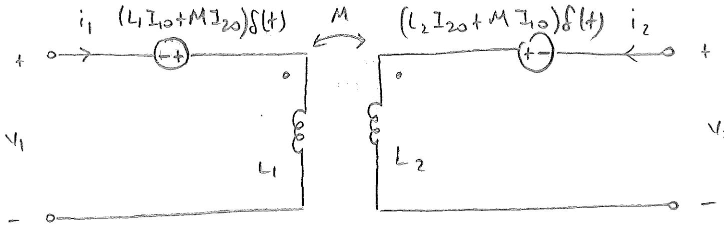
$\overset{(t>0)}{\equiv}$



(II)

$\hat{i}_1(0^-) = 0$, $\hat{i}_2(0^-) = 0$

$\overset{(t>0)}{\equiv}$



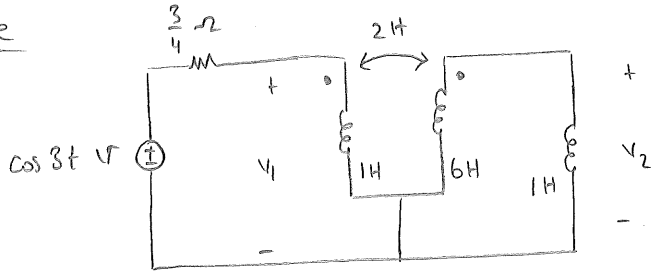
(III)

$i_1(0^-) = 0$, $i_2(0^-) = 0$

Exercise Verify that for all configurations I, II, III we have (for $t>0$)

$$\begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} I_{10} \\ I_{20} \end{bmatrix} + \int_{0^-}^{t} \begin{bmatrix} v_1(\tau) \\ v_2(\tau) \end{bmatrix} d\tau$$

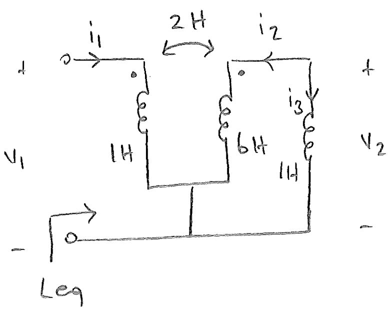
Example



Assume zero init. conditions.

$v_1(t)$, $v_2(t) = ?$ for $t \geq 0$.

Step 1 Find equiv. inductance



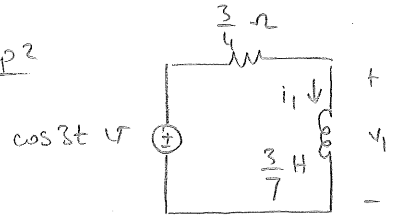
$v_1 = Di_1 + 2Di_2 \quad (1)$

$\left.\begin{aligned} v_2 &= 6Di_2 + 2Di_1 \\ v_2 &= Di_3 = -Di_2 \end{aligned}\right\} \quad -Di_2 = 6Di_2 + 2Di_1$

$\Rightarrow Di_2 = -\frac{2}{7}Di_1 \quad (2)$

$(1) \& (2) \Rightarrow v_1 = Di_1 - \frac{4}{7}Di_1 = \frac{3}{7}Di_1 \Rightarrow \boxed{L_{eq} = \frac{3}{7} H}$

Step 2



$$Di_1 = \frac{7}{3} v_1 = \frac{7}{3}\left\{\cos 3t - \frac{3}{4} i_1\right\}$$

$$\Rightarrow Di_1 + \frac{7}{4} i_1 = \frac{7}{3}\cos 3t$$

$$\Rightarrow \left(D + \frac{7}{4}\right) i_1 = \frac{7}{3}\cos 3t$$

$\Rightarrow \quad i_h(t) = K e^{-\frac{7}{4}t}$ (nat. freq. $s = -\frac{7}{4}$)

$i_p(t) = A\cos 3t + B\sin 3t$

$$Di_p + \frac{7}{4} i_p = \frac{7}{3}\cos 3t$$

$$\Rightarrow -3A\sin 3t + 3B\cos 3t + \frac{7}{4}\left\{A\cos 3t + B\sin 3t\right\} = \frac{7}{3}\cos 3t$$

$$\Rightarrow \underbrace{\left\{-3A + \frac{7}{4}B\right\}}_{=0}\sin 3t + \underbrace{\left\{3B + \frac{7}{4}A\right\}}_{=\frac{7}{3}}\cos 3t = \frac{7}{3}\cos 3t$$

$$\Rightarrow \begin{bmatrix} -3 & 7/4 \\ 7/4 & 3 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 7/3 \end{bmatrix} \Rightarrow \text{compute } A \,\&\, B$$

$$\Rightarrow \left.\begin{aligned} i_1(t) &= i_p(t) + i_h(t) \\ i_1(0) &= 0 \end{aligned}\right\} \quad K = -A \quad \Rightarrow \quad i_1(t) = A\cos 3t + B\sin 3t - Ae^{-\frac{7}{4}t} \text{ Amps}$$

$$v_1(t) = \frac{3}{7} Di_1(t) \quad \Rightarrow \quad \boxed{v_1(t) = \frac{3}{7}\left\{-3A\sin 3t + 3B\cos 3t + \frac{7A}{4} e^{-\frac{7}{4}t}\right\} \text{ V}}$$

Earlier we've obtained

$$\left.\begin{aligned} v_2 &= -Di_2 \\ Di_2 &= -\frac{2}{7} Di_1 \end{aligned}\right\} \quad v_2 = \frac{2}{7} Di_1 \qquad (3)$$

Also, $\quad v_1 = \frac{3}{7} Di_1 \quad (4)$

$$(3)\&(4) \Rightarrow \quad \boxed{v_2(t) = \frac{2}{3} v_1(t)}$$

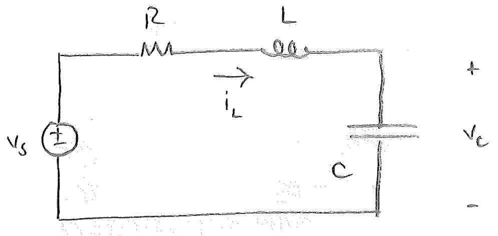
State Equation

For computational and theoretical convenience, engineers & physicists usually formulate a given LTI system in terms of state equation:

$$\dot{x} = Ax + Bw \quad \text{with} \quad x \in \mathbb{R}^n, \; w \in \mathbb{R}^m, \; A \in \mathbb{R}^{n \times n}, \; B \in \mathbb{R}^{n \times m}$$

where x is the state (vector), w is the input (vector) and A & B are constant matrices.

Example



$$C D v_C = i_L \Rightarrow D v_C = \frac{1}{C} i_L \quad (1)$$

$$R i_L + L D i_L + v_C = v_S \Rightarrow D i_L = -\frac{1}{L} v_C - \frac{R}{L} i_L + \frac{1}{L} v_S \quad (2)$$

let $x := \begin{bmatrix} v_C \\ i_L \end{bmatrix}$ be our state (input $w = v_S$)

Then (1) & (2) $\Rightarrow$ $\dot{x} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{B} v_S$ the state equation

Q: What is "state"?

A: A (minimal) set of data is said to be the state of a network if the following conditions are satisfied

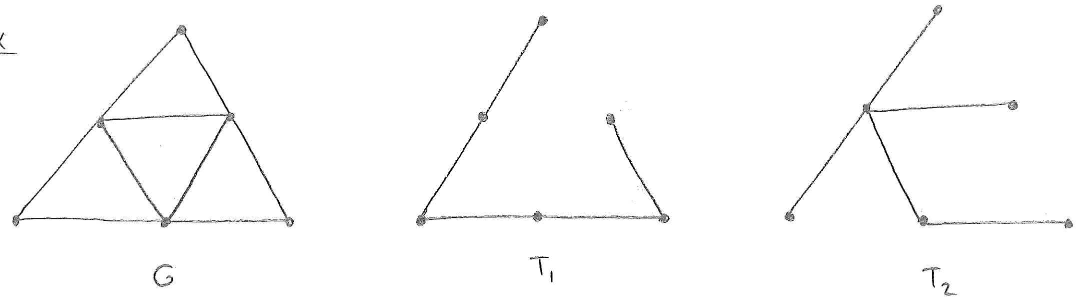
- → If we know $x(t_0)$ and $w(t)$ for $t \in [t_0, \infty)$ we can determine $x(t)$ for all $t \geq t_0$.
- → If we know $x(t)$ and $w(t)$ then we can compute any voltage or current $y(t)$ in the network.

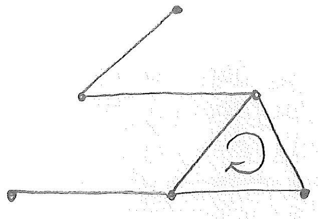
Remark: The choice of state is not unique. When possible, we will stick to the natural choice, namely, the capacitor voltages (or sometimes charges) and the inductor currents (or sometimes fluxes)

→ Graph Theory is a useful tool in obtaining the state equation systematically:

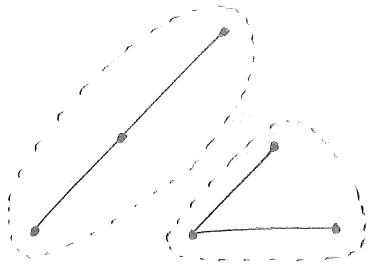
<u>Tree</u> Given a connected graph G, a tree T is a subgraph of G that contains all the nodes of G, contains no loops, and is connected.

<u>Ex</u>





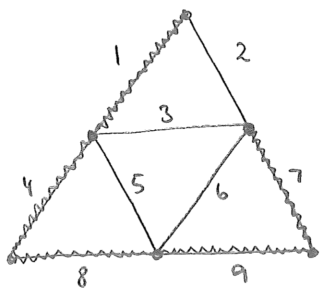
Not a tree, contains loop



Not a tree, disconnected

<u>Cotree</u> Given a tree, all the remaining branches form the associated cotree.

<u>Ex</u>



$T_1 = \{1, 4, 8, 9, 7\}$

$C_1 = \{2, 3, 5, 6\}$, cotree of T_1

<u>Cutset</u> Given a connected graph G, a cutset is a set of branches (of G) with the following properties:

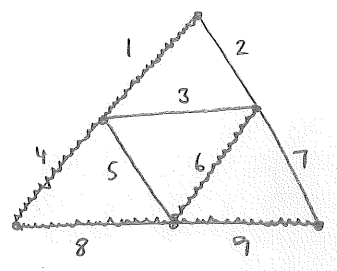
1) Removal of all the branches of the cutset renders the graph G unconnected.

2) After removal of all the branches except one, the graph G still stays connected.

<u>Fundamental Cutset</u> Let G be a connected graph, T be a tree of G, and C the associated cotree. Given a branch $b_1 \in T$, we can construct a cutset $\{b_1, b_2, \dots, b_k\}$ where all the complementary branches $b_2, \dots, b_k$ belong to C. This cutset is unique and it is called the fundamental cutset associated with $b_1 \in T$.

<u>Fundamental Loop</u> (Dual of fund. cutset) Given a branch $\tilde{b}_1 \in C$, we can construct a loop $\{\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_\ell\}$ where all the complementary branches $\tilde{b}_2, \dots, \tilde{b}_\ell$ belong to T. This loop is unique. We call it the fundamental loop associated with $\tilde{b}_1 \in C$.

<u>Ex</u>



(given) $T = \{1, 4, 8, 6, 9\}$

$\Rightarrow C = \{2, 3, 5, 7\}$

fund. cutsets:

$1 \to \{1, 2\}$

$4 \to \{4, 5, 3, 2\}$

$8 \to \{8, 5, 3, 2\}$

$6 \to \{6, 2, 3, 7\}$

$9 \to \{9, 7\}$

Fund. loops

$2 \to \{2, 6, 8, 4, 1\}$

$3 \to \{3, 6, 8, 4\}$

$5 \to \{5, 8, 4\}$

$7 \to \{7, 6, 9\}$

— o —

<u>Procedure to obtain state equation</u>

<u>Step 1</u> Choose a tree that is "proper." A proper tree satisfies:

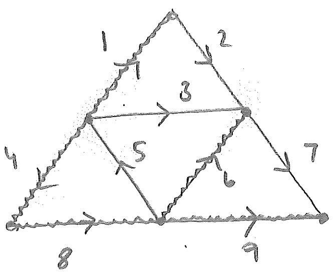
- → All the ind. voltage sources are in the tree. All ICS's are in the cotree.
- → If there is an ideal transformer, one and only one of its branches is in the tree.
- → Maximum number of capacitors are in the tree
- → Maximum number of inductors are in the cotree

<u>Step 2</u> Choose a state vector composed of capacitor voltages in the tree and the inductor currents in the cotree.

<u>Step 3</u> For each fund. cutset defined by a capacitor (in the tree) write KCL eqn.

<u>Step 4</u> For each fund. loop defined by an inductor (in the cotree) write KVL eqn.

Note :



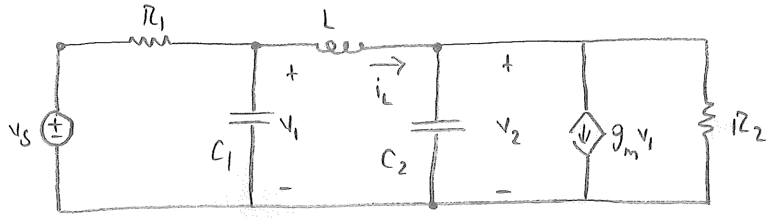
For the fund. cutset $\{8,5,3,7\}$

KCL $\Rightarrow$ $i_8 - i_5 + i_3 + i_2 = 0$

For the fund. loop $\{2,6,8,4,1\}$

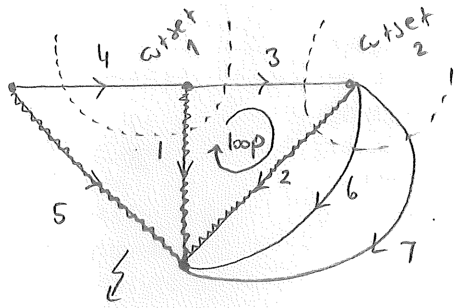
KVL $\Rightarrow$ $v_2 - v_6 - v_8 - v_4 + v_1 = 0$

Example :



obtain the state eqn.

Step 1



proper tree

cutset 1 $\Rightarrow$ $i_1 - i_4 + i_3 = 0$ (1)

cutset 2 $\Rightarrow$ $i_2 - i_3 + i_6 + i_7 = 0$ (2)

loop $\Rightarrow$ $v_3 + v_2 - v_1 = 0$ (3)

Step 2

$$x = \begin{bmatrix} v_1 \\ v_2 \\ i_L \end{bmatrix}$$

WANT: Find A, B matrices such that $\dot{x} = Ax + Bv_s$

Steps 3 & 4

(1) $\Rightarrow$ $C_1 Dv_1 - i_{R_1} + i_L = 0$ $\Rightarrow$ $C_1 Dv_1 - \dfrac{v_s - v_1}{R_1} + i_L = 0$ (4)

(2) $\Rightarrow$ $C_2 Dv_2 - i_L + g_m v_1 + \dfrac{v_2}{R_2} = 0$ (5)

(3) $\Rightarrow$ $L Di_L + v_2 - v_1 = 0$ (6)

(4) $\Rightarrow$ $Dv_1 = -\dfrac{1}{R_1C_1} v_1 - \dfrac{1}{C_1} i_L + \dfrac{1}{R_1C_1} v_s$

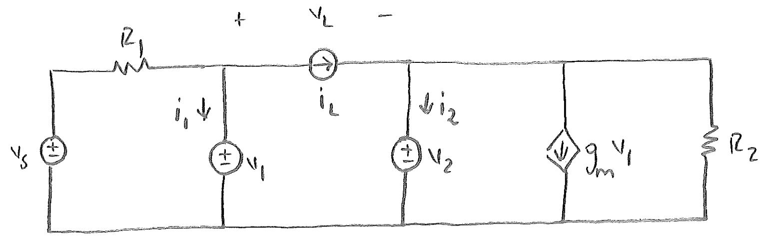
(5) $\Rightarrow$ $Dv_2 = -\dfrac{g_m}{C_2} v_1 - \dfrac{1}{R_2C_2} v_2 + \dfrac{1}{C_2} i_L$

(6) $\Rightarrow$ $Di_L = \dfrac{1}{L} v_1 - \dfrac{1}{L} v_2$

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{C_1} \\ -\frac{g_m}{C_2} & -\frac{1}{R_2C_2} & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \\ 0 \end{bmatrix}}_{B} v_s$$

Another way   Step 1 & Step 2 remain the same. Then apply substitution

thm: replace capacitors with IVS's & inductors with ICS's.



⇒ Note that this circuit is now LTI resistive with inputs v_s, v_1, v_2, i_L !

Express i_1, i_2, v_L in terms of the new inputs v_s, v_1, v_2, i_L

$$\left.\begin{aligned} i_1 &= \frac{v_s - v_1}{R_1} - i_L \\ i_2 &= i_L - g_m v_1 - \frac{v_2}{R_2} \\ v_L &= v_1 - v_2 \end{aligned}\right\} (*)$$

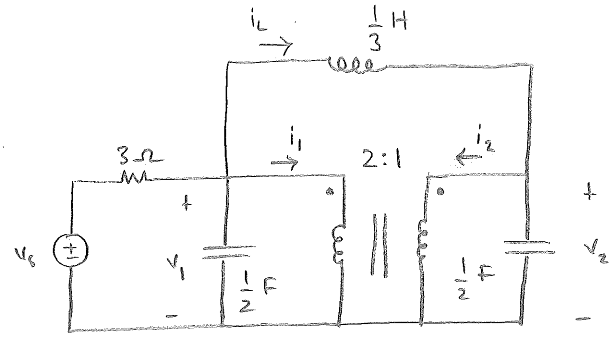
Recall the terminal eqn.'s

$$\left.\begin{aligned} i_1 &= C_1 D v_1 \\ i_2 &= C_2 D v_2 \\ v_L &= L D i_L \end{aligned}\right\} (**)$$

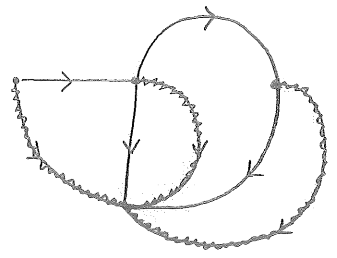
Combine $(*)$ & $(**)$ to obtain $\dot{x} = Ax + Bv_s$.

#5 22 Şubat 2016

<u>Example</u>



$\Rightarrow$



obtain the state eqn.

<u>Sol'n</u> state vector $x = \begin{bmatrix} i_L \\ v_2 \end{bmatrix}$. Note that v_1 does not belong to our proper tree!

cutset eqn : $\frac{1}{2}Dv_2 + i_2 - i_L = 0$ (1)

loop eqn : $\frac{1}{3}Di_L + v_2 - v_1 = 0$ (2)

$\Big\}$ $i_2, v_1 = ?$ (in terms of i_L, v_2, v_s)

$\frac{v_1}{2} = v_2 \Rightarrow v_1 = 2v_2$ (3)

$2i_1 + i_2 = 0 \Rightarrow i_2 = -2i_1 = 2\left\{\frac{1}{2}Dv_1 + \frac{v_1 - v_s}{3} + i_L\right\} = 2Dv_2 + \frac{4}{3}v_2 - \frac{2}{3}v_s + 2i_L$ (4)

(1) & (4) $\Rightarrow \frac{1}{2}Dv_2 + 2Dv_2 + \frac{4}{3}v_2 - \frac{2}{3}v_s + 2i_L - i_L = 0$

$\Rightarrow Dv_2 = -\frac{2}{5}i_L - \frac{8}{15}v_2 + \frac{4}{15}v_s$ (5)

(2) & (3) $\Rightarrow \frac{1}{3}Di_L + v_2 - 2v_2 = 0 \Rightarrow Di_L = 3v_2$ (6)

(5) & (6) $\Rightarrow$ $\boxed{\dot{x} = \begin{bmatrix} 0 & 3 \\ -\frac{2}{5} & -\frac{8}{15} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{4}{15} \end{bmatrix} v_s}$

<u>Remark</u> Given a circuit with state eqn. $\dot{x} = Ax + Bw$ with $x \in \mathbb{R}^n$,

the number "n" is the <u>order</u> of the circuit.

What's wrong with the following?

Ignore the proper tree and choose the "state" $z = \begin{bmatrix} i_L \\ v_2 \\ v_1 \end{bmatrix}$

Then, $v_1 = 2v_2 \Rightarrow \dot{v}_1 = 2\dot{v}_2 = -\frac{4}{15} i_L - \frac{16}{15} v_2 + \frac{8}{15} v_S$ (7)

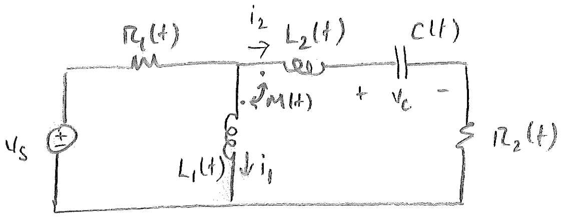
(5) & (6) & (7) $\Rightarrow$ $\dot{z} = \begin{bmatrix} 0 & 3 & 0 \\ -\frac{2}{5} & -\frac{8}{15} & 0 \\ -\frac{4}{5} & -\frac{16}{15} & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{4}{15} \\ \frac{8}{15} \end{bmatrix} v_S$ (*)

Answer: Although eqn. (*) is correct it is not a state equation because z is not a valid state since it is not minimal. (In fact, we cannot choose the initial condition $z(0) \in \mathbb{R}^3$ arbitrarily due to the constraint $v_1 = 2v_2$.)
Proper tree helps us choose a minimal state.

#1 29 Şubat 2016

MODELING CIRCUITS with LTV / NONLINEAR ELEMENTS

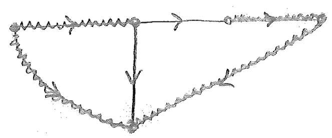
Example (time-varying elements) For LTV / nonlinear capacitor (inductor) it is easier to work with charge (flux) instead of voltage (current) as state variable.



$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \overbrace{\begin{bmatrix} L_1(t) & M(t) \\ M(t) & L_2(t) \end{bmatrix}}^{L(t)} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad \& \quad \dot{\phi}_k = v_k \ , \ k=1,2$$

$$q_c = C(t)\, v_c \quad \& \quad \dot{q}_c = i_c$$

$\Downarrow$



$\Rightarrow$ state vector : $x = [\phi_1 \ \ \phi_2 \ \ q_c]^T$

Let $L(t)^{-1} = \Gamma(t) = \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{12}(t) & \Gamma_{22}(t) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{12}(t) & \Gamma_{22}(t) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$

$\dot{\phi}_1 = ?$

$$\dot{\phi}_1 = v_1$$

$$v_1 = -v_{R_1} + v_s = -R_1(i_1+i_2) + v_s = -R_1\{(\Gamma_{11}+\Gamma_{12})\phi_1 + (\Gamma_{12}+\Gamma_{22})\phi_2\} + v_s \quad (1)$$

$\dot{\phi}_2 = ?$

$$\dot{\phi}_2 = v_2$$

$$v_2 = -v_c - R_2 i_2 - R_1(i_1+i_2) + v_s$$

$$= -\frac{q_c}{C} - R_2\{\Gamma_{12}\phi_1 + \Gamma_{22}\phi_2\} - R_1\{(\Gamma_{11}+\Gamma_{12})\phi_1 + (\Gamma_{12}+\Gamma_{22})\phi_2\} + v_s \quad (2)$$

$\dot{q}_c = ?$

$$\dot{q}_c = i_c$$

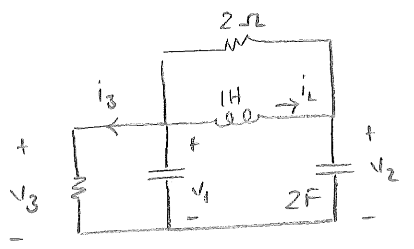
$$i_c = i_2 = \Gamma_{12}\phi_1 + \Gamma_{22}\phi_2 \quad (3)$$

(1)&(2)&(3) $\Rightarrow$

$$\dot{x} = \begin{bmatrix} -R_1(\Gamma_{11}+\Gamma_{12}) & -R_1(\Gamma_{12}+\Gamma_{22}) & 0 \\ -R_2\Gamma_{12} - R_1(\Gamma_{11}+\Gamma_{12}) & -R_2\Gamma_{22} - R_1(\Gamma_{12}+\Gamma_{22}) & -\frac{1}{C} \\ \Gamma_{12} & \Gamma_{22} & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} v_s$$

In general, LTV state eqn. reads $\dot{x}(t) = A(t)x(t) + B(t)w(t)$.

Example (Nonlinear Elements)



$i_3 = \tanh(v_3)$, nonlinear resistor

$v_1 = 2(q_1 + q_1^3)$; nonlinear capacitor

Let the state be $x = [\,q_1 \;\; v_2 \;\; i_L\,]^T$

$\dot{q}_1 = ?$

$$\dot{q}_1 = i_1$$

$$i_1 = -i_3 - i_L - \frac{v_1 - v_2}{2}$$

$$= -\tanh(v_1) - i_L - \frac{1}{2}v_1 + \frac{1}{2}v_2$$

$$= -\tanh(2q_1 + 2q_1^3) - i_L - (q_1 + q_1^3) + \frac{1}{2}v_2$$

$\dot{v}_2 = ?$

$$\dot{v}_2 = \frac{1}{2}i_2$$

$$i_2 = i_L + \frac{v_1 - v_2}{2} = i_L + q_1 + q_1^3 - \frac{1}{2}v_2$$

$\dot{i}_L = ?$

$$\dot{i}_L = v_L = v_1 - v_2 = 2(q_1 + q_1^3) - v_2$$

Hence,

$$\dot{\begin{bmatrix} q_1 \\ v_2 \\ i_L \end{bmatrix}} = \begin{bmatrix} -\tanh(2q_1 + 2q_1^3) - (q_1 + q_1^3) + \frac{1}{2}v_2 - i_L \\ \frac{1}{2}(q_1 + q_1^3) - \frac{1}{4}v_2 + \frac{1}{2}i_L \\ 2(q_1 + q_1^3) - v_2 \end{bmatrix}$$

In general, nonlinear state equation reads $\dot{x} = f(x, u)$

[In our case, we didn't have any input and had $\dot{x} = f(x)$ with $f : \mathbb{R}^3 \to \mathbb{R}^3$]

Solution to homogeneous state eqn. $\dot{x} = Ax$

Consider $\dot{x} = Ax$ with $x \in \mathbb{R}^n$ & $A \in \mathbb{R}^{n \times n}$ & $x(0) = x_0$

How to compute $x(t)$?

Assume there exists $M(t) \in \mathbb{R}^{n \times n}$, a matrix function, satisfying

c1) $M(0) = I$ (identity matrix)

c2) $\frac{d}{dt} M(t) = A M(t)$

Then $x(t) = M(t) x_0$ would be the solution because :

$\rightarrow$ $x(t)\big|_{t=0} = M(0)x_0 = I x_0 = x_0$ (initial cond. constraint)

$\rightarrow$ $\frac{d}{dt} x(t) = \frac{d}{dt}\{M(t) x_0\} = \frac{d}{dt}\{M(t)\} x_0$

$= AM(t)x_0 = Ax(t)$ (diff. eqn. constraint)

$M(t) = ?$

Guess : $M(t) := I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \dots$

Then $M(t)$ satisfies (c1) & (c2). $M(t)$ is called "matrix exponential" or the "state transition matrix" and usually denoted by e^{At}.

$$\boxed{e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k}$$ state transition matrix

solution to $\dot{x} = Ax$ with $x(0) = x_0$: $\boxed{x(t) = e^{At} x_0}$

#4 29 Şubat 2016

Properties of e^{At}

1) $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$ for all $t_1, t_2 \in \mathbb{R}$

2) $e^{A0} = I$

3) $[e^{At}]^{-1} = e^{-At}$ for all $t \in \mathbb{R}$. [Remark: e^{At} is nonsingular for all A & t.]

4) $Ae^{At} = e^{At}A$ for all $t \in \mathbb{R}$.

Computing e^{At} by Laplace Transform

$$\dot{x} = Ax \xrightarrow{\mathcal{L}} sX(s) - x(0) = AX(s)$$

$$\Rightarrow sX(s) - AX(s) = x(0)$$

$$\Rightarrow (sI - A)X(s) = x(0)$$

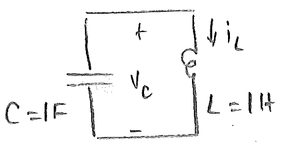
$$\Rightarrow X(s) = [sI - A]^{-1}x(0)$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\{[sI-A]^{-1}x(0)\}$$

$$= \underbrace{\mathcal{L}^{-1}\{[sI-A]^{-1}\}}_{\text{Hence, this must equal } e^{At}.} x(0) = e^{At}x(0)$$

$$\Rightarrow \boxed{e^{At} = \mathcal{L}^{-1}\{[sI-A]^{-1}\}}$$

Example (lossless LC circuit)



$$\left.\begin{aligned} Dv_c &= -i_L \\ Di_L &= v_c \end{aligned}\right\} (D^2+1)v_c = 0 \Rightarrow \text{char. poly. } d(s) = s^2+1$$

$$\Rightarrow \text{nat. freq } s_{1,2} = \mp j$$

$$\Rightarrow v_c(t) = k_1 \cos t + k_2 \sin t$$

$$\left.\begin{aligned} k_1 &= v_c(0), \\ k_2 &= Dv_c(0) = -i_L(0) \end{aligned}\right\} \left.\begin{aligned} &v_c(t) = v_c(0)\cos t - i_L(0)\sin t \\ &\Rightarrow i_L(t) = -Dv_c(t) = v_c(0)\sin t + i_L(0)\cos t \end{aligned}\right\} (*)$$

Let's obtain (*) by using state transition matrix

Let $x = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$. Then $\dot{x} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} x$

$$[sI - A] = \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \Rightarrow [sI - A]^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$$

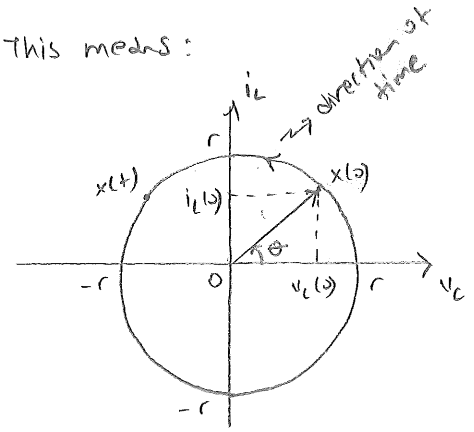
$$\Rightarrow \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \begin{bmatrix} \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} & \mathcal{L}^{-1}\{\frac{-1}{s^2+1}\} \\ \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} & \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = e^{At}$$

$$x(t) = e^{At}x(0) \Rightarrow \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \begin{bmatrix} v_C(0)\cos t - i_L(0)\sin t \\ v_C(0)\sin t + i_L(0)\cos t \end{bmatrix} \checkmark$$

$\|x(t)\|^2 = ?$

$$\|x(t)\|^2 = x(t)^T x(t) = x(0)^T [e^{At}]^T e^{At} x(0) = x(0)^T \underbrace{\begin{bmatrix} \cos^2 t + \sin^2 t & 0 \\ 0 & \cos^2 t + \sin^2 t \end{bmatrix}}_{I} x(0) = \|x(0)\|^2$$

This means:



→ the solution $x(t)$ traverses a circle with radius $r = \|x(0)\|$

→ e^{At} is a rotation matrix

→ Electrical energy is conserved because

$$E(t) = \frac{1}{2}v_C(t)^2 + \frac{1}{2}i_L(t)^2 = \frac{1}{2}\|x(t)\|^2 = \frac{1}{2}\|x(0)\|^2 = E(0)$$

Exercise Show that $\dot{\theta} = \frac{d}{dt}\arctan\left(\frac{i_L(t)}{v_C(t)}\right) = 1$ rad/sec.

Solution to $\dot{x} = Ax + Bw$

$$\dot{x} = Ax + Bw \overset{\mathcal{L}}{\Longrightarrow} sX(s) - x(0) = AX(s) + BW(s)$$

$$\Rightarrow [sI - A]X(s) = x(0) + BW(s)$$

$$\Rightarrow X(s) = [sI-A]^{-1}x(0) + [sI-A]^{-1}BW(s)$$

$$\overset{\mathcal{L}^{-1}}{\Longrightarrow} x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bw(\tau)\,d\tau$$

convolution integral

Therefore solution to nonhomogeneous state eqn. is

$$\boxed{x(t) = \underbrace{e^{At}x(0)}_{\text{zero-input solution}} + \underbrace{\int_0^t e^{A(t-\tau)}Bw(\tau)\,d\tau}_{\text{zero-state solution}} \qquad (1)}$$

Exercise : Show that (1) satisfies :

$$\rightarrow x(t)\Big|_{t=0} = x(0)$$

$$\rightarrow \dot{x}(t) = Ax(t) + Bw(t)$$

Example : Given the system $\dot{x} = Ax + Bw$ (with scalar input w) find the impulse response for the state.

Sol'n : $\dot{x} = Ax + B\delta(t)$ & $x(0^-) = 0$

$$\Rightarrow x(t) = e^{At}\cancel{x(0^-)}^{\,0} + \int_{0^-}^t e^{A(t-\tau)}B\delta(\tau)\,d\tau$$

$$= e^{At}\int_{0^-}^t e^{-A\tau}B\delta(\tau)\,d\tau$$

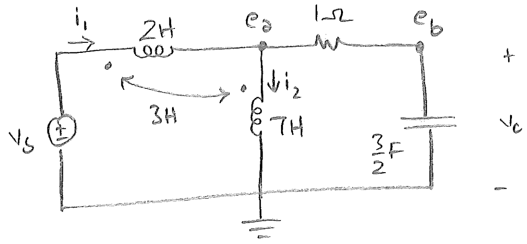
sifting property

$$= e^{At}\int_{0^-}^t \underbrace{\left[e^{-A\tau}B\right]_{\tau=0}}_{B} \cdot \delta(\tau)\,d\tau \qquad \Rightarrow \boxed{x(t) = e^{At}B}$$

Alternative Formulations of LTI circuits

Node analysis Write down KCL at each node in terms of node voltages.

Example



Recall $Df := \frac{d}{dt} f(t) = \dot{f}$

$D^{-1} f := \int_0^t f(\tau) d\tau$

Remark $D D^{-1} f = f(t)$

$D^{-1} D f = f(t) - f(0)$

Node ⓐ : $-i_1 + i_2 + \frac{e_a - e_b}{1} = 0$ (1)

Node ⓑ : $\frac{e_b - e_a}{1} + i_c = 0$ (2)

WANT Express i_1, i_2, i_c in terms of the formulation var. (e_a, e_b), inputs (v_s), and perhaps the init. conditions.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} \Rightarrow \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1.4 & -0.6 \\ -0.6 & 0.4 \end{bmatrix} \begin{bmatrix} v_s - e_a \\ e_a \end{bmatrix} \quad (3)$$

$$D^{-1}(3) \Rightarrow \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} - \begin{bmatrix} i_1(0) \\ i_2(0) \end{bmatrix} = \begin{bmatrix} 1.4 & -0.6 \\ -0.6 & 0.4 \end{bmatrix} \begin{bmatrix} D^{-1} v_s - D^{-1} e_a \\ D^{-1} e_a \end{bmatrix} \Rightarrow \begin{cases} i_1 = i_1(0) + 1.4 D^{-1} v_s - 2 D^{-1} e_a & (4) \\ i_2 = i_2(0) - 0.6 D^{-1} v_s + D^{-1} e_a & (5) \end{cases}$$

Also, $i_c = \frac{3}{2} D v_c = \frac{3}{2} D e_b$ (6)

$$\left.\begin{array}{ll} \text{Now, } (1),(4),(5) \Rightarrow & -i_1(0) + i_2(0) - 2D^{-1} v_s + 3D^{-1} e_a + e_a - e_b = 0 \\ \& \quad (2),(6) \Rightarrow & e_b - e_a + \frac{3}{2} D e_b = 0 \end{array}\right\} (*)$$

Rearranging $(*)$ $\Rightarrow$

$$\underbrace{\begin{bmatrix} 3D^{-1} + 1 & -1 \\ -1 & 1 + \frac{3}{2} D \end{bmatrix}}_{Y_n(D)} \underbrace{\begin{bmatrix} e_a \\ e_b \end{bmatrix}}_{e} = \underbrace{\begin{bmatrix} i_1(0) - i_2(0) + 2D^{-1} v_s \\ 0 \end{bmatrix}}_{i_{ns}}$$

The general form : $\boxed{Y_n(D)\,e = i_{ns}}$

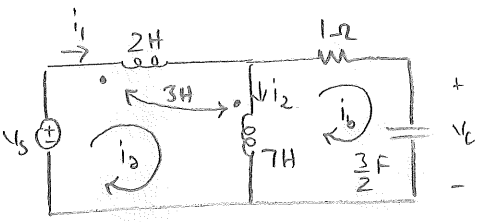
$Y_n(D)$: Node admittance matrix

e : Node voltage vector

i_{ns} : Node current source vector

Mesh Analysis Write down KVL at each mesh in terms of mesh currents.

Example



mesh ⓐ : $-v_s + v_1 + v_2 = 0$ (1)

mesh ⓑ : $-v_2 + i_b + v_c = 0$ (2)

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} Di_a \\ Di_a - Di_b \end{bmatrix} = \begin{bmatrix} 5Di_a - 3Di_b \\ 10Di_a - 7Di_b \end{bmatrix} \quad (3)$$

$$v_c = v_c(0) + \frac{2}{3}D^{-1}i_b \quad (4)$$

$$\left.\begin{array}{ll} (1),(3) \Rightarrow & -v_s + 15Di_a - 10Di_b = 0 \\ (2),(3),(4) \Rightarrow & -10Di_a + 7Di_b + i_b + v_c(0) + \frac{2}{3}D^{-1}i_b = 0 \end{array}\right\} (*)$$

$$(*) \Rightarrow \boxed{\underbrace{\begin{bmatrix} 15D & -10D \\ -10D & \frac{2}{3}D^{-1} + 1 + 7D \end{bmatrix}}_{Z_m(D)} \underbrace{\begin{bmatrix} i_a \\ i_b \end{bmatrix}}_{i_m} = \underbrace{\begin{bmatrix} v_s \\ -v_c(0) \end{bmatrix}}_{v_{ms}}}$$

The general form : $\boxed{Z_m(D)\,i_m = v_{ms}}$

$Z_m(D)$: Mesh impedance matrix

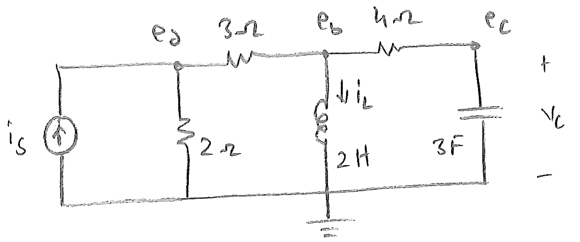
i_m : Mesh current vector

v_{ms} : Mesh voltage source vector.

#3 2 Mart 2016

<u>Modified node analysis</u> Augment the node voltage vector by inductor currents in order to avoid integral (D^{-1}) terms.

<u>Example</u>



Node ⓐ: $-i_s + \frac{e_a}{2} + \frac{e_a - e_b}{3} = 0$ (1)

Node ⓑ: $\frac{e_b - e_a}{3} + i_L(0) + \frac{1}{2}D^{-1}e_b + \frac{e_b - e_c}{4} = 0$ (2)

$\frac{e_b - e_a}{3} + i_L + \frac{e_b - e_c}{4} = 0$ (2')

Node ⓒ: $\frac{e_c - e_b}{4} + 3De_c = 0$ (3)

constraint: $e_b = 2Di_L \Rightarrow e_b - 2Di_L = 0$ (4')

<u>Standard node analysis</u> Formulation variables e_a, e_b, e_c (3 var. ⇒ 3 eqn.)

(1), (2), (3) ⇒
$$\begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{7}{12} + \frac{1}{2}D^{-1} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} + 3D \end{bmatrix} \begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} = \begin{bmatrix} i_s \\ -i_L(0) \\ 0 \end{bmatrix}$$

<u>Modified node analysis</u> Formulation variables e_a, e_b, e_c, i_L (4 var. ⇒ 4 eqn.)

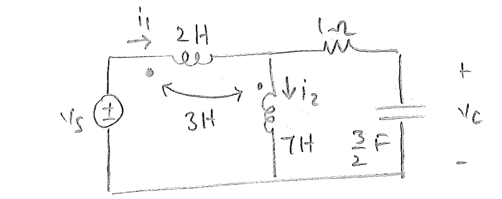
(1), (2'), (3), (4') ⇒
$$\underbrace{\begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{7}{12} & -\frac{1}{4} & 1 \\ 0 & -\frac{1}{4} & \frac{1}{4} + 3D & 0 \\ 0 & 1 & 0 & -2D \end{bmatrix}}_{\text{No } D^{-1} \text{ terms!}} \underbrace{\begin{bmatrix} e_a \\ e_b \\ e_c \\ i_L \end{bmatrix}}_{\text{augmented node voltage vector}} = \underbrace{\begin{bmatrix} i_s \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{}$$

→ No init. cond. terms!

#3 $\frac{1}{2}$ 2 Mart 2016

(13 Şubat 2020)

Example



a) obtain the state eqn.

b) Find the nat. freq.

c) Find the part. sol'n $v_{C,p}(t)$ for $v_s(t) = 2e^{-3t}$ V.

Sol'n a) Let $x = \begin{bmatrix} i_1 \\ i_2 \\ v_C \end{bmatrix}$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} \Rightarrow \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} = \begin{bmatrix} 7/5 & -3/5 \\ -3/5 & 2/5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad (1)$$

$$\left.\begin{aligned} v_2 &= v_R + v_C = 1(i_1 - i_2) + v_C \\ v_1 &= v_s - v_2 = v_s - i_1 + i_2 - v_C \end{aligned}\right\} \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ v_C \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_s \qquad (2)$$

$$(1)\&(2) \Rightarrow \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 7/5 & -3/5 \\ -3/5 & 2/5 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}} x + \underbrace{\begin{bmatrix} 7/5 & -3/5 \\ -3/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 7/5 \\ -3/5 \end{bmatrix}} v_s \qquad (3)$$

Also, $\frac{3}{2} Dv_C = i_C = i_1 - i_2 \Rightarrow Dv_C = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix} x + [0] v_s \qquad (4)$

$$(3)\&(4) \Rightarrow \quad \dot{x} = \underbrace{\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ \frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 7/5 \\ -3/5 \\ 0 \end{bmatrix}}_{B} v_s$$

b) $d(s) = \det(sI - A) = \begin{vmatrix} s+2 & -2 & 2 \\ -1 & s+1 & -1 \\ -\frac{2}{3} & \frac{2}{3} & s \end{vmatrix}$

$$= (s+2)(s+1)s - \frac{4}{3} - \frac{4}{3} + \frac{4}{3}(s+1) + \frac{2}{3}(s+2) - 2s$$

$$= (s+2)(s+1)s$$

$\Rightarrow$ eigenvalues of $A = \{-2, -1, 0\}$

c) For $v_s(t) = 2e^{-3t}$ we have $\left\{ \begin{array}{l} i_{1,p}(t) = a e^{-3t} \\ i_{2,p}(t) = b e^{-3t} \\ v_{c,p}(t) = c e^{-3t} \end{array} \right.$ for some a, b, c

$\Rightarrow$ Part. sol'n for the state therefore has to be $x_p(t) = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{v} e^{-3t}$

$x_p(t)$ solves the state eqn. : $\dot{x}_p = A x_p + B \cdot 2e^{-3t}$

$\Rightarrow -3v\cancel{e^{-3t}} = Av\cancel{e^{-3t}} + 2B\cancel{e^{-3t}}$

$\Rightarrow -3v = Av + 2B$

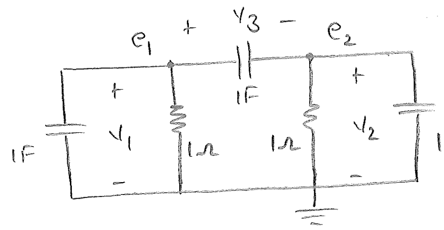
$\Rightarrow [-3I - A]v = 2B \quad \Rightarrow \quad v = 2[-3I - A]^{-1}B = \begin{bmatrix} -\frac{182}{45} \\ \frac{88}{45} \\ \frac{4}{3} \end{bmatrix}$ (by MATLAB)

$\Rightarrow \boxed{v_{c,p}(t) = \frac{4}{3}e^{-3t} \quad \checkmark}$

Natural Response

Nat. response is the response of a circuit to initial cond. only. (That is, it is another name for zero-input response.)

Example



$v_1(0) = V_{10}$, $v_2(0) = V_{20}$

a) Write node equations

b) Find $v_1(t), v_2(t), v_3(t)$.

Sol'n a)

Node ①: $De_1 + \frac{e_1}{1} + D(e_1 - e_2) = 0$

Node ②: $De_2 + \frac{e_2}{1} + D(e_2 - e_1) = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} 2D+1 & -D \\ -D & 2D+1 \end{bmatrix}}_{Y(D)} \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We've obtained $\boxed{Y(D)e = 0}$

b) $e(t) = ?$

Guess: $e(t) = e^{st}x$ where $s \in \mathbb{C}$ & $x \in \mathbb{C}^2$

$$\Rightarrow 0 = Y(D)e(t) = Y(D)\{e^{st}x\} = e^{st}Y(s)x \qquad De^{st} = se^{st}$$

$$\Rightarrow Y(s)x = 0$$

If $x \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $Y(s)x = 0 \Rightarrow Y(s)$ singular $\Rightarrow \boxed{\det Y(s) = 0}$

$$\det Y(s) = \begin{vmatrix} 2s+1 & -s \\ -s & 2s+1 \end{vmatrix} = (2s+1)^2 - s^2 = 3s^2 + 4s + 1 = (3s+1)(s+1)$$

Therefore, $\det Y(s) = 0 \Rightarrow \boxed{s = -\frac{1}{3} \text{ or } s = -1}$

Remark The roots $s_1 = -\frac{1}{3}$, $s_2 = -1$ are the natural frequencies (modes) of the circuit.

$s_1 = -\frac{1}{3}$ associated sol'n $e(t) = e^{s_1 t} x$, $x \in \mathbb{C}^2$.

$x_1 = ?$ Solve $Y(s)\big|_{s=-\frac{1}{3}} \cdot x_1 = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}}_{Y(-1/3)} x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ where c_1 is any constant

$$\Rightarrow \boxed{e(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-\frac{t}{3}}}$$

$s_2 = -1$ follow the procedure above $\Rightarrow \boxed{e(t) = c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}}$

Hence, general form of the solution : $\boxed{e(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t/3} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}}$ (*)

For any c_1, c_2 (*) satisfies the node eqn. $Y(D)e = 0$.

$c_1, c_2 = ?$ Apply init. cond. constraint

$$e(0) = \begin{bmatrix} e_1(0) \\ e_2(0) \end{bmatrix} = \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} = \begin{bmatrix} V_{10} \\ V_{20} \end{bmatrix} \quad (**)$$

$$(*), (**) \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} V_{10} \\ V_{20} \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} V_{10} \\ V_{20} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(V_{10} - V_{20}) \\ \frac{1}{2}(V_{10} + V_{20}) \end{bmatrix}$$

$$\Rightarrow \boxed{e(t) = \frac{V_{10} - V_{20}}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t/3} + \frac{V_{10} + V_{20}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}} \quad (1)$$

Finally, $v_1(t) = e_1(t)$, $v_2(t) = e_2(t)$, $v_3(t) = e_1(t) - e_2(t) = (V_{10} - V_{20}) e^{-t/3}$.

Exercise obtain $\dot{x} = Ax$ for $x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Compute $x(t) = e^{At} x(0)$ & verify (1).

Observations

1) Mode $s=-1$ does not appear in $v_3(t) = (V_{10} - V_{20})e^{-t/3}$ regardless of the init. conditions. In general, not all modes appear in every branch voltage/current.

2) By adjusting the init. cond. such that $V_{10} = V_{20}$ ($V_{10} = -V_{20}$) we can make the mode $s=-\frac{1}{3}$ ($s=-1$) disappear from $e_1(t)$ & $e_2(t)$ and, consequently, from the whole circuit. Hence, by choosing $V_{10} = V_{20}$ ($V_{10} = -V_{20}$) only the mode $s=-1$ ($s=-\frac{1}{3}$) is excited.

Determining the nat. frequencies (modes)

→ Node formulation $Y(D)e = i_{ns}$

obtain $\det Y(s) = \dfrac{num(s)}{den(s)}$

num(s) : numerator poly.

den(s) : denominator poly. It appears when we have integral (D^{-1}) terms.

nat. freq. = roots of num(s)

→ Mesh formulation $Z(D)i_m = v_{ms}$

obtain $\det Z(s) = \dfrac{num(s)}{den(s)}$

nat. freq. = roots of num(s)

→ State formulation $\dot{x} = Ax + Bw \Rightarrow (DI - A)x = Bw$

obtain $\det(sI - A) = d(s)$

nat. freq. = roots of $d(s)$ = eigenvalues of A.

Remark The nat. freq depend only on the circuit. They are independent of the formulation choice. However:

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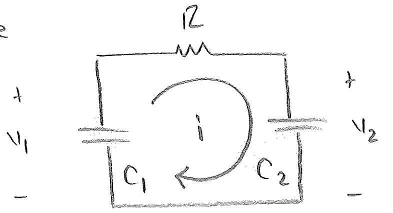
Example



find nat. frequencies.

Mesh formulation
$$-v_1(0) + \frac{1}{C_1}D^{-1}i + Ri + v_2(0) + \frac{1}{C_2}D^{-1}i = 0$$

$$\Rightarrow \underbrace{\left[\left(\frac{1}{C_1}+\frac{1}{C_2}\right)D^{-1} + R\right]}_{Z(D)} i = v_1(0) - v_2(0)$$

$$\det Z(s) = \left(\frac{1}{C_1}+\frac{1}{C_2}\right)\frac{1}{s} + R = \frac{Rs + \left(\frac{1}{C_1}+\frac{1}{C_2}\right)}{s} = \frac{num(s)}{den(s)}$$

root of num(s) : $s_1 = -\frac{1}{R}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)$ nat. frequency of the circuit

State formulation $x = [v_1 \;\; v_2]^T$

$$\left.\begin{aligned} C_1\dot{v}_1 + \frac{v_1 - v_2}{R} &= 0 \\ C_2\dot{v}_2 + \frac{v_2 - v_1}{R} &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} \dot{v}_1 &= -\frac{1}{RC_1}v_1 + \frac{1}{RC_1}v_2 \\ \dot{v}_2 &= \frac{1}{RC_2}v_1 - \frac{1}{RC_2}v_2 \end{aligned}\right\} \quad \dot{x} = \underbrace{\begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{RC_1} \\ \frac{1}{RC_2} & -\frac{1}{RC_2} \end{bmatrix}}_{A} x$$

$$\det(sI - A) = \begin{vmatrix} s + \frac{1}{RC_1} & -\frac{1}{RC_1} \\ -\frac{1}{RC_2} & s + \frac{1}{RC_2} \end{vmatrix} = s\left(s + \frac{1}{RC_1} + \frac{1}{RC_2}\right) =: d(s)$$

roots of $d(s)$: $s_1 = -\frac{1}{R}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)$ & $s_2 = 0$

$\Rightarrow$ this mode goes undetected by mesh formulation !

Remark Node & mesh formulations, if they contain integral (D^{-1}) terms, may sometimes fail to show the natural frequency at $s=0$.

Mode Excitation

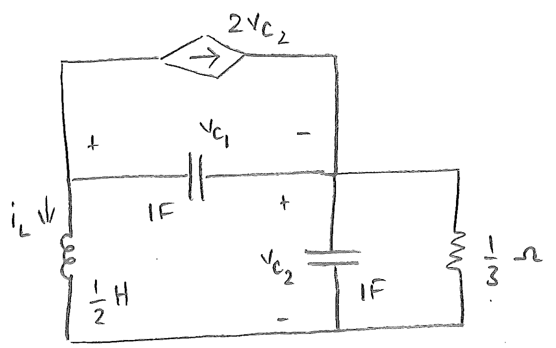
Def Given some circuit with no input (or all the ind. sources are killed) let $s=\lambda$ (with $\lambda \in \mathbb{R}$) be one of its modes. "Exciting the mode $s=\lambda$" means choose initial conditions (capacitor voltages, inductor currents) such that any branch current / voltage in the circuit is of the form $y(t) = ke^{\lambda t}$. ▨

Some useful facts:

Fact 1 Given a circuit with state formulation $\dot{x} = Ax$, if the solution is $x(t) = x(0)\, e^{\lambda t}$, then any branch current / voltage in the circuit will be of the form $y(t) = ke^{\lambda t}$. (WHY?)

Fact 2 Given $\dot{x} = Ax$, let $\lambda \in \mathbb{R}$ be an eigenvalue of A with the corresponding eigenvector $v \in \mathbb{R}^n$. That is, $Av = \lambda v$ or $[\lambda I - A]v = 0$. Then for $x(0) = v$ the solution is $x(t) = x(0)\, e^{\lambda t}$. (WHY?)

Example Consider the below circuit with $i_L(0) = -3A$.



a) obtain the state eqn.

b) Find the init. cap. voltages so that the response does not display any oscillations.

Sol'n a) [Exercise]

For $x = \begin{bmatrix} v_{C_1} \\ v_{C_2} \\ i_L \end{bmatrix}$ we have $\dot{x} = \underbrace{\begin{bmatrix} 0 & -2 & -1 \\ 0 & -3 & -1 \\ 2 & 2 & 0 \end{bmatrix}}_{A} x$

Nat. freq? Roots of char. poly. $d(s) = \det(sI - A)$

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$$d(s) = \det\left\{\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s\end{bmatrix} - \begin{bmatrix} 0 & -2 & -1 \\ 0 & -3 & -1 \\ 2 & 2 & 0\end{bmatrix}\right\} = \begin{vmatrix} s & 2 & 1 \\ 0 & s+3 & 1 \\ -2 & -2 & s\end{vmatrix}$$

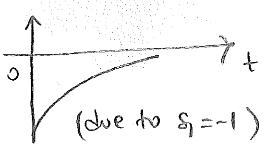
$$\Rightarrow d(s) = s^2(s+3) - 4 + 2(s+3) + 2s = s^3 + 3s^2 + 4s + 2 = (s+1)(s^2+2s+2)$$

$$\Rightarrow d(s) = (s+1)(s+1+j)(s+1-j)$$

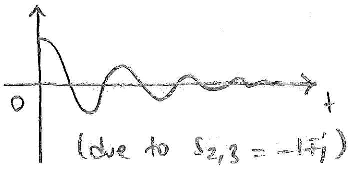
Nat. freq. (modes) of the circuit : $\boxed{s_1 = -1,\ s_{2,3} = -1 \mp j}$

Form of any branch voltage current ?

$$y(t) = \underbrace{k_1 e^{-t}}_{\text{non-oscillatory term}} + \underbrace{k_2 e^{-t}\cos(t+\phi_2)}_{\text{oscillatory term}}$$



(due to $s_1=-1$)



(due to $s_{2,3} = -1 \mp j$)

b) To avoid oscillations, excite the non-oscillatory mode $s_1=-1$. Let $v_1 \in \mathbb{R}^3$ be the eigenvector for $s_1=-1$. Then

$$[sI-A]\Big|_{s=s_1} \cdot v_1 = 0 \Rightarrow [-I-A]v_1 = \underbrace{\begin{bmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ -2 & -2 & -1\end{bmatrix}}_{[-I-A]} \underbrace{\begin{bmatrix} a \\ b \\ c\end{bmatrix}}_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix}$$

$$\left.\begin{aligned} -a + 2b + c &= 0 \\ 2b + c &= 0 \end{aligned}\right\} a = 0 \quad \left.\begin{aligned} & \\ -2a - 2b - c &= 0 \end{aligned}\right\} 2b + c = 0$$

Hence, $v_1 = \begin{bmatrix} 0 \\ b \\ -2b\end{bmatrix}$ is the eigenvector for $s_1=-1$ for any $b \neq 0$.

Now, we want $x(0) = v_1$ (because then $x(t) = v_1 e^{s_1 t} = v_1 e^{-t}$, i.e. no oscillations)

$$\Rightarrow \begin{bmatrix} 0 \\ b \\ -2b\end{bmatrix} = \begin{bmatrix} v_{c_1}(0) \\ v_{c_2}(0) \\ i_L(0)\end{bmatrix} = \begin{bmatrix} v_{c_1}(0) \\ v_{c_2}(0) \\ -3\end{bmatrix} \Rightarrow b = \frac{3}{2} \Rightarrow \boxed{v_{c_1}(0) = 0 \ \&\ v_{c_2}(0) = \frac{3}{2}\,V}$$

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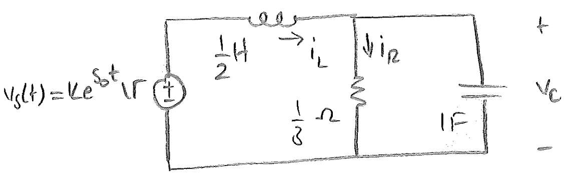
Particular solutions for complex exponential inputs

Consider the circuit



(C1)

Assume : $s_0 \in \mathbb{C}$ is not a natural freq. of the circuit

Question : What is the particular sol'n for the cap. voltage, $v_{C,p}(t) = ?$

Classic approach : Obtain the diff. eqn. $(D^2 + 3D + 2)\, v_C = 2 v_S$ & find $v_{C,p}(t)$.

— o —

New approach

We know $v_{C,p}(t) = \alpha e^{s_0 t}$ for some $\alpha \in \mathbb{C}$. In fact the particular sol'n for any branch current or voltage will have the same form:

$x_k(t) = X_k e^{s_0 t}$ for some $X_k \in \mathbb{C}$.

Fact Particular solutions satisfy terminal equations and obey KCL & KVL. (why?)

→ Since terminal equations hold :

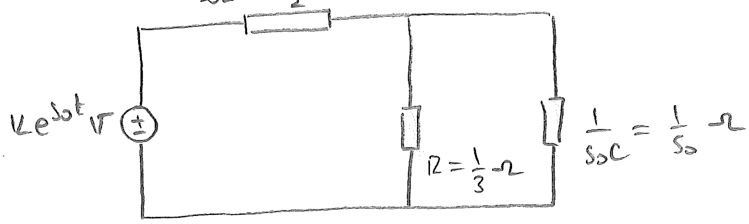
$$i_{C,p} = C D v_{C,p} = C\left\{\frac{d}{dt}\alpha e^{s_0 t}\right\} = s_0 C\left\{\alpha e^{s_0 t}\right\} = s_0 C\, v_{C,p}$$

$$\Rightarrow \frac{v_{C,p}}{i_{C,p}} = \left.\frac{1}{s_0 C}\right|_{C=1} = \frac{1}{s_0}.$$

That is, the ratio of voltage to current is constant. (Just like a resistor.)

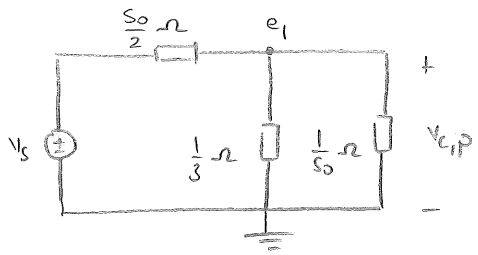
Likewise, $\frac{v_{L,p}}{i_{L,p}} = \left. s_0 L\right|_{L=\frac{1}{2}} = \frac{s_0}{2}$ & $\frac{v_{R,p}}{i_{R,p}} = \frac{1}{3}$

Hence, the particular sol'n for circuit (C1) will be identical to the solution of the LTI resistive circuit below



(C2)

→ since KCL / KVL hold:



Node eqn: $\dfrac{e_1 - v_s}{s_0/2} + \dfrac{e_1}{1/3} + \dfrac{e_1}{1/s_0} = 0$

$$\Rightarrow \left\{ \frac{2}{s_0} + 3 + s_0 \right\} e_1 = \frac{2}{s_0} v_s$$

$$\Rightarrow \frac{s_0^2 + 3s_0 + 2}{s_0} e_1 = \frac{2}{s_0} v_s$$

$$\Rightarrow e_1 = \frac{2}{s_0^2 + 3s_0 + 2} v_s .$$

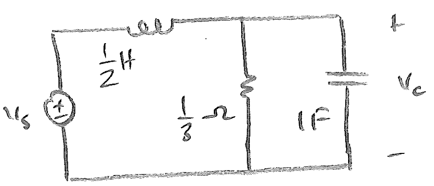
Hence, $$\boxed{v_{c,p}(t) = \frac{2}{s_0^2 + 3s_0 + 2} \cdot k e^{s_0 t}} \quad (1)$$

Examples:

$v_s(t)$	s_0	$v_{c,p}(t)$
e^{j2t}	$j2$	$\frac{1}{-1+j3} e^{j2t}$
e^t	1	$\frac{1}{3} e^t$
1	0	1
$3e^{j\frac{\pi}{6}} e^{-(2+j)t}$	$-2-j$	$3\sqrt{2}\, e^{-j\frac{7\pi}{12}} e^{-(2+j)t}$

Remark: Note that the denominator $den(s_0) = s_0^2 + 3s_0 + 2 = (s_0+1)(s_0+2)$ has roots at $s_0 = -1$ & $s_0 = -2$. Therefore when $v_s(t) = ke^{-t}$ or ke^{-2t} we cannot use (1) to compute the particular sol'n. WHY? (Hint: check the nat. frequencies of the circuit.)

Example: Find $v_{c,p}(t)$



$$v_s(t) = 6e^{-2t}\cos\left(t - \frac{\pi}{6}\right) \text{ V}$$

Sol'n

step 1: decompose $v_s = v_{s_1} + v_{s_2}$

step 2: find part sol'n v_{c,p_1}, v_{c,p_2}

step 3: superpose $v_{c,p} = v_{c,p_1} + v_{c,p_2}$

$$v_s(t) = 6e^{-2t}\cos\left(t - \frac{\pi}{6}\right)$$

$$= 6e^{-2t}\left\{ \frac{e^{j(t-\frac{\pi}{6})} + e^{-j(t-\frac{\pi}{6})}}{2} \right\}$$

$$= \underbrace{3e^{-j\frac{\pi}{6}} e^{-(2-j)t}}_{\downarrow} + \underbrace{3e^{j\frac{\pi}{6}} e^{-(2+j)t}}_{\downarrow}$$

$$v_{c,p}(t) = \overbrace{3\sqrt{2}\, e^{j\frac{7\pi}{12}} e^{-(2-j)t}} + \overbrace{3\sqrt{2}\, e^{-j\frac{7\pi}{12}} e^{-(2+j)t}}$$

$$= 3\sqrt{2}\, e^{-2t}\left\{ e^{j(t+\frac{7\pi}{12})} + e^{-j(t+\frac{7\pi}{12})} \right\}$$

$$\boxed{= 6\sqrt{2}\, e^{-2t}\cos\left(t + \frac{7\pi}{12}\right) \text{ V}}$$

LAPLACE TRANSFORM

$$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s) \qquad F(s) = \mathcal{L}\{f(t)\} \,, \quad f(t) = \mathcal{L}^{-1}\{F(s)\}$$

The transformation involves two domains :

1) The time domain, where the signal is represented by its waveform $f(t)$

2) The complex frequency domain, where the signal is represented by it transform $F(s)$.

The symbol s stands for the complex frequency variable. The Laplace transform is defined by the integral

$$\boxed{F(s) := \int_{0^-}^{\infty} f(t) e^{-st} \, dt}$$

The integral (hence $F(s)$) exists if $f(t)$ satisfies :

→ $f(t)$ is piecewise continuous

→ $f(t)$ is of exponential order, i.e., $\exists \gamma > 0$ s.t. $\lim_{t \to \infty} f(t) e^{-\gamma t} = 0$

We will always be working with $f(t)$ for which $F(s)$ exists. In particular, we assume $\lim_{t\to\infty} f(t) e^{-st} = 0$.

Example $f(t) = u(t)$ (unit step function). $F(s) = ?$

$$F(s) = \int_{0^-}^{\infty} u(t) e^{-st} dt = \underbrace{\int_{0^-}^{0^+} u(t) e^{-st} dt}_{=0} + \int_{0^+}^{\infty} 1 \cdot e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{t=0}^{t=\infty} = \frac{1}{s}$$

$$\Rightarrow \boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}}$$

Example $f(t) = e^{-\alpha t}$, $\alpha \in \mathbb{C}$

$$F(s) = \int_0^{\infty} e^{-\alpha t} e^{-st} dt = \int_0^{\infty} e^{-(s+\alpha)t} dt = -\frac{1}{s+\alpha} e^{-(s+\alpha)t} \Big|_0^{\infty} = \frac{1}{s+\alpha}$$

$$\Rightarrow \boxed{\mathcal{L}\{e^{-\alpha t}\} = \frac{1}{s+\alpha}}$$

Example $f(t) = \delta(t)$ (unit impulse function)

$$F(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) e^{-s \cdot 0} dt = 1 \quad \Rightarrow \boxed{\mathcal{L}\{\delta(t)\} = 1}$$

linearity property

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

Example $\mathcal{L}\{\sin wt\} = ?$ $\quad \sin wt = \dfrac{e^{jwt} - e^{-jwt}}{j2}$

$$\Rightarrow \mathcal{L}\{\sin wt\} = \frac{1}{j2}\left(\mathcal{L}\{e^{jwt}\} - \mathcal{L}\{e^{-jwt}\}\right)$$

$$= \frac{1}{j2}\left(\frac{1}{s-jw} - \frac{1}{s+jw}\right) = \frac{1}{\cancel{j2}} \cdot \frac{\cancel{j2}w}{s^2+w^2}$$

$$\Rightarrow \boxed{\mathcal{L}\{\sin wt\} = \frac{w}{s^2+w^2}}$$

Integration property

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

Proof let $g(t) = \int_{0^-}^{t} f(\tau)\,d\tau \Rightarrow dg = f\,dt$

let $v(t) = -\frac{1}{s}e^{-st} \Rightarrow dv = e^{-st}dt$

Now, $\mathcal{L}\{g(t)\} = \int_{0^-}^{\infty} g(t)\underbrace{e^{-st}dt}_{dv} = gv - \int v\,dg$ (integration by parts)

$$= g(t)\left\{-\frac{1}{s}e^{-st}\right\}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty}\left\{-\frac{1}{s}e^{-st}\right\}f(t)\,dt$$

$$= \underbrace{\frac{g(0^-)}{s}}_{0} + \frac{1}{s}\underbrace{\int_{0^-}^{\infty} f(t)e^{-st}dt}_{F(s)} = \frac{F(s)}{s}$$

Example Consider unit ramp $r(t) = t$

$$\mathcal{L}\{r(t)\} = \mathcal{L}\left\{\int_0^t u(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{u(t)\} = \frac{1}{s^2}$$

Hence, $\mathcal{L}\{\delta(t)\} = 1 \Rightarrow \mathcal{L}\{u(t)\} = \frac{1}{s} \Rightarrow \mathcal{L}\{r(t)\} = \frac{1}{s^2} \Rightarrow \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3} \Rightarrow \cdots$

Complex Differentiation

$$\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s) \quad \text{(WHY?)}$$

$$\Rightarrow \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

Differentiation in time domain

$$\mathcal{L}\{\dot{f}(t)\} = \int_{0^-}^{\infty} e^{-st} \underbrace{\dot{f}(t)\,dt}_{df}$$

integration by parts

$$= f(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\{-s e^{-st}\}dt$$

$$= -f(0^-) + s\underbrace{\int_{0^-}^{\infty} f(t)e^{-st}dt}_{F(s)}$$

$$\Rightarrow \mathcal{L}\{\tfrac{d}{dt}f(t)\} = sF(s) - f(0^-)$$

$$\Rightarrow \mathcal{L}\{\tfrac{d^n}{dt^n}f(t)\} = s^nF(s) - \{s^{n-1}f(0) + s^{n-2}\dot{f}(0) + s^{n-3}\ddot{f}(0) + \cdots + s\,f^{(n-2)}(0) + f^{(n-1)}(0)\}$$ (WHY?)

Example $\mathcal{L}\{\cos wt\} = ?$

we have $\rightarrow \mathcal{L}\{\sin wt\} = \frac{w}{s^2+w^2}$

$\rightarrow \cos wt = \frac{1}{w}\frac{d}{dt}\sin wt$

$$\Rightarrow \mathcal{L}\{\cos wt\} = \mathcal{L}\{\tfrac{1}{w}\tfrac{d}{dt}\sin wt\}$$

$$= \frac{1}{w}\{s\mathcal{L}\{\sin wt\} - \underbrace{\sin wt\big|_{t=0}}_{0}\}$$

$$= \frac{1}{w}\cdot\frac{sw}{s^2+w^2}$$

$$\Rightarrow \boxed{\mathcal{L}\{\cos wt\} = \frac{s}{s^2+w^2}}$$

S-domain translation property

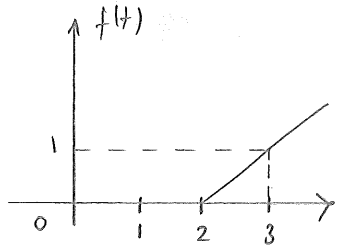
$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha) \qquad (\text{WHY?})$$

Example $\quad f(t) = e^{-\alpha t}\cos\omega t \Rightarrow F(s) = \dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Time-domain translation property

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s) \quad \text{for } a \geq 0 \qquad (\text{WHY?})$$

Example



$$\Rightarrow F(s) = \frac{e^{-2s}}{s^2}$$

Scale change

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) \quad \text{for } a > 0 \qquad (\text{WHY?})$$

Initial & Final Value Theorem

Initial Value Thm (IVT) : Let $f(t)$ be bounded for $t \in [0^-, 0^+]$. Then

$$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)}$$

Final Value Thm (FVT) : Suppose $\lim_{t\to\infty} f(t)$ exists. Then

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

Exercise Prove IVT & FVT. Hint : Recall $sF(s) - f(0^-) = \int_{0^-}^{\infty} \dot{f}(t)e^{-st}\,dt$.

Pole-zero diagram

We will be interested in signals whose transforms have the form

$$F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n} \quad \text{where} \quad a_i, b_i \text{ are real numbers.}$$

Expressed in factorized form

$$F(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \quad \text{where} \quad K = b_0 \text{ is called the } \underline{\text{scale factor}}$$

Denominator roots $\{p_1, p_2, \dots, p_n\}$ are called poles.

Numerator roots $\{z_1, z_2, \dots, z_m\}$ are called zeros.

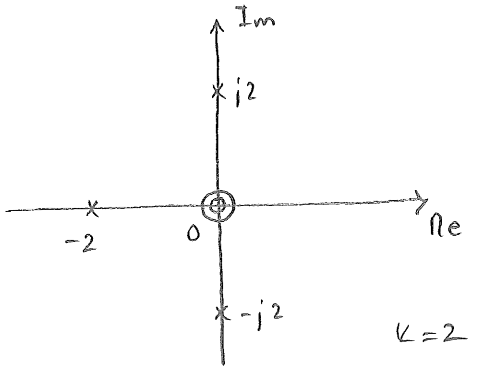
Example Find the poles & zeros of $f(t) = e^{-2t} + \cos 2t - \sin 2t$

$$F(s) = \frac{1}{s+2} + \frac{s}{s^2+4} - \frac{2}{s^2+4} = \frac{1}{s+2} + \frac{s-2}{s^2+4} = \frac{s^2+4+s^2-4}{(s+2)(s^2+4)} = 2\frac{s^2}{(s+2)(s+j2)(s-j2)}$$

$\Rightarrow$ poles : $\{-2, -j2, j2\}$

zeros : $\{0, 0\}$ (repeated zeros at 0)

pole-zero diagram of $F(s)$:



pole is denoted by ×

zero is denoted by ∘

Complex Plane

Inverse Laplace Transform

Problem: Given $F(s)$, find $f(t)$.

Let $F(s) = \dfrac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$ $\quad (b_0 \neq 0)$

If $n > m$ then $F(s)$ is called a proper rational function.

Special case: simple poles. If the denominator poly. of $F(s)$ has no repeated roots then $F(s)$ is said to have simple poles.

i.e. $F(s) = \dfrac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{(s-p_1)(s-p_2)\cdots(s-p_n)}$ & $p_i \neq p_j$ for $i \neq j$

Let F be proper with simple poles. Then we can write

$$F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}$$

The k_i's associated with each pole are called residues.

Note that $\boxed{k_i = (s-p_i)F(s)\Big|_{s=p_i}}$ (WHY?)

Example $F(s) = \dfrac{2s+6}{s^3+3s^2+2s} = \dfrac{2s+6}{s(s+1)(s+2)} = \dfrac{k_1}{s} + \dfrac{k_2}{s+1} + \dfrac{k_3}{s+2}$, $k_i = ?$

$$k_1 = \left(k_1 + \frac{k_2 s}{s+1} + \frac{k_3 s}{s+2}\right)_{s=0} = sF(s)\Big|_{s=0} = \frac{2s+6}{(s+1)(s+2)}\Big|_{s=0} = \frac{6}{2} = 3$$

$$k_2 = (s+1)F(s)\Big|_{s=-1} = \frac{2s+6}{s(s+2)}\Big|_{s=-1} = \frac{4}{-1} = -4$$, likewise $k_3 = 1$

$$\Rightarrow F(s) = \frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+2} \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \boxed{3 - 4e^{-t} + e^{-2t}}$$

Example $F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)} = \frac{20(s+3)}{(s+1)(s+1+j2)(s+1-j2)} = \frac{k_1}{s+1} + \frac{k_2}{s+1+j2} + \frac{k_3}{s+1-j2}$

$$k_1 = (s+1)F(s)\Big|_{s=-1} = \frac{20(s+3)}{s^2+2s+5}\Big|_{s=-1} = \frac{40}{4} = 10$$

$$k_2 = (s+1+j2)F(s)\Big|_{s=-1-j2} = \frac{20(s+3)}{(s+1)(s+1-j2)}\Big|_{s=-1-j2} = \frac{20(2-j2)}{-j2(-j4)} = -5(1-j)$$

$$k_3 = (s+1-j2)F(s)\Big|_{s=-1+j2} = \frac{20(s+3)}{(s+1)(s+1+j2)}\Big|_{s=-1+j2} = \frac{20(2+j2)}{j2(j4)} = -5(1+j)$$

Note that $k_3 = k_2^*$. This is not a coincidence. Residues of complex conjugate poles are complex conjugates of each other since $F(s)$ has real coefficients.

Now, $F(s) = \frac{10}{s+1} - \frac{5(1-j)}{s+1+j2} - \frac{5(1+j)}{s+1-j2}$

$$\Rightarrow f(t) = 10e^{-t} - 5(1-j)\,e^{-(1+j2)t} - 5(1+j)\,e^{-(1-j2)t}$$

Let us write $f(t)$ in a prettier form:

observe: $-5+j5 = 5\sqrt{2}\,e^{j\frac{3\pi}{4}}$ & $-5-j5 = 5\sqrt{2}\,e^{-j\frac{3\pi}{4}}$

$$\Rightarrow f(t) = 10e^{-t} + 5\sqrt{2}\,e^{-t}\,e^{-j(2t-\frac{3\pi}{4})} + 5\sqrt{2}\,e^{-t}\,e^{j(2t-\frac{3\pi}{4})}$$

$$= 10e^{-t} + 5\sqrt{2}\,e^{-t}\underbrace{\left\{ e^{-j(2t-\frac{3\pi}{4})} + e^{j(2t-\frac{3\pi}{4})} \right\}}_{2\cos(2t-\frac{3\pi}{4})}$$

$$\boxed{\Rightarrow f(t) = 10e^{-t} + 10\sqrt{2}\,e^{-t}\cos\left(2t - \frac{3\pi}{4}\right)}$$

SUMMARY

→ Inverse Laplace transform : "Given $F(s)$, find $f(t)$."

$$F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{(s-p_1)(s-p_2)\dots(s-p_n)}, \quad n > m \quad (\text{i.e. } F(s) \text{ proper})$$

→ when poles $(p_1, p_2, \dots, p_n)$ are simple ($p_i \neq p_j$ for $i \neq j$) we have

$$F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \dots + \frac{k_n}{s-p_n}$$

& $\boxed{k_i = (s-p_i) F(s) \Big|_{s=p_i}}$

then $f(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \dots + k_n e^{p_n t}$

Example (revisited)

$$F(s) = \frac{20s+60}{(s+1)(s^2+2s+5)} = \frac{20s+60}{(s+1)(s+1+j2)(s+1-j2)} = \frac{k_1}{s+1} + \frac{k_2}{s+1+j2} + \frac{k_2^*}{s+1-j2}$$

$k_1 = 10$, $k_2 = -5+j5 = 5\sqrt{2} e^{j\frac{3\pi}{4}}$

$$\Rightarrow f(t) = 10e^{-t} + \underbrace{5\sqrt{2} e^{j\frac{3\pi}{4}} e^{-(1+j2)t}}_{z} + \underbrace{5\sqrt{2} e^{-j\frac{3\pi}{4}} e^{-(1-j2)t}}_{z^*}$$

$z + z^* = 2\,\mathrm{Re}\{z\} = 2\,\mathrm{Re}\{z^*\}$

$$= 10e^{-t} + 2\,\mathrm{Re}\left\{5\sqrt{2} e^{-t} e^{j(2t - \frac{3\pi}{4})}\right\}$$

$$= 10e^{-t} + 10\sqrt{2} e^{-t} \,\mathrm{Re}\left\{e^{j(2t - \frac{3\pi}{4})}\right\}$$

$$\boxed{= 10e^{-t} + 10\sqrt{2} e^{-t} \cos\left(2t - \frac{3\pi}{4}\right)}$$

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[Another way]

$$F(s) = \frac{20s+60}{(s+1)(s^2+2s+5)} = \frac{10}{s+1} + \frac{As+B}{s^2+2s+5} = \frac{10(s^2+2s+5) + (As+B)(s+1)}{(s+1)(s^2+2s+5)}$$

$$\Rightarrow 10(s^2+2s+5) + (As+B)(s+1) = 20s+60 \Rightarrow A = -10 \;\&\; B = 10$$

$$\Rightarrow F(s) = \frac{10}{s+1} + \frac{-10s+10}{s^2+2s+5}$$ ⇝ decompose this into sine & cosine

$$\frac{-10s+10}{s^2+2s+5} = \frac{-10s+10}{(s+1)^2+2^2} = C\,\frac{s+1}{(s+1)^2+2^2} + D\,\frac{2}{(s+1)^2+2^2} \Rightarrow C(s+1)+2D = -10s+10$$

$$\Rightarrow C = -10, \; D = 10$$

Hence, $$F(s) = \frac{10}{s+1} - 10\,\frac{s+1}{(s+1)^2+2^2} + 10\,\frac{2}{(s+1)^2+2^2}$$

$$\Rightarrow \boxed{f(t) = 10e^{-t} - 10e^{-t}\cos 2t + 10e^{-t}\sin 2t}$$

— o —

General case (poles need not be simple)

$$F(s) = \frac{\cdots}{(s-p_1)^{n_1}(s-p_2)^{n_2}\cdots(s-p_\ell)^{n_\ell}} = \frac{A_1 s^{n_1-1}+\cdots}{(s-p_1)^{n_1}} + \cdots + \frac{A_\ell s^{n_\ell-1}+\cdots}{(s-p_\ell)^{n_\ell}}$$

$$\& \quad \frac{A_i s^{n_i-1}+\cdots}{(s-p_i)^{n_i}} = \frac{k_{i,0}}{(s-p_i)^{n_i}} + \frac{k_{i,1}}{(s-p_i)^{n_i-1}} + \cdots + \frac{k_{i,n_i-1}}{s-p_i}$$

$$k_{i,0}\frac{t^{n_i-1}}{(n_i-1)!}e^{p_it} + k_{i,1}\frac{t^{n_i-2}}{(n_i-2)!}e^{p_it} + \cdots + k_{i,n_i-1}e^{p_it}$$

→ how to find $k_{i,m}$?

$$k_{i,0} = (s-p_i)^{n_i}F(s)\Big|_{s=p_i}, \quad k_{i,1} = \frac{d}{ds}\left\{(s-p_i)^{n_i}F(s)\right\}\Big|_{s=p_i}, \quad \boxed{k_{i,m} = \frac{1}{m!}\frac{d^m}{ds^m}\left\{(s-p_i)^{n_i}F(s)\right\}\Big|_{s=p_i}}$$

Example

$$F(s) = \frac{1}{(s+1)^3 s^2}$$

$$\Rightarrow F(s) = \frac{k_{10}}{(s+1)^3} + \frac{k_{11}}{(s+1)^2} + \frac{k_{12}}{s+1} + \frac{k_{20}}{s^2} + \frac{k_{21}}{s}$$

$$k_{10} = (s+1)^3 F(s)\Big|_{s=-1} = \frac{1}{s^2}\Big|_{s=-1} = 1$$

$$k_{11} = \frac{d}{ds}\left\{\frac{1}{s^2}\right\}\Big|_{s=-1} = -\frac{2}{s^3}\Big|_{s=-1} = 2$$

$$k_{12} = \frac{1}{2}\frac{d^2}{ds^2}\left\{\frac{1}{s^2}\right\}\Big|_{s=-1} = \frac{1}{2}\cdot\frac{6}{s^4}\Big|_{s=-1} = 3$$

$$k_{20} = s^2 F(s)\Big|_{s=0} = \frac{1}{(s+1)^3}\Big|_{s=0} = 1$$

$$k_{21} = \frac{d}{ds}\left\{\frac{1}{(s+1)^3}\right\}\Big|_{s=0} = -\frac{3}{(s+1)^4}\Big|_{s=0} = -3$$

$$\Rightarrow F(s) = \frac{1}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{3}{s+1} + \frac{1}{s^2} - \frac{3}{s}$$

$$\Rightarrow \boxed{f(t) = \left(\frac{t^2}{2} + 2t + 3\right)e^{-t} + t - 3}$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$

& $e^{-\alpha t} f(t) \updownarrow F(s+\alpha)$

Initial & Final Value Theorems

Initial Value Thm (IVT) : Let $f(t)$ be bounded for $t \in [0^-, 0^+]$ then

$$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)}$$

Final Value Thm (FVT) : Suppose $\lim_{t\to\infty} f(t)$ exists. Then

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

Exercise : Prove IVT & FVT. Hint : Recall $sF(s) - f(0^-) = \int_{0^-}^{\infty} \dot{f}(t) e^{-st} dt$.

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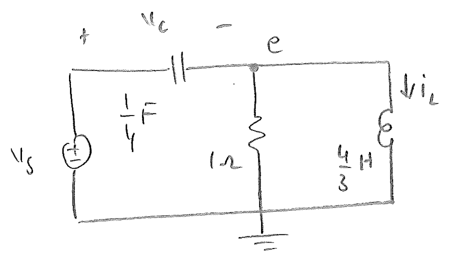
CIRCUIT RESPONSE USING LAPLACE TRANSFORM

Example



For $v_s(t) = \frac{4}{3} e^{-2t} u(t)$ V

find zero-state response for $i_L(t)$.

by Node analysis:

$$KCL \Rightarrow \frac{1}{4} D(e - v_s) + \frac{e}{1} + i_L(0^-) + \frac{3}{4} D^{-1} e = 0$$

$$\Rightarrow \frac{1}{4} De + 1e + \frac{3}{4} D^{-1} e = \frac{1}{4} D v_s - i_L(0^-) \quad (1)$$

$$\underbrace{\left[\frac{1}{4} D + 1 + \frac{3}{4} D^{-1}\right]}_{Y(D)} e = \frac{1}{4} D v_s - i_L(0^-)$$

$$Y(s) = \frac{1}{4} s + 1 + \frac{3}{4s} = \frac{s^2 + 4s + 3}{4s}$$

$$= \frac{(s+1)(s+3)}{4s}$$

$\det Y(s) = 0 \Rightarrow s = -1$ & $s = -3$

$\Rightarrow$ nat. freq. : $s_1 = -1$, $s_2 = -3$

$$\mathcal{L}\{(1)\} \Rightarrow \frac{1}{4}\{sE - e(0^-)\} + E + \frac{3}{4} \cdot \frac{E}{s} = \frac{1}{4}\{sV_s - v_s(0^-)\} - \frac{i_L(0^-)}{s}$$

$$e(0^-) = v_s(0^-) - v_c(0^-)$$

$$\Rightarrow \left(\frac{s}{4} + 1 + \frac{3}{4s}\right) E = \frac{s}{4} V_s - \frac{1}{4} v_c(0^-) - \frac{i_L(0^-)}{s}$$

$$\Rightarrow E = \frac{s^2}{s^2+4s+3} V_s - \frac{s}{s^2+4s+3} v_c(0^-) - \frac{4}{s^2+4s+3} i_L(0^-) \quad (2)$$

$$i_L = i_L(0^-) + \frac{3}{4} D^{-1} e \Rightarrow I_L = \frac{i_L(0^-)}{s} + \frac{3}{4s} E \quad (3)$$

$$(2) \,\&\, (3) \Rightarrow I_L = \underbrace{\frac{3}{4} \frac{s}{s^2+4s+3} V_s}_{\text{zero-state response}} \underbrace{- \frac{3}{4} \frac{1}{s^2+4s+3} v_c(0^-) + \frac{s+4}{s^2+4s+3} i_L(0^-)}_{\text{zero-input response}} \quad (4)$$

zero-state response for $V_s = \mathcal{L}\left\{ \frac{4}{3} e^{-2t} u(t) \right\} = \frac{4}{3} \frac{1}{s+2}$:

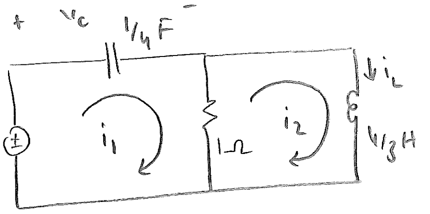
$$I_L = \frac{3}{4} \cdot \frac{s}{s^2+4s+3} \left(\frac{4}{3} \cdot \frac{1}{s+2} \right) = \frac{s}{(s+1)(s+2)(s+3)} = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3}$$

$$k_1 = (s+1) I_L \Big|_{s=-1} = \frac{s}{(s+2)(s+3)} \Big|_{s=-1} = -\frac{1}{2}$$

$$k_2 = \frac{s}{(s+1)(s+3)} \Big|_{s=-2} = 2 \quad \& \quad k_3 = \frac{s}{(s+1)(s+2)} \Big|_{s=-3} = -\frac{3}{2}$$

$$\Rightarrow \boxed{ i_L(t) = \underbrace{2e^{-2t}}_{\text{particular sol'n}} \underbrace{- \frac{1}{2} e^{-t} - \frac{3}{2} e^{-3t}}_{\text{hmg. sol'n}} \ A } \quad \text{for } t > 0$$

by Mesh analysis :



$$\left. \begin{aligned} &\text{mesh ①} : -v_s + v_c(0^-) + 4D^{-1} i_1 + i_1 - i_2 = 0 \\ &\text{mesh ②} : i_2 - i_1 + \frac{4}{3} D i_2 = 0 \end{aligned} \right\} (*)$$

$$\Rightarrow \underbrace{\begin{bmatrix} 4D^{-1}+1 & -1 \\ -1 & 1+\frac{4}{3}D \end{bmatrix}}_{Z(D)} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} v_s - v_c(0^-) \\ 0 \end{bmatrix}$$

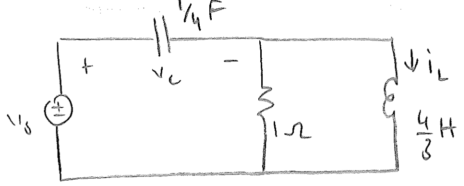
$$\mathcal{L}\{(*)\} : \left. \begin{aligned} & -V_s + \frac{v_c(0^-)}{s} + \frac{4}{s} I_1 + I_1 - I_2 = 0 \\ \& \quad & I_2 - I_1 + \frac{4}{3} \left(s I_2 - i_L(0^-) \right) = 0 \end{aligned} \right\} \begin{aligned} i_L(0^-) &= 0 \\ v_c(0^-) &= 0 \end{aligned} \quad \Rightarrow \underbrace{\begin{bmatrix} \frac{4}{s}+1 & -1 \\ -1 & 1+\frac{4}{3}s \end{bmatrix}}_{Z(s)} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

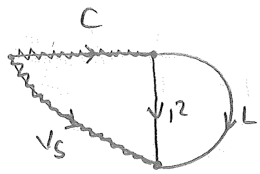
$$\det Z(s) = \left(\frac{4}{s} + 1 \right) \left(1 + \frac{4}{3} s \right) - 1 = \frac{4s^2+16s+12}{3s} = \frac{4(s+1)(s+3)}{3s} \Rightarrow \text{nat. freq. } s_1 = -1, s_2 = -3$$

$$\Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{3s}{4(s+1)(s+3)} \begin{bmatrix} 1+\frac{4}{3}s & 1 \\ 1 & \frac{4}{s}+1 \end{bmatrix} \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

$$\Rightarrow I_L = I_2 = \frac{3s}{4(s+1)(s+3)} V_s \quad \text{(zero-state sol'n)} \quad \text{as expected.}$$

by state equation:





$$\left.\begin{aligned} \frac{1}{4} Dv_c &= i_{1\Omega} + i_L = \frac{v_s - v_c}{1} + i_L \\ \frac{4}{3} Di_L &= v_s - v_c \end{aligned}\right\} \quad \begin{aligned} Dv_c &= -4v_c + 4i_L + 4v_s \\ Di_L &= -\frac{3}{4}v_c + \frac{3}{4}v_s \end{aligned} \qquad \text{let } x = \begin{bmatrix} v_c \\ i_L \end{bmatrix}$$

$$\Rightarrow \quad \dot{x} = \underbrace{\begin{bmatrix} -4 & 4 \\ -\frac{3}{4} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 4 \\ \frac{3}{4} \end{bmatrix}}_{B} v_s$$

Laplace transform: $sX - x(0^-) = AX + BV_s$

$$\Rightarrow [sI - A]X = x(0^-) + BV_s$$

$$\Rightarrow X = \underbrace{[sI-A]^{-1}x(0^-)}_{\text{zero-input sol'n}} + \underbrace{[sI-A]^{-1}BV_s}_{\text{zero-state sol'n}}$$

$$|sI - A| = \begin{vmatrix} s+4 & -4 \\ \frac{3}{4} & s \end{vmatrix} = s(s+4) + 3 = s^2 + 4s + 3 = (s+1)(s+3) \Rightarrow \text{nat. freq. } s_1 = -1, s_2 = -3$$

$$[sI - A]^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s & 4 \\ -\frac{3}{4} & s+4 \end{bmatrix}$$

zero-state response:

$$\underbrace{\begin{bmatrix} V_c \\ I_L \end{bmatrix}}_{X} = [sI-A]^{-1}BV_s = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s & 4 \\ -\frac{3}{4} & s+4 \end{bmatrix} \begin{bmatrix} 4 \\ \frac{3}{4} \end{bmatrix} V_s$$

$$\Rightarrow I_L = \frac{1}{(s+1)(s+3)} \begin{bmatrix} -\frac{3}{4} & s+4 \end{bmatrix} \begin{bmatrix} 4 \\ \frac{3}{4} \end{bmatrix} V_s = \frac{3}{4} \frac{s}{(s+1)(s+3)} V_s \quad \text{(zero-state sol'n)}$$

as expected,

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SINUSOIDAL STEADY STATE ANALYSIS

Consider an LTI circuit such that

→ Each natural freq. $\lambda_i = \alpha_i + jw_i$ satisfies $Re\{\lambda_i\} = \alpha_i < 0$. This implies that the hmg. sol'n always decays to zero, regardless of the init. cond.

→ The inputs are sinusoidal signals at the same frequency w rad/sec.

In such a circuit the response converges to the particular sol'n which is also a sinusoid with frequency w (the freq. of the inputs). This solution is called the sinusoidal steady state (SSS) solution.

Some useful identities

1) $e^{j\phi} = \cos\phi + j\sin\phi$

2) $Re\{z_1 + z_2\} = Re\{z_1\} + Re\{z_2\}$

3) $D\, Re\{f(t)\} = Re\{D f(t)\}$

4) Let $w \neq 0$ be real. If $Re\{X_1 e^{jwt}\} = Re\{X_2 e^{jwt}\}$ for all $t \in \mathbb{R}$

then $X_1 = X_2$.

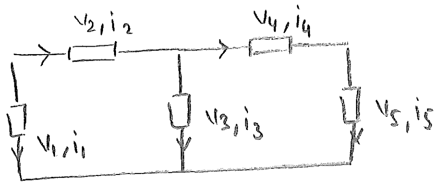
proof: $t = 0 \Rightarrow Re\{X_1\} = Re\{X_2\}$ (*)

$t = \frac{3\pi}{2w} \Rightarrow Re\{-jX_1\} = Re\{-jX_2\} \Rightarrow Im\{X_1\} = Im\{X_2\}$ (**)

(*) & (**) $\Rightarrow X_1 = X_2$

— o —

Consider a circuit in sinusoidal steady state



(input frequency : w rad/sec)

Since the circuit is in SSS, each voltage / current signal should look like

$x_k(t) = A_k \cos(wt + \phi_k)$ for some magnitude A_k and phase ϕ_k.

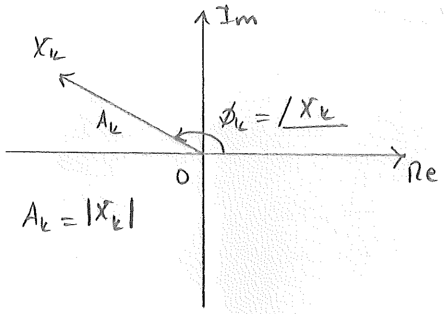
<u>Definition</u> For a signal $x_k(t) = A_k \cos(\omega t + \phi_k)$, the complex number $X_k := A_k e^{j\phi_k}$ is called the <u>phasor</u> of $x_k(t)$.

Note that given the pair (X_k, ω) we can reconstruct the signal $x_k(t)$ as

(X_k: phasor of $x_k(t)$; ω: the operating freq. of the circuit)

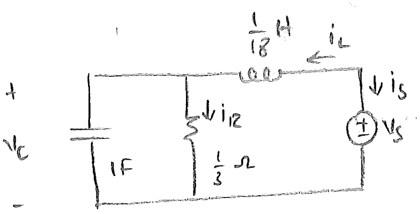
$$x_k(t) = \operatorname{Re}\{X_k e^{j\omega t}\} = A_k \cos(\omega t + \phi_k)$$

<u>Phasor diagram</u> contains the vector representation of X_k on the complex plane:



$A_k = |X_k|$

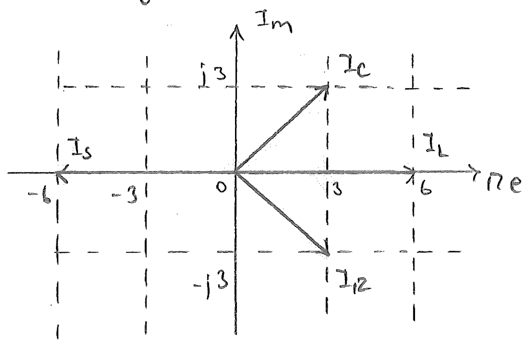
<u>Example</u> [Circuit in SSS]



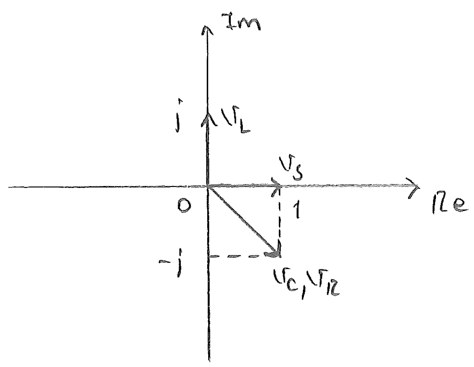
$v_s(t) = \cos 3t$ V

<u>Signals</u>	<u>Phasors</u>
$i_L(t) = 6\cos 3t$ A	$I_L = 6e^{j0} = 6$ A
$i_R(t) = 3\sqrt{2}\cos(3t - \frac{\pi}{4})$ A	$I_R = 3\sqrt{2}e^{-j\frac{\pi}{4}} = 3 - j3$ A
$i_C(t) = 3\sqrt{2}\cos(3t + \frac{\pi}{4})$ A	$I_C = 3\sqrt{2}e^{j\frac{\pi}{4}} = 3 + j3$ A
$i_s(t) = -6\cos 3t$ A	$I_s = 6e^{j\pi} = -6$ A
$v_L(t) = -\sin 3t = \cos(3t + \frac{\pi}{2})$ V	$V_L = e^{j\frac{\pi}{2}} = j$ V
$v_R(t) = \sqrt{2}\cos(3t - \frac{\pi}{4})$ V	$V_R = \sqrt{2}e^{-j\frac{\pi}{4}} = 1 - j$ V
$v_C(t) = \sqrt{2}\cos(3t - \frac{\pi}{4})$ V	$V_C = \sqrt{2}e^{-j\frac{\pi}{4}} = 1 - j$ V
$v_s(t) = \cos 3t$ V	$V_s = e^{j0} = 1$ V

<u>Phasor Diagram</u>



Note: $I_L = I_C + I_R$ (KCL?)



Note: $V_s = V_C + V_L$ (KVL?)

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Claim Phasors of current obey KCL

proof Let some currents satisfy KCL: $i_1(t) + i_2(t) + \cdots + i_n(t) = 0$

$$\Rightarrow \operatorname{Re}\{I_1 e^{j\omega t}\} + \operatorname{Re}\{I_2 e^{j\omega t}\} + \cdots + \operatorname{Re}\{I_n e^{j\omega t}\} = 0$$

$$\Rightarrow \operatorname{Re}\{I_1 e^{j\omega t} + \cdots + I_n e^{j\omega t}\} = 0$$

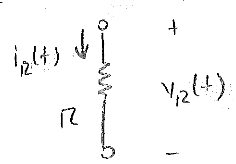
$$\Rightarrow \operatorname{Re}\{(I_1 + I_2 + \cdots + I_n) e^{j\omega t}\} = 0$$

$$\Rightarrow I_1 + I_2 + \cdots + I_n = 0 \quad \blacksquare$$

Likewise, phasors of voltages obey KVL.

Terminal Equations in Phasors

Resistor



(belongs to a circuit in SSS at freq. ω rad/sec)

$v_R(t) = \operatorname{Re}\{V_R e^{j\omega t}\}$, V_R: phasor of $v_R(t)$

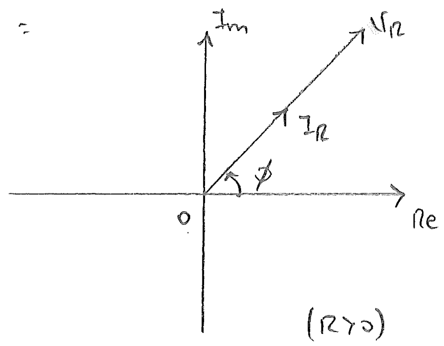
$i_R(t) = \operatorname{Re}\{I_R e^{j\omega t}\}$, I_R: phasor of $i_R(t)$

$V_R \sim I_R$?

$$v_R(t) = R\, i_R(t) \Rightarrow \operatorname{Re}\{V_R e^{j\omega t}\} = R \cdot \operatorname{Re}\{I_R e^{j\omega t}\}$$

$$= \operatorname{Re}\{R I_R e^{j\omega t}\} \Rightarrow \boxed{V_R = R I_R}$$

Phasor diagram:



$(R > 0)$

Both V_R & I_R have the same phase (ϕ). Such signals are said to be in phase.

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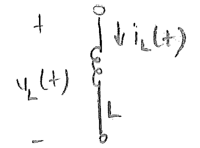
Inductor



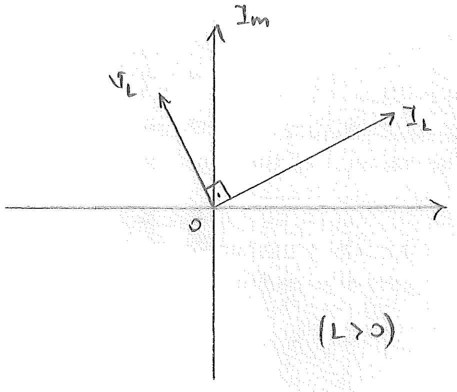
$$v_L(t) = \mathrm{Re}\{V_L e^{j\omega t}\}$$

$$i_L(t) = \mathrm{Re}\{I_L e^{j\omega t}\}$$

$$v_L(t) = L D i_L(t) \quad \Rightarrow \quad \mathrm{Re}\{V_L e^{j\omega t}\} = L D\, \mathrm{Re}\{I_L e^{j\omega t}\}$$

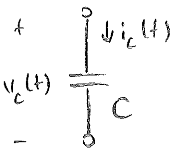
$$= \mathrm{Re}\{L I_L D e^{j\omega t}\}$$

$$= \mathrm{Re}\{j\omega L I_L e^{j\omega t}\} \quad \Rightarrow \quad \boxed{V_L = j\omega L I_L}$$



The current phasor I_L <u>lags</u> the voltage phasor V_L by $90°$. (or voltage <u>leads</u> current by $90°$)

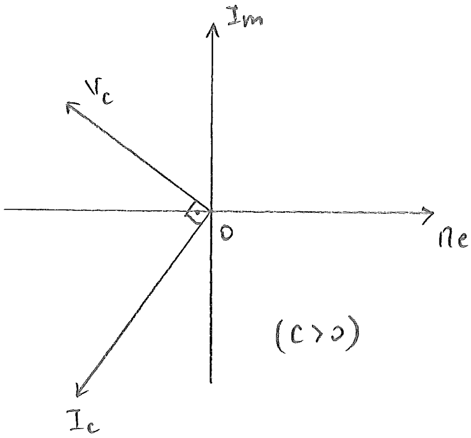
Capacitor



$$v_c(t) = \mathrm{Re}\{V_c e^{j\omega t}\}$$

$$i_c(t) = \mathrm{Re}\{I_c e^{j\omega t}\}$$

Dual of inductor $\Rightarrow$ $\boxed{I_c = j\omega C V_c}$



The current leads the voltage by $90°$.
(or, voltage lags current by $90°$.)

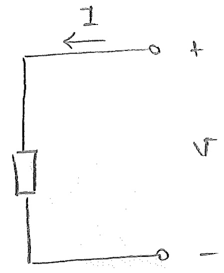
Impedance & Admittance

Consider a one-port that

→ is part of a circuit in SSS

→ contains no independent sources

→ contains the control variables of its dependent sources



(I, v: phasors)

We define:

impedance $\quad Z := \frac{V}{I} = |Z| e^{j\theta_Z} = |Z| \angle \theta_Z \quad$ (measured in ohms)

admittance $\quad Y := \frac{I}{V} = \frac{1}{Z} \quad$ (measured in mhos ℧)

Remark Impedance is the generalization of resistance in phasor domain. It is a function of the operating frequency ω.

Impedance Z in rectangular form: $Z = R + jX$

$R = \mathrm{Re}\{Z\}$: resistance

$X = \mathrm{Im}\{Z\}$: reactance

Likewise, $Y = G + jB$

$G = \mathrm{Re}\{Y\}$: conductance

$B = \mathrm{Im}\{Y\}$: susceptance

Warning $\quad R \neq \frac{1}{G} \;\; \& \;\; X \neq \frac{1}{B}$ in general !

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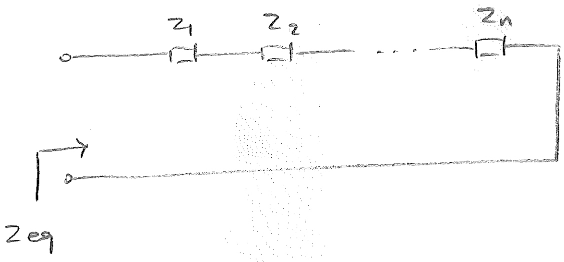
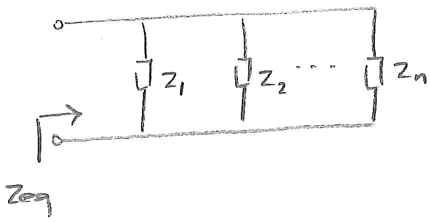
we have

Component	Impedance	Admittance
R	$Z_R = R$	$Y_R = \frac{1}{R}$
L	$Z_L = j\omega L$	$Y_L = \frac{1}{j\omega L}$
C	$Z_C = \frac{1}{j\omega C}$	$Y_C = j\omega C$

Series & parallel connections

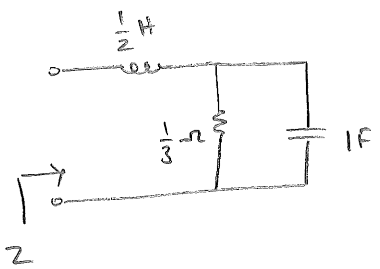


$$Z_{eq} = Z_1 + Z_2 + \cdots + Z_n$$



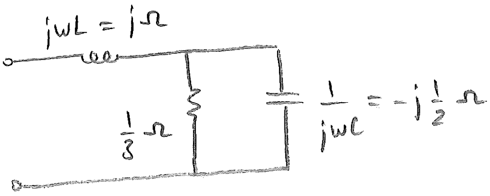
$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$$

Example

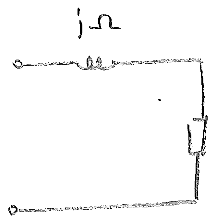


find Z for $\omega = 2$ rad/sec.

Sol'n

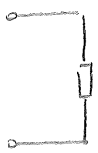


$\equiv$



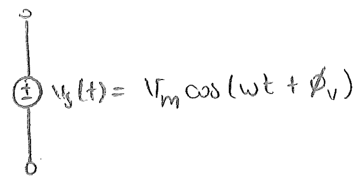
$$\frac{1}{3} \,//\, -j\frac{1}{2} = \frac{1}{3+j2} = \frac{3}{13} - j\frac{2}{13}\,\Omega$$

$\equiv$

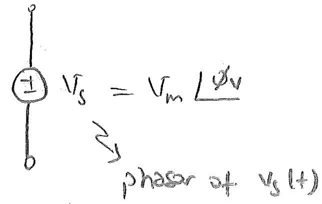


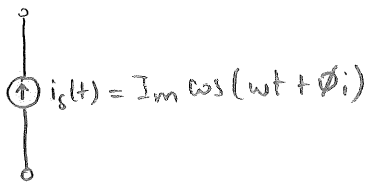
$$j + \frac{3}{13} - j\frac{2}{13} = \boxed{\frac{3}{13} + j\frac{11}{13}\,\Omega}$$

Independent sources (in SSS)

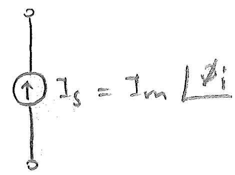


in phaser domain $\longrightarrow$

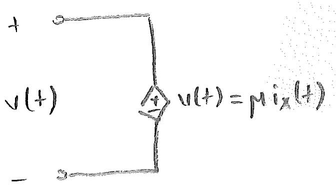




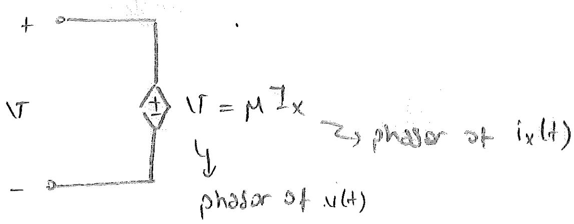
P.D. $\longrightarrow$



Dependent sources

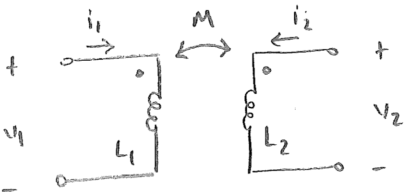


P.D. $\longrightarrow$

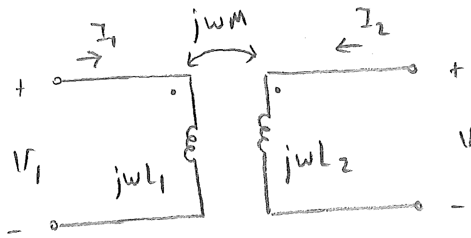


i_x : control current

Coupled Inductors



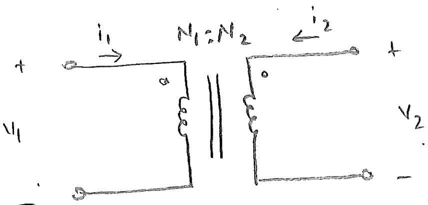
P.D. $\longrightarrow$



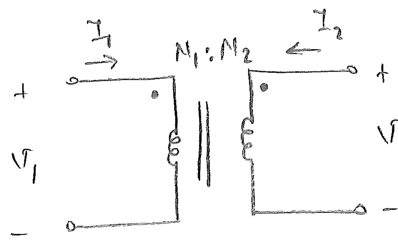
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Ideal Transformer



P.D. $\longrightarrow$



$\frac{v_1}{N_1} = \frac{v_2}{N_2}$ & $N_1 i_1 + N_2 i_2 = 0$

$\frac{V_1}{N_1} = \frac{V_2}{N_2}$ & $N_1 I_1 + N_2 I_2 = 0$

PHASOR DOMAIN ANAYSIS

[Goal: Solve the circuit in sinusoidal steady state.]

Step 1: Transform the circuit into phasor domain by representing the input and circuit variables as phasors and passive circuit elements by their impedances.

Step 2: Solve the phasor domain circuit treating it like an LTI resistive circuit. That is, obtain the phasors of the asked variables.

Step 3: Transform the phasors (obtained in step 2) into their time domain representation.

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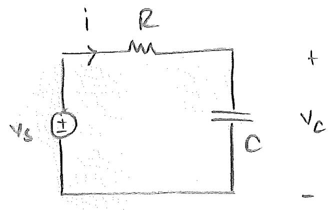
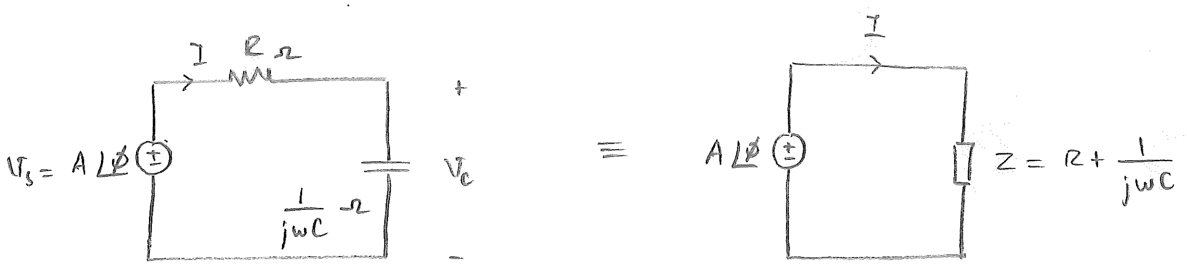
Example:



$v_s(t) = A\cos(\omega t + \phi)$

Find $i(t)$, $v_c(t)$ in steady state.

Sol'n: Phasor domain circuit:



$$Z = R - j\frac{1}{\omega C} = \left[R^2 + \frac{1}{(\omega C)^2}\right]^{1/2} \angle -\tan^{-1}\left\{\frac{1/\omega C}{R}\right\} = \frac{\sqrt{(\omega RC)^2 + 1}}{\omega C} \angle -\tan^{-1}\frac{1}{\omega RC}$$

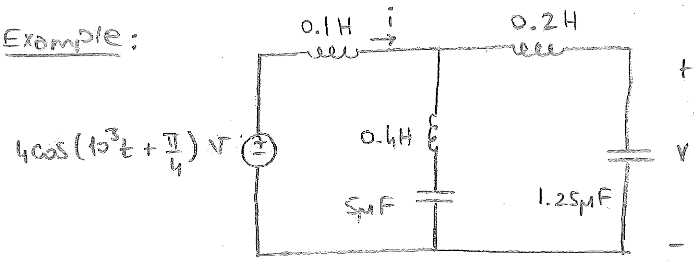
$$\Rightarrow I = \frac{A\angle\phi}{Z} = \frac{A\omega C}{\sqrt{1+\omega^2R^2C^2}} \angle \phi + \tan^{-1}\frac{1}{\omega RC} \quad (1)$$

$$\Rightarrow V_c = \frac{1}{j\omega C}\cdot I = \frac{1}{\omega C}\angle -\frac{\pi}{2} \cdot I = \frac{A}{\sqrt{1+\omega^2R^2C^2}} \angle \phi + \tan^{-1}\frac{1}{\omega RC} - \frac{\pi}{2} \quad (2)$$

$$(1) \Rightarrow i(t) = \frac{A\omega C}{\sqrt{1+\omega^2R^2C^2}} \cos\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega RC}\right)$$

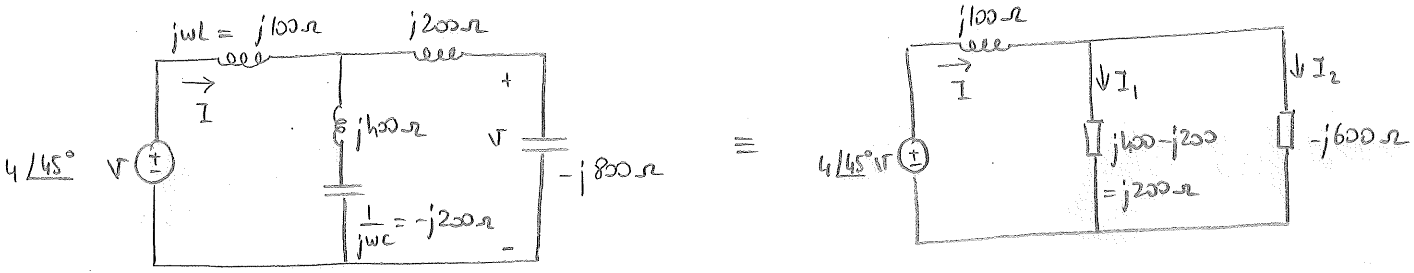
$$(2) \Rightarrow v_c(t) = \frac{A}{\sqrt{1+\omega^2R^2C^2}} \cos\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega RC} - \frac{\pi}{2}\right)$$

Example:



Find the particular solutions $i_p(t)$ & $v_p(t)$.

Sol'n: Equivalent circuit in phasor domain at freq. $\omega = 10^3$ rad/sec :



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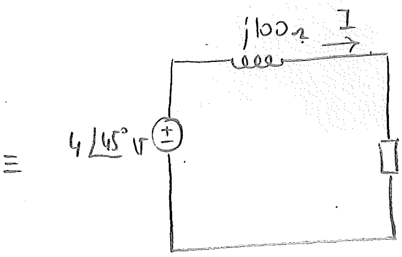


$$Z = \left(\frac{1}{j200} + \frac{1}{-j600}\right)^{-1} = \frac{j200(-j600)}{j200-j600} = \frac{200\times 600}{-j400} = j300\,\Omega$$

$$\Rightarrow I = \frac{4\angle 45^\circ}{j100+j300} = \frac{4\angle 45^\circ}{400\angle 90^\circ} = \frac{1}{100}\angle -45^\circ \text{ A} \quad (1)$$

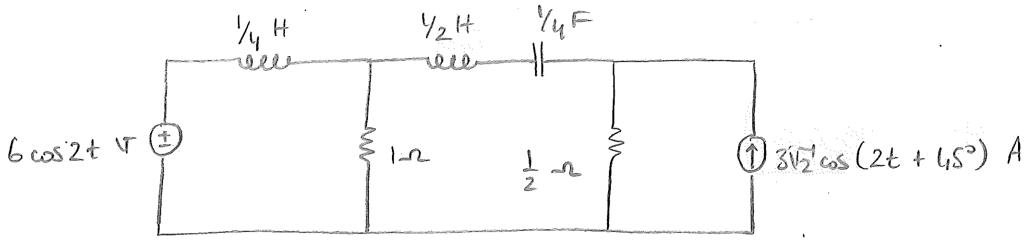
Current division $\Rightarrow$ $I_2 = \dfrac{j200}{j200-j600} I = -\dfrac{1}{2} I = -\dfrac{1}{200}\angle -45^\circ = \dfrac{1}{200}\angle 135^\circ$ A

$$\Rightarrow V = -j800\, I_2 = 800\angle -90^\circ \; \frac{1}{200}\angle 135^\circ = 4\angle 45^\circ \text{ volts} \quad (2)$$

(1) $\Rightarrow$ $i_p(t) = Re\{I e^{j\omega t}\} = \boxed{10\cos(1000t - \frac{\pi}{4})\text{ mA}}$

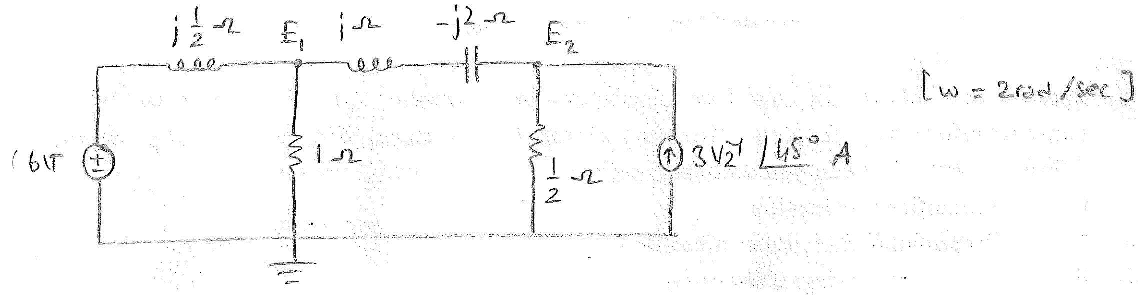
(2) $\Rightarrow$ $\boxed{v_p(t) = 4\cos(1000t + \frac{\pi}{4})\text{ V}}$

Example [Node analysis]



Find the steady state voltage across the 1Ω-resistor.

Sol'n :



[w = 2 rad/sec]

Node ①: $\dfrac{E_1-6}{j\,1/2} + \dfrac{E_1}{1} + \dfrac{E_1-E_2}{-j} = 0$ (1)

Node ②: $\dfrac{E_2-E_1}{-j} + \dfrac{E_2}{1/2} - 3\sqrt{2}\angle 45° = 0$ (2)

(1) $\Rightarrow \{-j2+1+j\}E_1 - jE_2 = -j12 \Rightarrow (1-j)E_1 - jE_2 = -j12 \Rightarrow E_2 = -(1+j)E_1 + 12$ (3)

(2) $\Rightarrow -jE_1 + (2+j)E_2 = 3+j3$ (4)

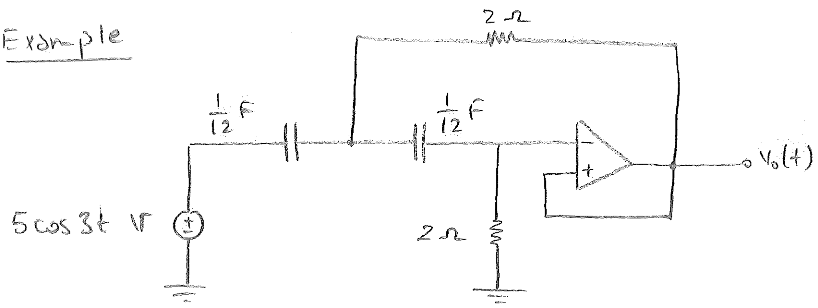
(3) & (4) $\Rightarrow -jE_1 + (2+j)(12-(1+j)E_1) = 3+j3$

$\Rightarrow -(j + (1+j)(2+j))E_1 = -12(2+j) + 3 + j3$

$\Rightarrow -(1+j4)E_1 = -21 - j9 \Rightarrow E_1 = \dfrac{21+j9}{1+j4} = \dfrac{57-j75}{17} = \dfrac{3}{17}(19-j25)$ (5)

Finally, (5) $\Rightarrow$ $\boxed{e_1(t) = \dfrac{3}{17}\sqrt{19^2+25^2}\,\cos\left(2t - \tan^{-1}(25/19)\right)\ V}$

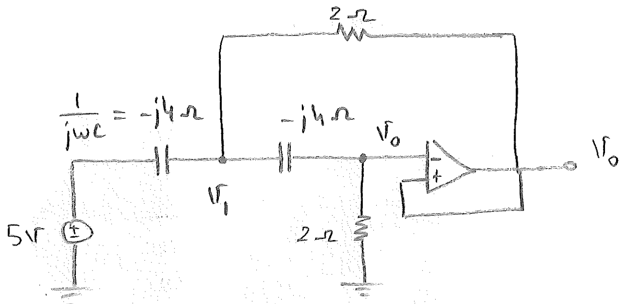
Example



Assume : OPAMP is in lin. region & circuit is in SSS.

Find $v_o(t)$.

Sol'n



[w = 3rad/sec]

$$\frac{V_1-5}{-j4}+\frac{V_1-V_o}{2}+\frac{V_1-V_o}{-j4}=0 \Rightarrow \left(\frac{1}{2}+j\frac{1}{2}\right)V_1-\left(\frac{1}{2}+j\frac{1}{4}\right)V_o=j\frac{5}{4} \quad (1)$$

$$\frac{V_o-V_1}{-j4}+\frac{V_o}{2}=0 \Rightarrow \left(\frac{1}{2}+j\frac{1}{4}\right)V_o-j\frac{1}{4}V_1=0 \Rightarrow V_1=(1-j2)V_o \quad (2)$$

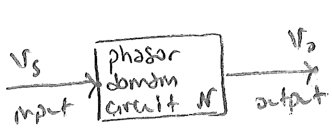
$$(1)\,\&\,(2) \Rightarrow \left[\left(\frac{1}{2}+j\frac{1}{2}\right)(1-j2)-\left(\frac{1}{2}+j\frac{1}{4}\right)\right]V_o=j\frac{5}{4} \Rightarrow \left(1-j\frac{3}{4}\right)V_o=j\frac{5}{4}$$

$$\Rightarrow \frac{5}{4}\,\underline{/-\tan^{-1}3/4}\;V_o=\frac{5}{4}\,\underline{/90^\circ} \Rightarrow V_o=1\,\underline{/90^\circ+\tan^{-1}(3/4)}\;V \quad (3)$$

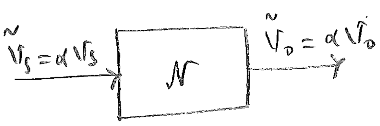
$$(3) \Rightarrow \boxed{v_o(t)=\cos\left(3t+90^\circ+\tan^{-1}\frac{3}{4}\right)\,V}$$

Question : What if $\tilde{v}_s(t)=5\cos(3t+23^\circ)$ V ?

Answer :



linearity $\Rightarrow$



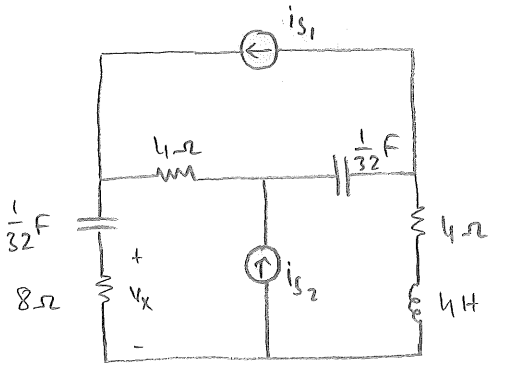
Now, $\tilde{v}_s(t)=5\cos(3t+23^\circ)V \Rightarrow \tilde{V}_s=\overbrace{[1\,\underline{/23^\circ}]}^{\alpha}\cdot V_s$

$$\Rightarrow \tilde{V}_o=[1\,\underline{/23^\circ}]\cdot V_o=1\,\underline{/90^\circ+\tan^{-1}\frac{3}{4}+23^\circ}$$

$$\Rightarrow \tilde{v}_o(t)=\cos\left(3t+113^\circ+\tan^{-1}\frac{3}{4}\right)V$$

▨

Example [Mesh Analysis]



Given :

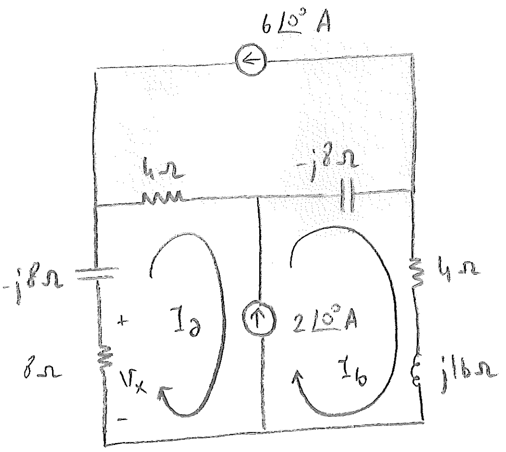
→ $i_{s_1}(t) = 6\cos 4t$ A

→ $i_{s_2}(t) = 2\cos 4t$ A

→ circuit is in SSS

Find $v_x(t)$.

Sol'n [$w = 4$ rad/sec]



Supermesh:

$$(8-j8)\,I_a + 4\,(I_a+6) - j8\,(I_b+6) + (4+j16)\,I_b = 0 \qquad (1)$$

Constraint : $I_b - I_a = 2 \;\Rightarrow\; I_b = I_a + 2$ (2)

$$(1)\&(2) \Rightarrow (8-j8+4)\,I_a + (-j8+4+j16)\,(I_a+2) = -24+j48$$

$$\Rightarrow (12-j8)\,I_a + (4+j8)\,I_a = -24+j48 - 2\,(4+j8)$$

$$\Rightarrow 16 I_a = -32+j32 \Rightarrow I_a = -2+j2 \text{ A}$$

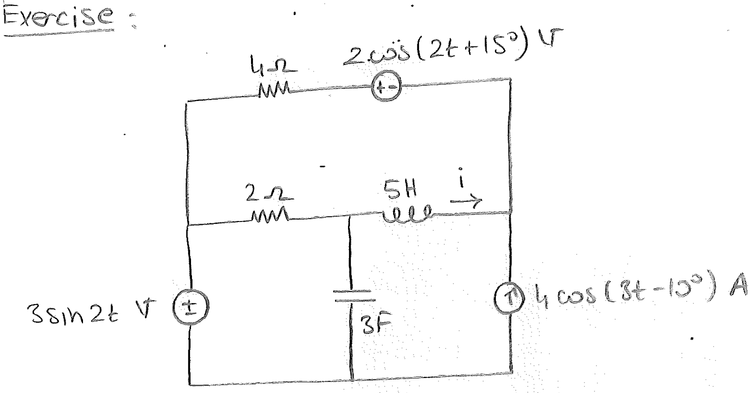
$$\Rightarrow V_x = -8 I_a = 16-j16 = 16\sqrt{2}\,\angle -45^\circ \text{ V}$$

$$\Rightarrow \boxed{v_x(t) = 16\sqrt{2}\cos\left(4t - \frac{\pi}{4}\right) \text{ V}}$$

Superposition

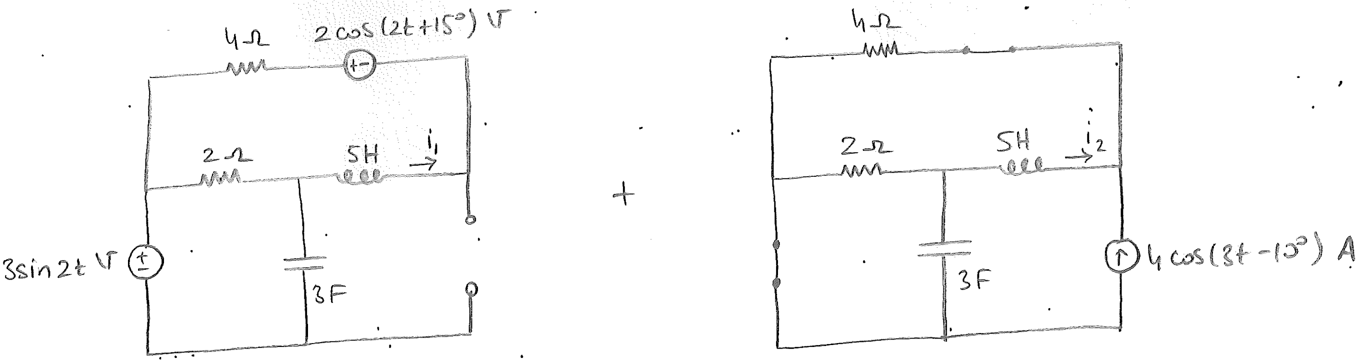
In phasor domain, superposition is possible / meaningful only for sources operating at the same frequency w. For sources of different frequencies $\therefore$ superposition should be applied to time-domain signals.

Exercise:

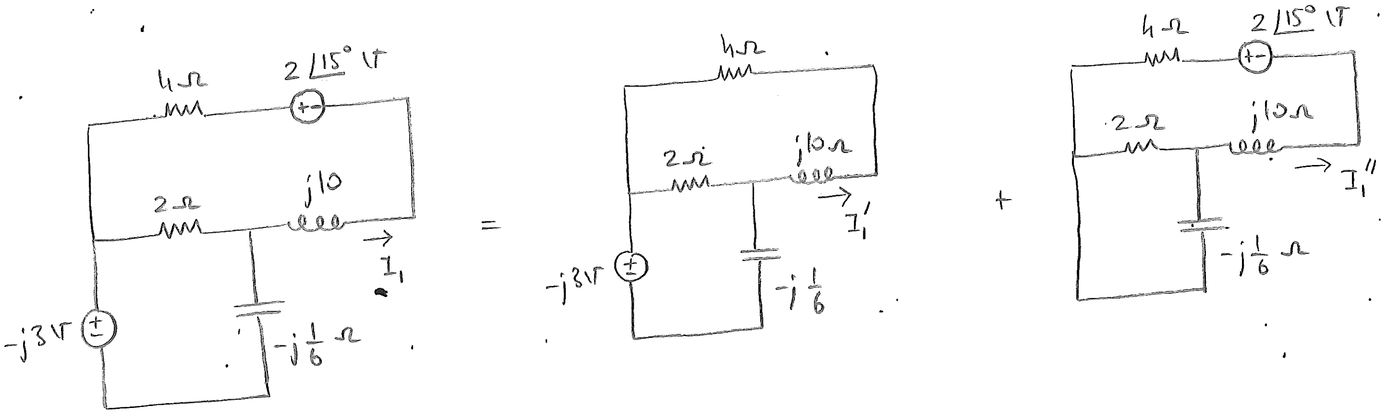


Find $i(t)$ in the steady state.

Superposition in time domain: $i(t) = i_1(t) + i_2(t)$ where



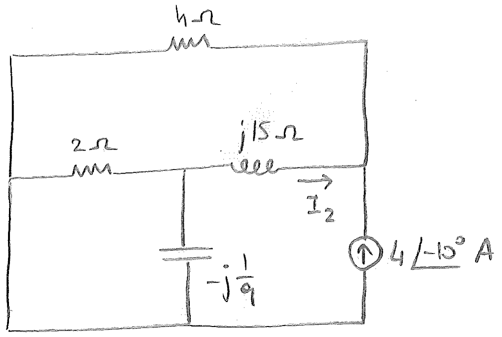
Superposition in phasor domain: $I_1 = I_1' + I_1''$



$$\left.\begin{aligned} I_1' &= 0.2509 + j\,0.1276 \text{ A} \\ I_1'' &= 0.1138 - j\,0.1500 \text{ A} \end{aligned}\right\} \quad I_1 = 0.3654 \angle -3.506° \text{ A}$$

$$\left.\begin{aligned} I_1' &\cong 0.25 + j\,0.13 \text{ A} \\ I_1'' &\cong 0.11 - j\,0.15 \text{ A} \end{aligned}\right\} \quad I_1 \cong 0.37 \angle -4° \text{ A}$$

To find $i_2(t)$:

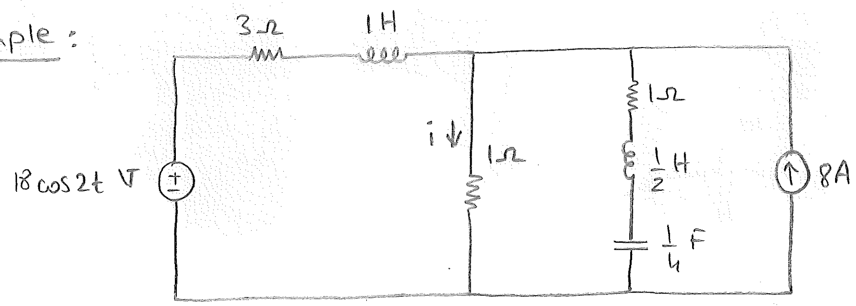


$I_2 = 1.0377\angle{95.^\circ0596}$ A

$I_2 \cong 1.04\angle{95^\circ}$ A

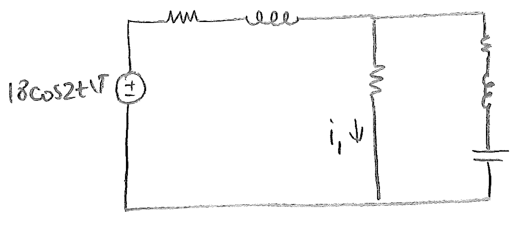
Hence, $\boxed{i(t) \cong 0.37\cos(2t-4^\circ) + 1.04\cos(3t+95^\circ) \quad A}$

Example:

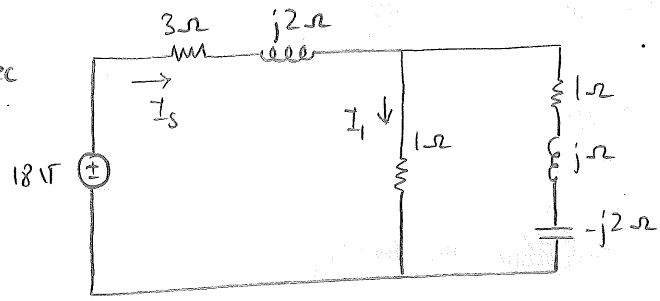


Find $i(t)$ in the steady state.

Kill DC source:



$\omega = 2$ rad/sec

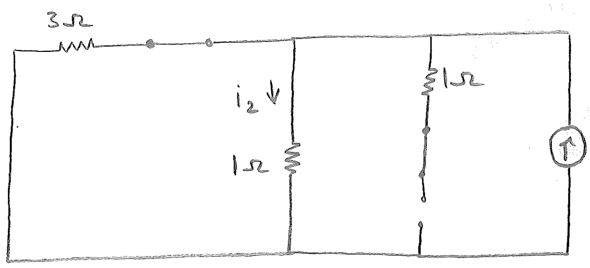


$$I_s = \frac{18}{(3+j2) + (1\,/\!/\,(1-j))} = \frac{18}{3.6 + j1.8} = 4 - j2$$

$$I_1 = \frac{1-j}{1+(1-j)} \cdot I_s = \frac{3-j}{5} \cdot (4-j2) = 2-j2 = 2\sqrt{2}\angle{-45^\circ} \text{ A}$$

$\Rightarrow i_1(t) = 2\sqrt{2}\cos(2t-45^\circ)$ A

Now, kill AC source :

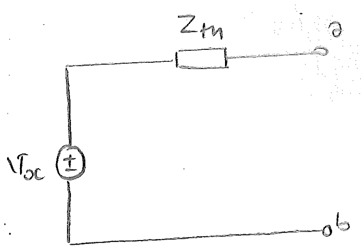


$i_2(t) = 6A$

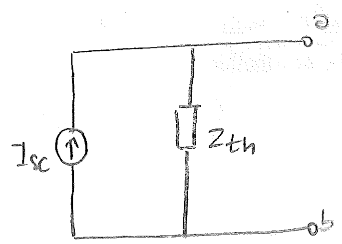
By superposition, $i(t) = i_1(t) + i_2(t) = \boxed{6 + 2\sqrt{2}\cos(2t - 45°)\ A}$

Thevenin / Norton Equivalent Circuits

Just like in the LTI resistive case.



$\equiv$

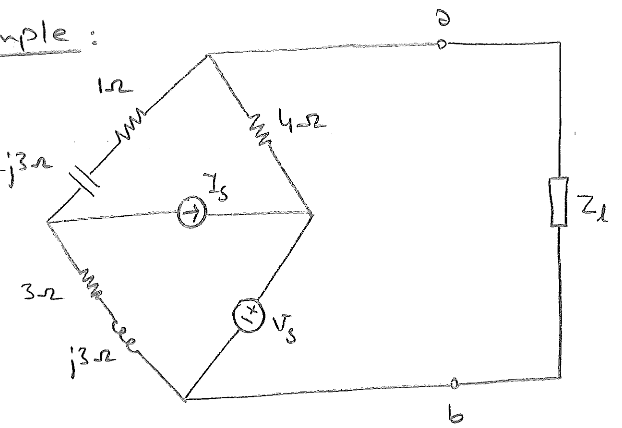


$Z_{th} = \frac{V_{oc}}{I_{sc}}$

Thevenin Equivalent in Phasor domain

Norton Equivalent

Example :

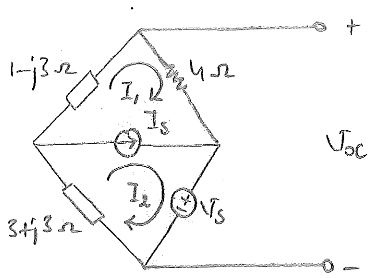


$I_s = \sqrt{2}\angle{-45°}\ A$

$V_s = 8\angle{0°}\ V$

Find the Thevenin equiv. circuit seen by the load Z_l.

Sol'n: First, find V_{oc}



Supermesh: $(1-j3)I_1 + 4I_1 + V_s + (3+j3)I_2 = 0$

$$\Rightarrow (5-j3)I_1 + (3+j3)I_2 = -8 \quad (1)$$

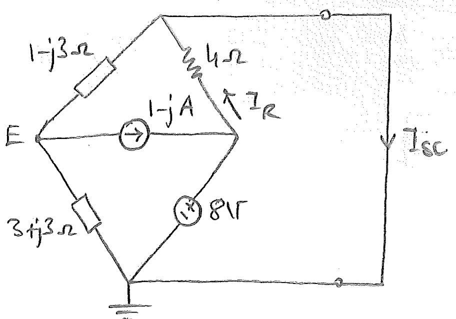
Constraint: $I_2 - I_1 = I_s \Rightarrow I_2 = I_1 + (1-j) \quad (2)$

$$(1)\ \&\ (2) \Rightarrow (5-j3)I_1 + (3+j3)\{I_1 + (1-j)\} = -8$$

$$\Rightarrow 8I_1 = -8 - (3+j3)(1-j) = -14 \Rightarrow I_1 = -\frac{7}{4}\ A$$

$$\Rightarrow V_{oc} = 4I_1 + V_s = 4\left(-\frac{7}{4}\right) + 8 = \boxed{1V}$$

Now, compute I_{sc}



$$\frac{E}{3+j3} + 1-j + \frac{E}{1-j3} = 0$$

$$\Rightarrow \left(\frac{1}{3+j3} + \frac{1}{1-j3}\right)E = -1+j$$

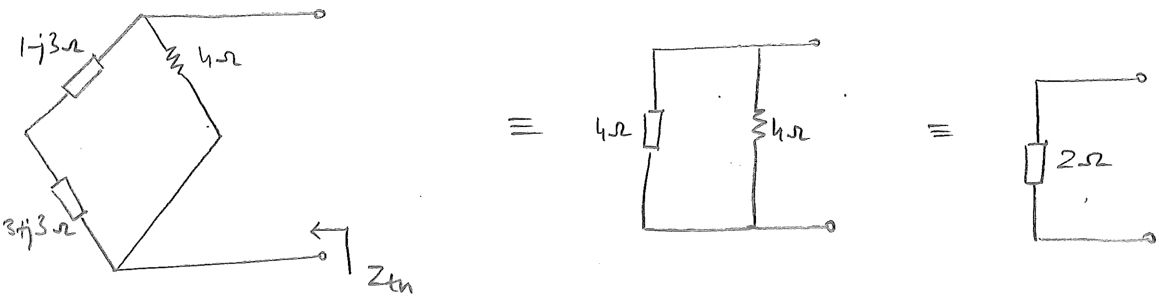
$$\Rightarrow \frac{4+j2}{15}E = -1+j \Rightarrow E = -\frac{3}{2} + j\frac{9}{2}\ V$$

$$I_R = \frac{8}{4} = 2A$$

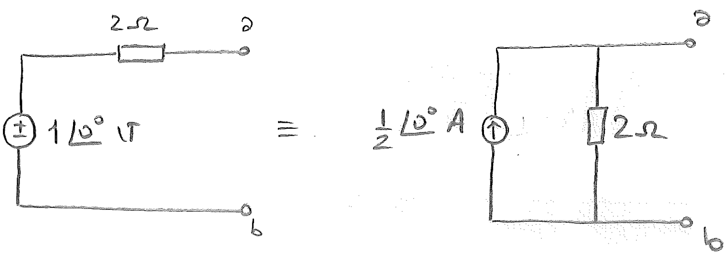
$$\Rightarrow I_{sc} = \frac{E}{1-j3} + I_R = -\frac{3}{2}(1-j3)\frac{1}{1-j3} + 2 = \boxed{\frac{1}{2}A}$$

Hence, $Z_{th} = \dfrac{V_{oc}}{I_{sc}} = \dfrac{1}{1/2} = \boxed{2\,\Omega}$

Alternatively, we can compute Z_{th} by killing the sources:



Finally, Z_ℓ sees:



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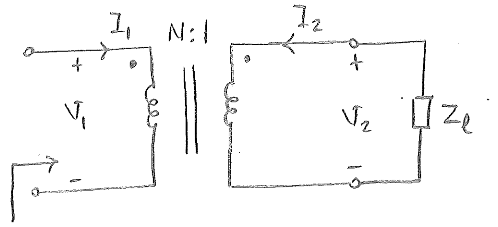
Example :

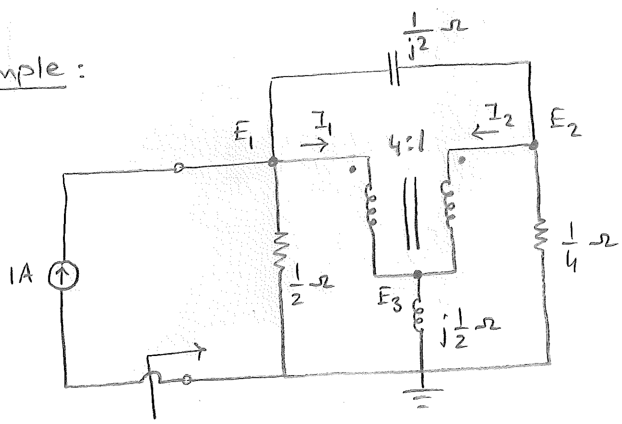


$Z_{in} = ?$

$$Z_{in} = \frac{V_1}{I_1} = \frac{N V_2}{-I_2/N} = \frac{N\{-Z_\ell I_2\}}{-I_2/N} = N^2 Z_\ell$$

$$\Rightarrow \boxed{Z_{in} = N^2 Z_\ell}$$

Example :



$Z_{in} = ?$

Node ①: $-1 + \frac{E_1}{1/2} + I_1 + \frac{E_1 - E_2}{1/j2} = 0$

$\Rightarrow (2+j2)E_1 - j2E_2 + I_1 = 1 \quad (1)$

Node ②: $I_2 + \frac{E_2}{1/4} + \frac{E_2 - E_1}{1/j2} = 0$

$\Rightarrow -j2E_1 + (4+j2)E_2 + I_2 = 0 \quad (2)$

Node ③: $-j2E_3 - I_1 - I_2 = 0 \quad (3)$

voltage constraint: $\frac{E_1 - E_3}{4} = E_2 - E_3 \Rightarrow E_1 - 4E_2 + 3E_3 = 0 \quad (4)$

current constraint: $4I_1 + I_2 = 0 \quad (5)$

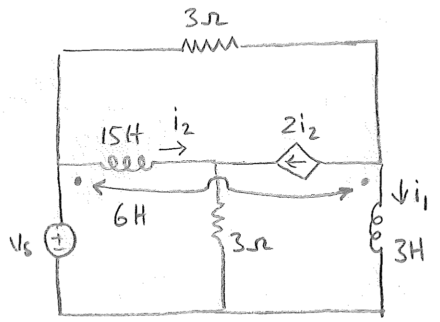
(1),(2),(3),(4)',(5) $\Rightarrow$

$$\begin{bmatrix} 2+j2 & -j2 & 0 & 1 & 0 \\ -j2 & 4+j2 & 0 & 0 & 1 \\ 0 & 0 & -j2 & -1 & -1 \\ 1 & -4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

MATLAB & (6) $\Rightarrow E_1 = \frac{47}{180} - j\frac{39}{180}$ V

$$\Rightarrow Z_{in} = \frac{E_1}{1} = \boxed{\frac{47}{180} - j\frac{39}{180}\ \Omega}$$

Example :



a) obtain the state eqn. for $x = [i_1 \ i_2]^T$.

b) Show that this circuit is stable.

c) For $V_s(t) = 30\cos(t+\Psi)$ V find : the steady state sol'n $x_{ss}(t)$

d) Find the init. inductor currents so that $x(t) = x_{ss}(t)$.

Sol'n a) [Exercise]

Answer : $\dot{x} = \begin{bmatrix} -5 & -4 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1/3 \end{bmatrix} V_s$

b) [Exercise] Recall : Nat. freq. = eigenvalues of A.

c) In steady state $x(t) = \begin{bmatrix} r_1 \cos(\omega t + \phi_1) \\ r_2 \cos(\omega t + \phi_2) \end{bmatrix}$ where $\omega = 1$ rad/sec

Define phasors $X_1 = r_1 e^{j\phi_1}$, $X_2 = r_2 e^{j\phi_2}$, and $V_s = 30 e^{j\Psi}$

Let $X = [X_1 \ X_2]^T$

Then $x(t) = \mathrm{Re}\{X e^{j\omega t}\}$ which should satisfy the state eqn. $\dot{x} = Ax + BV_s$

$\Rightarrow \frac{d}{dt} \mathrm{Re}\{X e^{j\omega t}\} = A\,\mathrm{Re}\{X e^{j\omega t}\} + B\,\mathrm{Re}\{V_s e^{j\omega t}\}$

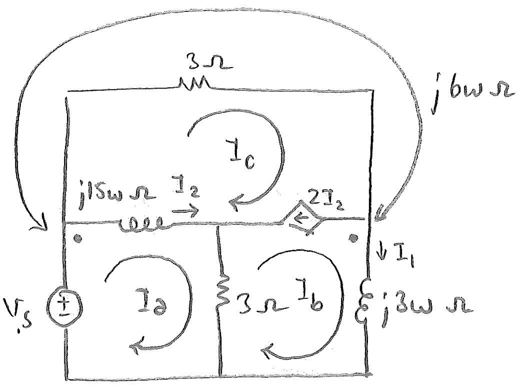
$\Rightarrow \mathrm{Re}\{j\omega X e^{j\omega t}\} = \mathrm{Re}\{(AX + BV_s)e^{j\omega t}\}$

$\Rightarrow j\omega X = AX + BV_s \Rightarrow X = (j\omega I - A)^{-1} B V_s = \begin{bmatrix} 7+j \\ -1-j3 \end{bmatrix} e^{j\Psi}$

Hence, $x_{ss}(t) = \begin{bmatrix} \sqrt{50}\cos(t + \Psi + \tan^{-1} 1/7) \\ \sqrt{10}\cos(t + \Psi + \pi + \tan^{-1} 3) \end{bmatrix}$

d) $x(0) = x_{ss}(0)$.

Exercise Verify part (c) by mesh analysis



$$\begin{pmatrix} V_1 = j3\omega I_1 + j6\omega I_2 \\ V_2 = j15\omega I_2 + j6\omega I_1 \end{pmatrix}$$

Mesh ⓐ : $-V_s + j15\omega(I_a - I_c) + j6\omega I_b + 3(I_a - I_b) = 0$

Supermesh ⓑ+ⓒ : $j15\omega(I_c - I_a) - j6\omega I_b + 3I_c + j3\omega I_b - j6\omega(I_c - I_a) + 3(I_b - I_a) = 0$

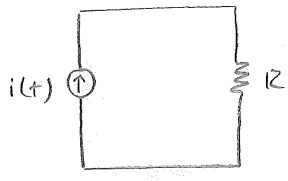
Constraint : $I_c - I_b = 2(I_a - I_c)$

Exercise How about node formulation?

#1 16 Mart 2014

Effective (RMS) Value

Consider the two circuits

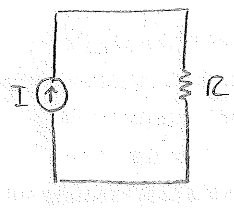


N_{AC}

$i(t+T) = i(t)$, periodic signal

instantaneous power dissipated on the resistor: $p(t) = R\, i(t)^2$

average power: $P_{av} = \frac{1}{T}\int_0^T p(t)\,dt$



N_{DC}

$I > 0$, constant

$p(t) = P_{av} = RI^2$

Suppose : The average powers are equal for N_{AC} and N_{DC}.

Question : what does the above condition imply?

$$\Rightarrow \frac{1}{T}\int_0^T \cancel{R}\, i(t)^2\,dt = \cancel{R} I^2 \quad \Rightarrow \quad \frac{1}{T}\int_0^T i(t)^2\,dt = I^2$$

$$\Rightarrow \sqrt{\frac{1}{T}\int_0^T i(t)^2\,dt} = I$$

This motivates the following definition.

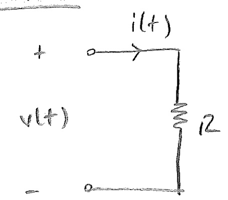
Definition : Given a periodic signal $i(t) = i(t+T)$, the effective or RMS (root mean square) value $I_{eff} > 0$ is defined via

$$\boxed{I_{eff}^2 = \frac{1}{T}\int_0^T i(t)^2\,dt}$$

Exercise : Show that for $i(t) = A\cos(\omega t + \phi)$ we have $I_{eff} = \frac{A}{\sqrt{2}}$.

Instantenous & average power in SSS

Resistor



Let $i(t) = I_m \cos(wt + \phi)$; period $T = \frac{2\pi}{w}$ sec. , $I_{eff} = \frac{I_m}{\sqrt{2}}$

$$\Rightarrow p(t) = i(t)v(t)$$
$$= R I_m \cos(wt + \phi) I_m \cos(wt + \phi)$$
$$= \frac{1}{2} R I_m^2 (1 + \cos[2(wt + \phi)])$$

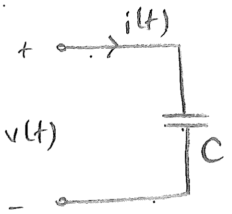
$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$

Hence, inst. power : $p(t) = \frac{1}{2} R I_m^2 (1 + \cos(2wt + 2\phi))$, periodic with T

aver. power : $P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2} R I_m^2 \left\{ \underbrace{\frac{1}{T}\int_0^T dt}_{=1} + \underbrace{\frac{1}{T}\int_0^T \cos(2wt + 2\phi)dt}_{=0} \right\}$

$$\Rightarrow P_{av} = \frac{1}{2} R I_m^2 = \boxed{R I_{eff}^2}$$

Capacitor



Let $v(t) = V_m \cos(wt + \phi)$, period $T = 2\pi/w$ sec , $V_{eff} = \frac{V_m}{\sqrt{2}}$

$$\Rightarrow I = jwC V = jwC V_m \angle\phi = wC V_m \angle \phi + \frac{\pi}{2}$$

$$\Rightarrow i(t) = wC V_m \cos(wt + \phi + \frac{\pi}{2})$$

$$\Rightarrow p(t) = i(t)v(t) = wC V_m^2 . \cos(wt + \phi)\cos(wt + \phi + \frac{\pi}{2})$$
$$= \frac{1}{2} wC V_m^2 \cos(2wt + 2\phi + \frac{\pi}{2})$$

$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$

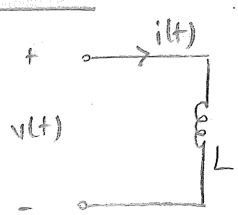
$\Rightarrow \boxed{P_{av} = 0}$ because p(t) is purely sinusoidal.

How about stored energy?

$e(t) = \frac{1}{2} C v(t)^2 = \frac{1}{2} C V_m^2 \cos^2(wt + \phi)$, periodic with T

$$\Rightarrow E_{av} = \frac{1}{T}\int_0^T e(t)dt = \frac{1}{4} C V_m^2 = \boxed{\frac{1}{2} C V_{eff}^2}$$

Inductor :



$i(t) = I_m \cos(\omega t + \phi)$, period $T = 2\pi/\omega$, $I_{eff} = \frac{I_m}{\sqrt{2}}$

By duality we can write

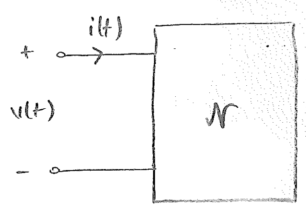
$$p(t) = \frac{1}{2}\omega L I_m^2 \cos[2(\omega t + \phi + \frac{\pi}{4})]$$

$$\boxed{P_{av} = 0}$$

$$e(t) = \frac{1}{2} L I_m^2 \cos^2(\omega t + \phi)$$

$$E_{av} = \frac{1}{4} L I_m^2 = \boxed{\frac{1}{2} L I_{eff}^2}$$

Power into one-port



Let $i(t) = I_m \cos(\omega t + \phi_i)$

$v(t) = V_m \cos(\omega t + \phi_v)$

$$\Rightarrow p(t) = i(t)v(t)$$

$$= I_m V_m \cos(\omega t + \phi_i)\cos(\omega t + \phi_v)$$

$$= \frac{1}{2} I_m V_m \cos(\phi_v - \phi_i) + \underbrace{\frac{1}{2} I_m V_m \cos(2\omega t + \phi_i + \phi_v)}_{\text{purely sinusoidal term, i.e., has zero average}}$$

$$\Rightarrow P_{av} = \frac{1}{2} I_m V_m \cos(\phi_v - \phi_i) = \boxed{I_{eff} V_{eff} \cos(\phi_v - \phi_i)}$$

Term "$\cos(\phi_v - \phi_i)$" is called the power factor. when N is passive and ind. source free.

Now suppose N contains no ind. sources and therefore can be represented by an impedance Z. That is, $V = ZI$.

Observe : $Z = \frac{V_m \angle \phi_v}{I_m \angle \phi_i} = \frac{V_m}{I_m} \angle \phi_v - \phi_i = |Z| \angle \phi_v - \phi_i$

Then, $P_{av} = \frac{1}{2} V_m I_m \cos(\phi_v - \phi_i) = \frac{1}{2} I_m \{I_m |Z|\} \cos(\phi_v - \phi_i) = \frac{1}{2} I_m^2 \{|Z| \cos(\phi_v - \phi_i)\}$

Hence, $P_{av} = \frac{1}{2} I_m^2 \, Re\{Z\} = \boxed{I_{eff}^2 \, Re\{Z\}}$

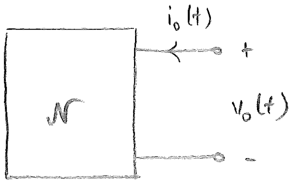
(Likewise, $P_{av} = \frac{1}{2} V_m^2 \, Re\{Y\} = \boxed{V_{eff}^2 \, Re\{Y\}}$)

Question :

"Does $I_{eff}^2 \, Im\{Z\}$ mean anything?"

Classification of Passive One-Ports

Let the one-port below consist of [LTI passive] resistors, inductors, capacitors ONLY. Suppose it is in SSS.

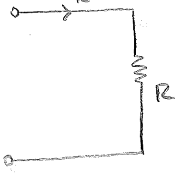


$v_o(t) = V_m \cos(\omega t + \phi_v) \Rightarrow V_o = V_m e^{j\phi_v}$

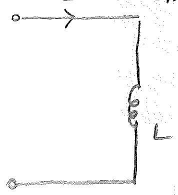
$i_o(t) = I_m \cos(\omega t + \phi_i) \Rightarrow I_o = I_m e^{j\phi_i}$

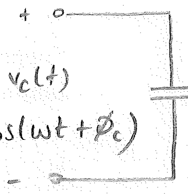
Recall:

$i_R(t) = I_{R_m} \cos(\omega t + \phi_R)$



$i_L(t) = I_{Lm} \cos(\omega t + \phi_L)$





$v_c(t) = V_{Cm} \cos(\omega t + \phi_c)$

$P_{av,R} = \frac{1}{2} R I_{Rm}^2$ $\qquad E_{av,L} = \frac{1}{4} L I_{Lm}^2$ $\qquad E_{av,C} = \frac{1}{4} C V_{Cm}^2$

Define:

P_{av} := the sum of average powers delivered to the resistors in $\mathcal{N}$.

$$\Rightarrow P_{av} = \sum_{k \text{ over resistors}} \frac{1}{2} R_k I_{R_k m}^2$$

E_{av}^L := the sum of average stored energies in the inductors in $\mathcal{N}$.

$$\Rightarrow E_{av}^L = \sum_{k \text{ over inductors}} \frac{1}{4} L_k I_{L_k m}^2$$

E_{av}^C := the sum of average stored energies in the capacitors in $\mathcal{N}$.

$$\Rightarrow E_{av}^C = \sum_{k \text{ over capacitors}} \frac{1}{4} C_k V_{C_k m}^2$$

Passivity $\Rightarrow$ $P_{av} \geq 0$, $E_{av}^L \geq 0$, $E_{av}^C \geq 0$.

Suppose there are N elements in $\mathcal{N}$,

Voltage phasors : $V_0, V_1, \dots, V_N$ (satisfy KVL)

Current phasors : $I_0, I_1, \dots, I_N$ (satisfy KCL)

Tellegens Theorem $\Rightarrow$ $-V_0 I_0^* + \sum_{k=1}^{N} V_k I_k^* = 0$ (1) [I_k^* : conjugate of I_k]

<u>Exercise</u> : why is eq. (1) true?

Now, we write

$$\sum_{k=1}^{N} V_k I_k^* = \underbrace{\sum V_{R,k} I_{R,k}^*}_{\text{over resistors}} + \underbrace{\sum V_{L,k} I_{L,k}^*}_{\text{over inductors}} + \underbrace{\sum V_{C,k} I_{C,k}^*}_{\text{over capacitors}} \qquad (2)$$

Note that $V_{R,k} = R_k I_{R,k}$; $V_{L,k} = j\omega L_k I_{L,k}$; $I_{C,k} = j\omega C_k V_{C,k}$ (3)

By (1), (2), & (3)

$$V_0 I_0^* = \sum R_k |I_{R,k}|^2 + \sum j\omega L_k |I_{L,k}|^2 - \sum j\omega C_k |V_{C,k}|^2$$

$$= 2 P_{av} + j4\omega \left[E_{av}^{L} - E_{av}^{C} \right] \qquad (4)$$

Let $Z = \frac{V_0}{I_0}$ (impedance of $\mathcal{N}$) & $E_{av}^{\Delta} := E_{av}^{L} - E_{av}^{C}$

Then, (4) $\Rightarrow$ $Z |I_0|^2 = 2 P_{av} + j4\omega E_{av}^{\Delta}$

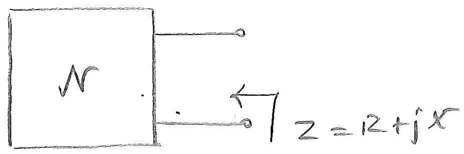
$$\Rightarrow \boxed{Z\, I_{0,eff}^2 = P_{av} + j\, 2\omega E_{av}^{\Delta}} \quad (*)$$

(*) $\Rightarrow$ $I_{0,eff}^2 \, \mathrm{Re}\{Z\} = P_{av}$ (obtained earlier)

(*) $\Rightarrow$ $I_{0,eff}^2 \, \mathrm{Im}\{Z\} = 2\omega E_{av}^{\Delta}$ (this is NEW!)

<u>Remark</u> : Note that $P_{av} \geq 0$ but E_{av}^{Δ} may have any sign!

Definition: Given LTI passive one-port in SSS

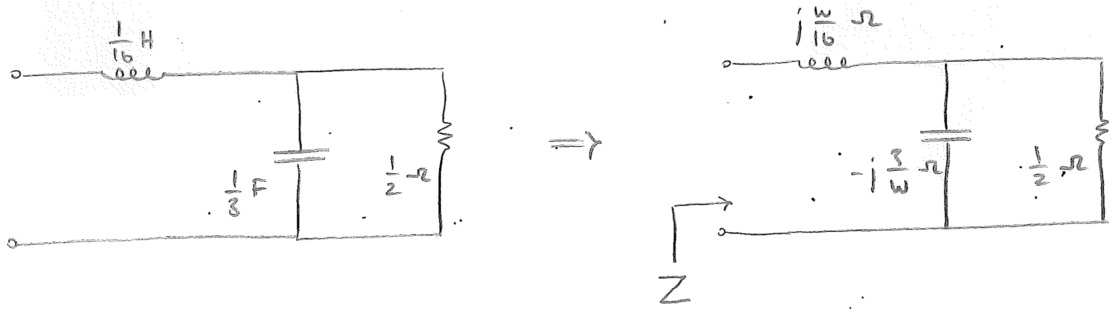


Different cases are named as follows:

	$X=0$	$X<0$	$X>0$
$R=0$		purely capacitive	purely inductive
$R>0$	resistive	capacitive	inductive

The $R=0$ case is sometimes called "purely reactive".

Example:

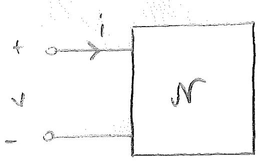


$$Z = j\frac{w}{16} + \left(\frac{1}{1/2} + \frac{1}{-j\,3/w}\right)^{-1} = j\frac{w}{16} + \left(2 + j\frac{w}{3}\right)^{-1} = j\frac{w}{16} + \frac{2 - j\frac{w}{3}}{4 + \frac{w^2}{9}}$$

$$= j\frac{w}{16} + \frac{18 - j3w}{36 + w^2} = \frac{j36w + jw^3 + 288 - j48w}{16(36 + w^2)} = \frac{288 + jw(w^2 - 12)}{16(w^2 + 36)}$$

Hence, Z is
$$\begin{cases} \text{resistive} & \text{for } w = 0 \ \& \ w = \sqrt{12} \text{ rad/sec} \\ \text{inductive} & \text{for } w > \sqrt{12} \\ \text{capacitive} & \text{for } 0 < w < \sqrt{12} \end{cases}$$

COMPLEX POWER



($\mathcal{N}$ is in SSS)

Complex power delivered to $\mathcal{N}$ is defined as

$\boxed{S := \frac{1}{2} V I^*}$ where V & I are phasors of $v(t)$ & $i(t)$.

Let $\left.\begin{array}{l} V = V_m e^{j\phi_v} \\ I = I_m e^{j\phi_i} \end{array}\right\} \Rightarrow S = \frac{1}{2} V_m I_m e^{j(\phi_v - \phi_i)}$

$$= \frac{1}{2} V_m I_m \cos(\phi_v - \phi_i) + j \frac{1}{2} V_m I_m \sin(\phi_v - \phi_i)$$

$$= V_{rms} I_{rms} \cos(\phi_v - \phi_i) + j V_{rms} I_{rms} \sin(\phi_v - \phi_i)$$

Note that $Re\{S\} = V_{rms} I_{rms} \cos(\phi_v - \phi_i)$ = average power delivered to $\mathcal{N}$.

Hence we write $\boxed{S = P + jQ}$

→ S: complex power (this is not a phasor, just a complex number).

→ P: real power (active power), measured in watts (W)

$$P = Re\{S\} = V_{rms} I_{rms} \cos(\phi_v - \phi_i)$$

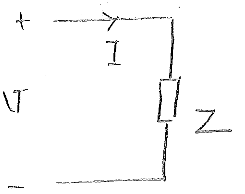
→ Q: reactive power, measured in volt-ampere-reactive (VAR)

$$Q = Im\{S\} = V_{rms} I_{rms} \sin(\phi_v - \phi_i)$$

→ |S|: apparent power, measured in volt-ampere (VA)

$$|S| = \frac{1}{2} V_m I_m = V_{rms} I_{rms} = \sqrt{P^2 + Q^2}$$

Consider a one-port containing V & passive R, L, C only



$$\left.\begin{array}{l} V = V_m e^{j\phi_v} \\ I = I_m e^{j\phi_i} \end{array}\right\} \Rightarrow Z = \frac{V}{I} = \frac{V_m}{I_m} e^{j(\phi_v - \phi_i)} = |Z| e^{j(\phi_v - \phi_i)} =: R + jX$$

complex power ? $\quad S = \frac{1}{2} V I^* = \frac{1}{2}(ZI)I^* = \frac{1}{2} Z I_m^2 = Z I_{rms}^2 \qquad \left(\text{Note: } S = \frac{V_{rms}^2}{Z^*}\right)$

$$= R I_{rms}^2 + jX I_{rms}^2$$

$$= P + jQ$$

$\Rightarrow \quad P = \mathrm{Re}\{Z\} I_{rms}^2 \quad , \quad Q = \mathrm{Im}\{Z\} I_{rms}^2 \quad , \quad |S| = |Z| I_{rms}^2$

$\qquad = R I_{rms}^2 \qquad\qquad = X I_{rms}^2 \qquad\qquad = \sqrt{R^2 + X^2}\, I_{rms}^2$

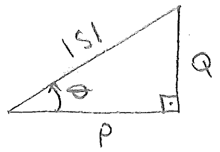
(real power) (reactive power) (apparent power)

power factor , $pf = \cos(\phi_v - \phi_i)$

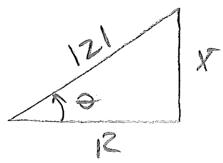
Note that $\quad Z = R + jX = |Z| \cos(\phi_v - \phi_i) + j |Z| \sin(\phi_v - \phi_i)$

$\qquad\qquad S = P + jQ = |S| \cos(\phi_v - \phi_i) + j |S| \sin(\phi_v - \phi_i)$

Let $\theta := \phi_v - \phi_i$



and



<u>Cases</u> $\quad Z = R + jX = |Z| e^{j\theta}$

1) $\underline{\theta = 0}$: One-port is resistive.

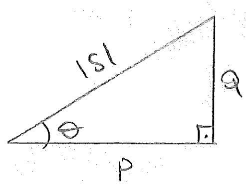
$R > 0 \, , \; X = 0 \, , \; pf = \cos\theta = 1$

$P > 0 \, , \; Q = 0$ (zero reactive power)

2) $\underline{0 < \theta < \frac{\pi}{2}}$: One-port is inductive. Such one-port is said to have <u>lagging</u> power factor. (current lags voltage.)

$R > 0 \, , \; X > 0 \, ,$ pf lagging

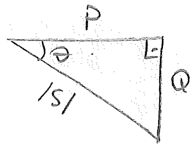
$P > 0 \, , \; Q > 0$



3) $-\frac{\pi}{2} < \theta < 0$: One-port is capacitive.

$R > 0$, $X < 0$, pf leading

$P > 0$, $Q < 0$



4) $\theta = \frac{\pi}{2}$: One-port is purely inductive

$R = 0$, $X > 0$, $pf = 0$ lagging

$P = 0$, $Q > 0$

5) $\theta = -\frac{\pi}{2}$: One-port is purely capacitive

$R = 0$, $X < 0$, $pf = 0$ leading

$P = 0$, $Q < 0$

Example : 10kVA load at 0.8 power factor, leading means:

$|S| = 10{,}000$ VA

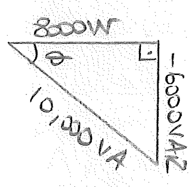
$\cos\theta = 0.8$

pf leading $\Rightarrow \theta < 0$

$\sin^2\theta = 1 - \cos^2\theta = 0.36 \Rightarrow \sin\theta = -0.6$

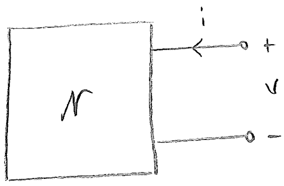
real power $P = |S|\cos\theta = 8000W$

reactive power $Q = |S|\sin\theta = -6000$ VAR



Example :

(N in SSS)



$v(t) = 3\cos(4t + 30^\circ)$ V

$i(t) = 6\cos(4t + \theta_i)$ A

a) $\theta_i = -20^\circ$:

$V = 3\angle 30^\circ$ volts

$I = 6\angle -20^\circ$ amps

$S = \frac{1}{2} V I^* = 9\angle 50^\circ$ VA

$\Rightarrow |S| = 9$VA (apparent power)

$P = |S|\cos 50^\circ \cong 5.8$ W (real power)

$Q = |S|\sin 50^\circ \cong 6.9$ VAR (reactive power)

b) $\theta_i = 140^\circ$:

$V = 3\angle 30^\circ$ volts

$I = 6\angle 140^\circ$ amps

$S = 9\angle -110^\circ$ VA

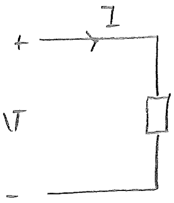
$\Rightarrow |S| = 9$VA

$P = |S|\cos(-110^\circ) = -3.1$ W

$Q = |S|\sin(-110^\circ) = -8.5$ VAR

Since $P < 0$, N is active (there is an independent source inside, for instance.)

Example :



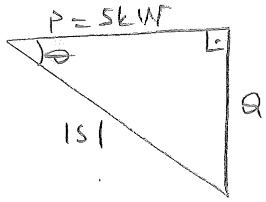
$V_{rms} = 100V$

$P = 5kW$

$pf = 0.8$ leading

Find $S, |S|, Q, Z, I_{rms}$

Sol'n :



$\cos\theta = 0.8 \Rightarrow \theta = -\arctan\frac{3}{4}$

$|S| = \frac{P}{\cos\theta} = \frac{5}{0.8} = \boxed{6.25 \text{ kVA}}$

$Q = |S|\sin\theta = 6.25(-0.6) = \boxed{-3.75 \text{ kVAR}}$

$S = P + jQ = \boxed{5 - j3.75 \text{ kVA}}$

$|S| = V_{rms} I_{rms} \Rightarrow I_{rms} = \frac{6250}{100} = \boxed{62,5 \text{ A}}$

$S = Z I_{rms}^2 \Rightarrow Z = \frac{5000 - j3750}{(62,5)^2} = 1.6 \angle -\arctan 3/4 = \boxed{1.28 - j0.96 \ \Omega}$

—o—

Conservation of Power

Given a circuit in SSS whose graph has N branches, let

$V_1, V_2, \ldots, V_N$ be voltage phasors & $V := [V_1 \ V_2 \ \cdots \ V_N]^T$

$I_1, I_2, \ldots, I_N$ be current phasors & $I := [I_1 \ I_2 \ \cdots \ I_N]^T$

KVL $\Rightarrow V = A^T E$ where $\begin{cases} E : \text{the node voltage vector} \\ A : \text{the reduced incidence matrix} \end{cases}$

KCL $\Rightarrow AI = 0 \Rightarrow (AI)^* = 0 \Rightarrow AI^* = 0$ (because A is real)

Total complex power of the circuit:

$$S_{total} = \sum_{k=1}^{N} \frac{1}{2} V_k I_k^* = \frac{1}{2} V^T I^* = \frac{1}{2} \overbrace{E^T}^{V^T} \underbrace{A I^*}_{0} = 0$$

Hence $S_{total} = 0$ and the complex power is conserved.

Now, $S_k = \frac{1}{2} V_k I_k^* = P_k + j Q_k$ (complex power of the k^{th} branch)

$$\Rightarrow \quad 0 = S_{total} = \sum_{k=1}^{N} S_k = \sum_{k=1}^{N} P_k + j \sum_{k=1}^{N} Q_k$$

$$\Rightarrow \quad \sum_{k=1}^{N} P_k = 0 \quad \& \quad \sum_{k=1}^{N} Q_k = 0$$

That is, both real & reactive powers are conserved!

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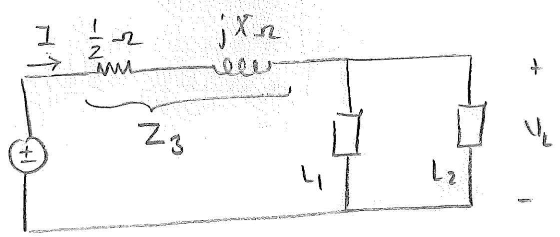
<u>Example</u> (VPS IV_1)



first load L_1 : 4W, resistive

second load L_2 : 5VA, $pf_2 = \frac{3}{5}$ leading

$S_s = 9 + j7$ VA (complex power supp. by source)

Find I_{eff}, X, V_{Leff}, V_{seff}.

<u>Sol'n</u>: $S_1 = 4 + j0$ VA, $S_2 = 5\left(\frac{3}{5} - j\frac{4}{5}\right) = 3 - j4$ VA

$S_s = S_1 + S_2 + S_3 \Rightarrow 9 + j7 = 4 + 3 - j4 + S_3 \Rightarrow S_3 = 2 + j11$ VA

$S_3 = Z_3 I_{eff}^2 = \frac{1}{2} I_{eff}^2 + jX I_{eff}^2 = 2 + j11 \Rightarrow \boxed{I_{eff} = 2A}$ & $\boxed{X = \frac{11}{4}\,\Omega}$

$V_{Leff} I_{eff} = |S_1 + S_2| \Rightarrow V_{Leff} = \frac{|7 - j4|}{2} = \boxed{\frac{\sqrt{65}}{2}\,V}$

$V_{seff} I_{eff} = |S_s| \Rightarrow V_{seff} = \frac{|9 + j7|}{2} = \boxed{\frac{\sqrt{130}}{2}\,V}$

Superposition in power calculations

Suppose that in an LTI circuit in SSS there are two sinusoidal sources. Applying the superposition principle, the current through some resistor R is $i(t) = i_1(t) + i_2(t)$, where i_1 is due to the first source & i_2 is due to the second. Let's write the instantaneous power:

$$p(t) = R\,i(t)^2 = R\,(i_1(t)+i_2(t))^2$$
$$= R\,i_1(t)^2 + R\,i_2(t)^2 + 2R\,i_1(t)\,i_2(t)$$
$$= p_1(t) + p_2(t) + 2R\,i_1(t)\,i_2(t)$$

Hence $p(t) \neq p_1(t) + p_2(t)$. That is, inst. power does not obey superposition.

How about the average power?

$$P_{av} = \frac{1}{T}\int_0^T p(t)\,dt \qquad \text{for } p(t) \text{ that is periodic with period } T.$$

[Note: if $p(t)$ is not periodic we can take $P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_0^T p(t)\,dt$]

$$\Rightarrow P_{av} = \frac{1}{T}\int_0^T \{p_1(t) + p_2(t) + 2R\,i_1(t)\,i_2(t)\}\,dt$$

$$= P_{1,av} + P_{2,av} + \underbrace{\frac{2R}{T}\int_0^T i_1(t)\,i_2(t)\,dt}_{P_{12,av}}$$

Now, let $i_1(t) = I_{1m}\cos(\omega_1 t + \phi_1)$

$i_2(t) = I_{2m}\cos(\omega_2 t + \phi_2)$

Then, $$P_{12,av} = \frac{2R}{T}\int_0^T I_{1m} I_{2m}\cos(\omega_1 t + \phi_1)\cos(\omega_2 t + \phi_2)\,dt$$

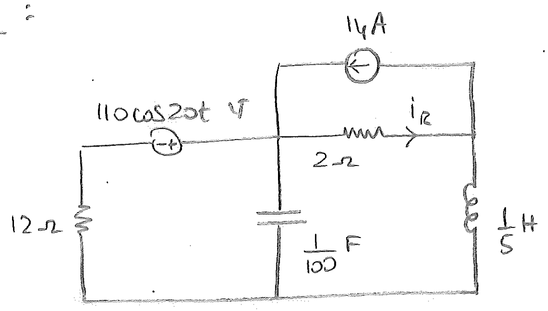
$$= \frac{R I_{1m} I_{2m}}{T}\int_0^T \{\cos[(\omega_1+\omega_2)t + \phi_1+\phi_2] + \cos[(\omega_1-\omega_2)t + \phi_1-\phi_2]\}\,dt$$

Hence, $$P_{12,av} = \begin{cases} R I_{1m} I_{2m}\cos(\phi_1-\phi_2) & \text{if } \omega_1 = \omega_2 \\ 0 & \text{if } \omega_1 \neq \omega_2 \end{cases}$$

Conclusion: The average power delivered to a one-port by several sinusoidal sources, acting together, is equal to the sum of the average powers delivered by each source acting alone, provided that no two sources have the same freq.

[Question: How about the average stored energy (for L & C)?]

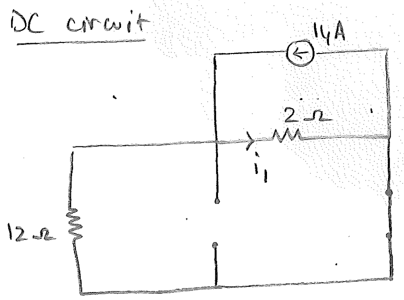
Example:



Find the SS $i_R(t)$ & the average power delivered to the 2Ω resistor.

Sol'n: Apply superposition

DC circuit

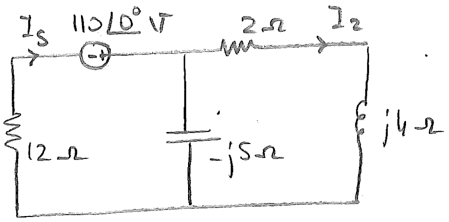


$$i_1 = 14 \cdot \frac{12}{2+12} = 12\,A$$

$$P_{1,av} = 2 \cdot i_1^2 = 288\,W$$

AC circuit in phasor domain

$$I_s = \frac{110}{12 + (-j5 \,//\, (2+j4))} = \frac{110}{12 + \frac{20-j10}{2-j}} = 5\,A$$



$$\Rightarrow I_2 = \frac{-j5}{-j5+2+j4} I_s = \frac{-j5(2+j)}{5}\, 5 = 5 - j10\,A$$

$$\Rightarrow I_2 = \sqrt{125}\angle -\tan^{-1}2 \;A \quad \Rightarrow \quad i_2(t) = 5\sqrt{5}\cos(20t - \tan^{-1}2)\;A$$

$$\& \quad P_{2,av} = R I_{2,rms}^2 = 2\left(\frac{5\sqrt{5}}{\sqrt{2}}\right)^2 = 125\,W$$

Finally, $i_R(t) = i_1(t) + i_2(t) = \boxed{12 + 5\sqrt{5}\cos(20t - \tan^{-1}2)\;A}$

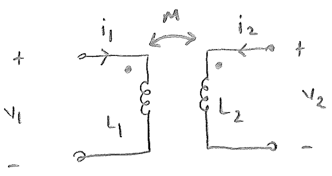
$$P_{R,av} = P_{1,av} + P_{2,av} = \boxed{413\,W}$$

Exercise: Find the average stored energy in the inductor. Answer: $E^L_{av} = \frac{2}{5} + \frac{25}{4} = \frac{133}{20}\,J$

$i_L(t) = -2 + 5\sqrt{5}\cos(20t - \tan^{-1}2)\;A$

#8 18 Mart 2014

Average Stored Energy of Coupled Inductors



$$i_1(t) = I_{1m}\cos(\omega t + \phi_1) \qquad I_1 = I_{1m}e^{j\phi_1} \qquad i(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}, \quad I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$i_2(t) = I_{2m}\cos(\omega t + \phi_2) \qquad I_2 = I_{2m}e^{j\phi_2}$$

instanteneous stored energy: $e(t) = \frac{1}{2} i(t)^T L\, i(t) = \frac{1}{2}L_1 i_1^2 + M i_1 i_2 + \frac{1}{2}L_2 i_2^2$

average stored energy: $E_{av} = \frac{1}{T}\int_0^T e(t)\,dt = ?$

$$e(t) = \frac{1}{2}L_1 I_{1m}^2\cos^2(\omega t+\phi_1) + M I_{1m} I_{2m}\cos(\omega t+\phi_1)\cos(\omega t+\phi_2) + \frac{1}{2}L_2 I_{2m}^2\cos^2(\omega t+\phi_2)$$

$$= \frac{1}{4}L_1 I_{1m}^2\left\{1+\cos(2\omega t+2\phi_1)\right\} + \frac{1}{2}M I_{1m} I_{2m}\left\{\cos(\phi_1-\phi_2) + \cos(2\omega t+\phi_1+\phi_2)\right\}$$
$$+ \frac{1}{4}L_2 I_{2m}^2\left\{1+\cos(2\omega t+2\phi_2)\right\}$$

$$\Rightarrow E_{av} = \frac{1}{4}L_1 I_{1m}^2 + \frac{1}{2}M I_{1m}I_{2m}\cos(\phi_1-\phi_2) + \frac{1}{4}L_2 I_{2m}^2$$

$$= \frac{1}{4}L_1 I_1 I_1^* + \frac{1}{2}M\,\mathrm{Re}\{I_1 I_2^*\} + \frac{1}{4}L_2 I_2 I_2^*$$

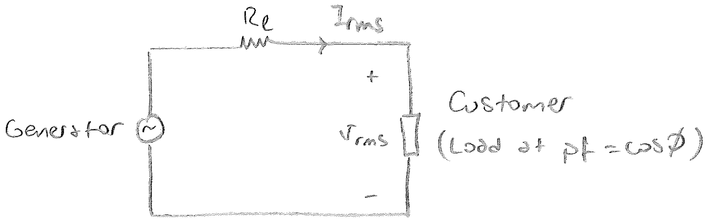
$$= \frac{1}{4}L_1 I_1 I_1^* + \frac{1}{4}M\left(I_1 I_2^* + I_1^* I_2\right) + \frac{1}{4}L_2 I_2 I_2^*$$

$$= \frac{1}{4}\underbrace{\begin{bmatrix} I_1^* & I_2^* \end{bmatrix}}_{I^*}\begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \boxed{\frac{1}{4}I^* L I}$$

where I^* : conjugate transpose

POWER FACTOR CORRECTION

Consider the following scenario



In general the curstomer requires power at a specified voltage. Hence let us take the load voltage V_{rms} & the average power delivered to the load P_L as fixed.

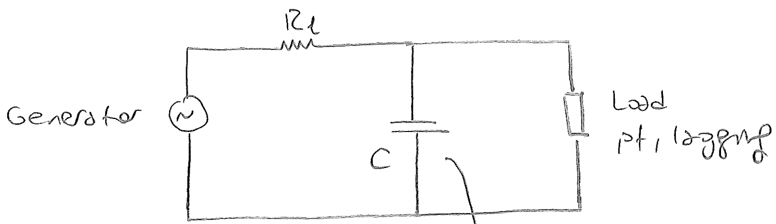
R_ℓ : resistance of the transmission lines

$$\Rightarrow \quad P_L = I_{rms} V_{rms}\, pf \quad \Rightarrow \quad I_{rms} = \frac{P_L}{V_{rms} \cdot pf}$$

power loss during transmission? $\quad P_{loss} = I^2_{rms} R_\ell = \left(\frac{P_L}{V_{rms} \cdot pf}\right)^2 R_\ell \quad (1)$

(1) $\Rightarrow$ the smaller the power factor, the larger the loss. Therefore, the larger the power factor ($pf \approx 1$) the better it is.

In industrial applications the power factor of loads is usually lagging due to the windings of the electric motors. A common way to increase pf is therefore to connect a parallel capacitor to the load.



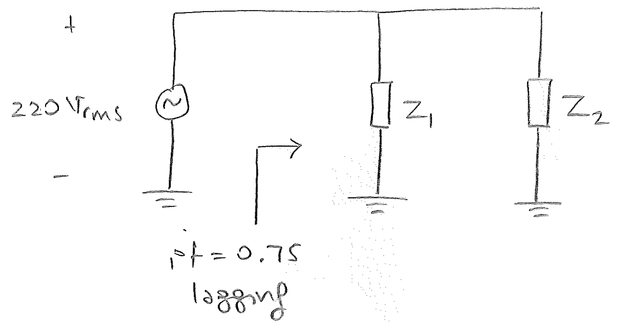
to decrease the total reactive power and increase pf.

<u>Example</u>: A generator supplies two parallel loads at 220 V_{rms}, 50 Hz with a net pf of 0.75 lagging. One load is 4800 VA at a pf of 0.85 lagging. The second load absorbs 4 kW average power.

a) What are the apparent power & pf of the second load?

b) Determine the kVAR rating and the capacitance value of the capacitor that should be connected in parallel with the two loads to improve the net pf to 0.95 lagging.

<u>Sol'n</u>:



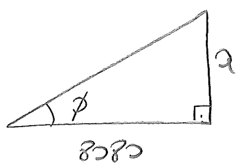
Load 1: $\cos\phi_1 = 0.85$, $\sin\phi_1 > 0$ since lagging

$\Rightarrow S_1 = 4800\left(0.85 + j\,(1-0.85^2)^{1/2}\right)$

$= 4080 + j2530$ VA

Load 2: $S_2 = 4000 + jQ_2$

total complex power: $S = S_1 + S_2 = 8080 + j\underbrace{(2530 + Q_2)}_{Q}$ pf = 0.75 lag.



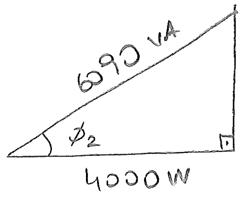
$\cos\phi = 0.75$

$\Rightarrow Q = 8080 \tan\phi = 8080 \tan(\arccos 0.75) = 8080 \cdot \sqrt{7}/3$

$= 7125$ VAR

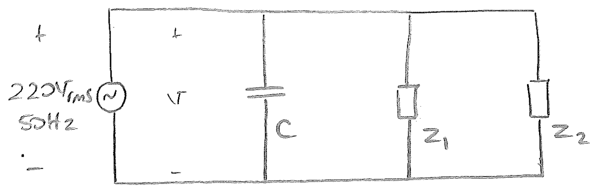
$\Rightarrow 7125 = 2530 + Q_2 \Rightarrow Q_2 = 4595$ VAR

Hence $S_2 = 4000 + j4595$ VA $\Rightarrow \boxed{|S_2| = 6090 \text{ VA}}$



$\Rightarrow \cos\phi_2 = \dfrac{4000}{6090} \cong 0.66$

$\Rightarrow \boxed{pf_2 = 0.66 \text{ lagging}}$

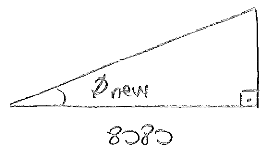


Now, let the power of C be $S_3 = P_3 + jQ_3$

We know that $P_3 = 0$ & $Q_3 < 0$.

Then the new total power becomes:

$$S_{new} = S_1 + S_2 + S_3 = 8080 + j(7125 + Q_3)$$



Want: $\cos\phi_{new} = 0.95$

$$\Rightarrow \frac{7125+Q_3}{8080} = \tan(\arccos 0.95)$$

$$\Rightarrow 7125 + Q_3 = 2655 \Rightarrow Q_3 = -4470 \text{ VAR}$$

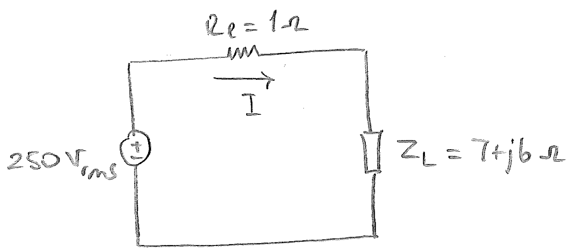
Hence, $S_3 = -j4470$ VA

$$S_3 = \frac{V_{eff}^2}{Z_3^*} = \frac{V_{eff}^2}{\frac{1}{-jwC}} = -jwC V_{eff}^2 = -j4470 \Rightarrow wC V_{eff}^2 = 4470$$

$$\left.\begin{array}{l} V_{eff} = 220 \text{ volts} \\ w = 2\pi f \Big|_{f=50Hz} = 100\pi \text{ rad/sec} \end{array}\right\} \quad C = \frac{4470}{100\pi \cdot 220^2} = 294 \times 10^{-6} \text{ F}$$

$$\Rightarrow \boxed{C = 294 \mu F}$$

Example (Efficiency)



Definition: Percent efficiency (η)

$$\eta = \frac{\text{Real power absorbed by load}}{\text{Real power supplied by source}} \times 100\%$$

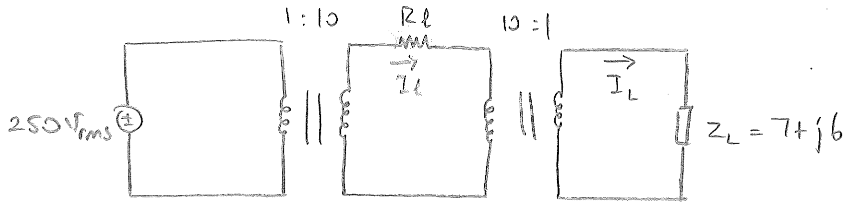
$$Z_{eq} = 8 + j6$$

$$\Rightarrow I_{rms} = \frac{250}{|Z_{eq}|} = 25A$$

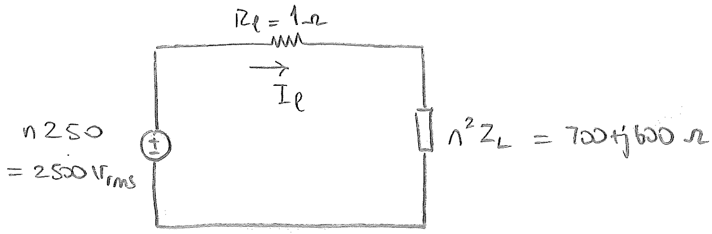
$$\left.\begin{array}{l} \Rightarrow P_L = 7 \cdot I_{rms}^2 = 4375 W \\ P_e = 1 \cdot I_{rms}^2 = 625 W \end{array}\right\} \quad \eta = \frac{P_L}{P_L + P_e} \times 100\%$$

$$= \frac{4375}{4375+625} = \frac{4375}{5000} = \boxed{87.5\%}$$

Now consider



$\Downarrow$



$$\Rightarrow I_{\ell,rms} = \frac{2500}{|701 + j600|} = 2.71\text{ A}$$

$$\Rightarrow I_{L,rms} = n\, I_{\ell,rms} = 27.1\text{ A}$$

$$\left.\begin{aligned} \Rightarrow P_\ell &= 1\cdot(2.71)^2 = 7.34\text{ W} \\ P_L &= 7\cdot(27.1)^2 = 5138\text{ W} \end{aligned}\right\} \quad \boxed{\eta_{new} = 99.85\,\%}$$

<u>Remark</u> : Note that $\eta = \dfrac{R_L}{R_L + R_\ell}$ & $\eta_{new} = \dfrac{R_L}{R_L + \frac{R_\ell}{n^2}}$.

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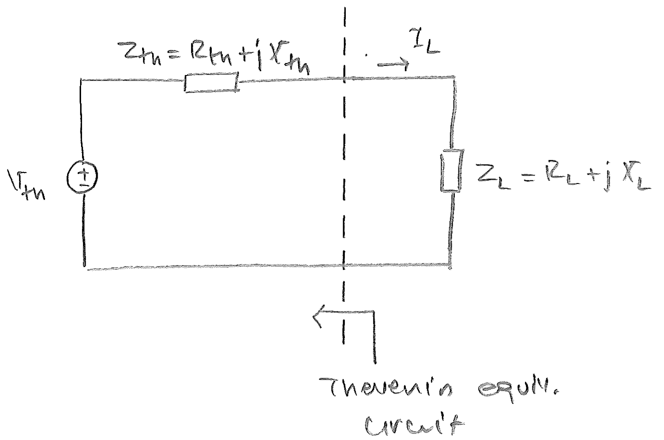
<u>MAXIMUM POWER TRANSFER</u>



Fixed: V_{th}, Z_{th}

Design var: Z_L

WANT: maximize P_L

$$P_L = \frac{1}{2} |I_L|^2 R_L$$

$$I_L = \frac{V_{th}}{(R_{th} + R_L) + j(X_{th} + X_L)}$$

$$\Rightarrow P_L = \frac{1}{2} \frac{|V_{th}|^2 R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$

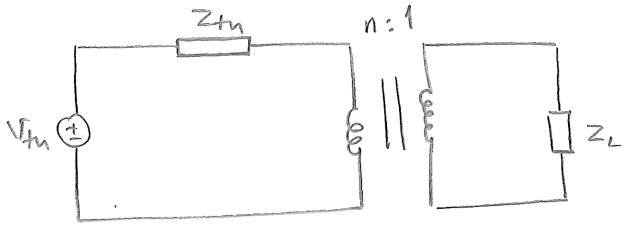
Note that the term $X_{th} + X_L$ can be eliminated by letting $X_L = -X_{th}$. Then:

$$\left.\frac{dP_L}{dR_L}\right|_{X_L = -X_{th}} = \frac{|V_{th}|^2}{2} \frac{d}{dR_L} \left\{ \frac{R_L}{(R_{th} + R_L)^2} \right\} = 0 \Rightarrow R_L = R_{th}$$

Therefore, $Z_L = Z_{th}^*$ for max. power transfer in our case. How about different cases?

Remark: when the design variable is different, formulate P_L in terms of the design parameter (say α) then solve for $\frac{d}{d\alpha} P_L(\alpha) = 0$.

Exercise:



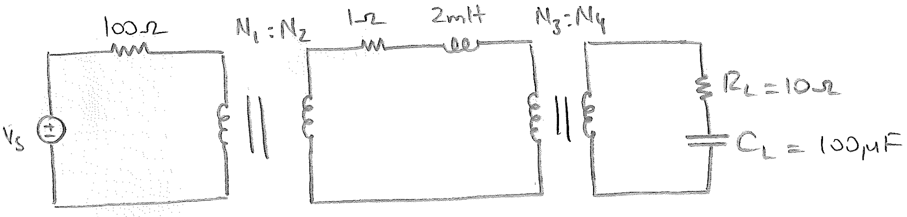
fixed: V_{th}, Z_{th}, Z_L

design var: turns ratio n

Find n for max P_L.

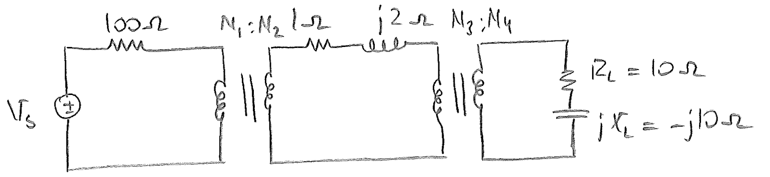
(Answer: $n = \sqrt{|Z_{th}|/|Z_L|}$)

Example:



R_L & C_L are chosen such that maximum power is extracted from the rest of the circuit by R_L at $w = 1000$ rad/sec. Find turns ratios of the transformers.

Sol'n: phasor domain circuit



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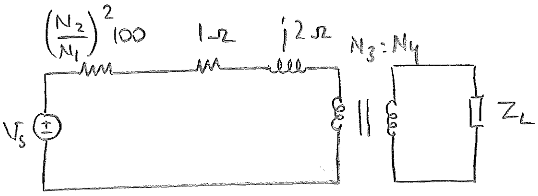
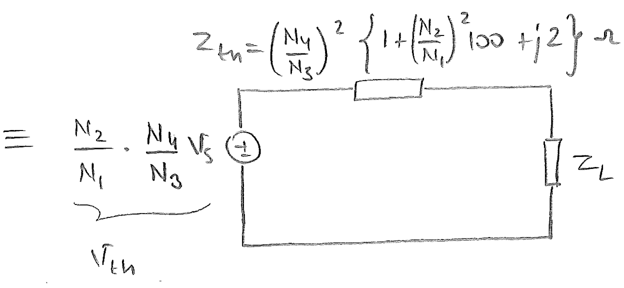




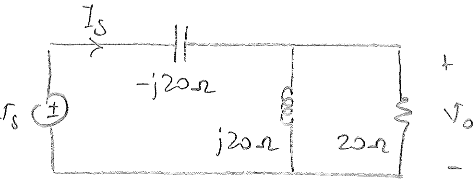
For max power $Z_L = Z_{th}^*$

$$\Rightarrow \underbrace{10 - j10}_{Z_L} = \underbrace{\left(\frac{N_4}{N_3}\right)^2 \left\{1 + \left(\frac{N_2}{N_1}\right)^2 100\right\} - j\left(\frac{N_4}{N_3}\right)^2 2}_{Z_{th}^*}$$

$$\Rightarrow \left(\frac{N_4}{N_3}\right)^2 2 = 10 \Rightarrow \boxed{\frac{N_4}{N_3} = \sqrt{5}}$$

$$\Rightarrow \left(\frac{N_4}{N_3}\right)^2 \left\{1 + \left(\frac{N_2}{N_1}\right)^2 100\right\} = 10 \Rightarrow 1 + \left(\frac{N_2}{N_1}\right)^2 100 = 2 \Rightarrow \boxed{\frac{N_2}{N_1} = \frac{1}{10}}$$

Example



The average power delivered to the resistor is 500 W.

Find : $V_{o,rms}$, $I_{s,rms}$, $V_{s,rms}$, pf seen by the source

Sol'n

$$\underbrace{S_R}_{500+j0} = \frac{V_{o,rms}^2}{\underbrace{Z_R^*}_{20}} \Rightarrow V_{o,rms}^2 = 500 \times 20 \Rightarrow \boxed{V_{o,rms} = 100\ V}$$

$$S_L = \frac{V_{o,rms}^2}{Z_L^*} = \frac{10^4}{-j20} = j500\ VA \quad , \quad I_{s,rms} = \frac{|S_R + S_L|}{V_{o,rms}} = \frac{|500+j500|}{100}$$

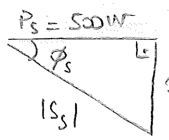
$$\Rightarrow \boxed{I_{s,rms} = 5\sqrt{2}\ A}$$

$$S_C = I_{s,rms}^2 Z_C = -j20 \times (5\sqrt{2})^2 = -j1000\ VA$$

$$V_{s,rms} = \frac{|S_R + S_L + S_C|}{I_{s,rms}} = \frac{|500 - j500|}{5\sqrt{2}} \Rightarrow \boxed{V_{s,rms} = 100\ V}$$

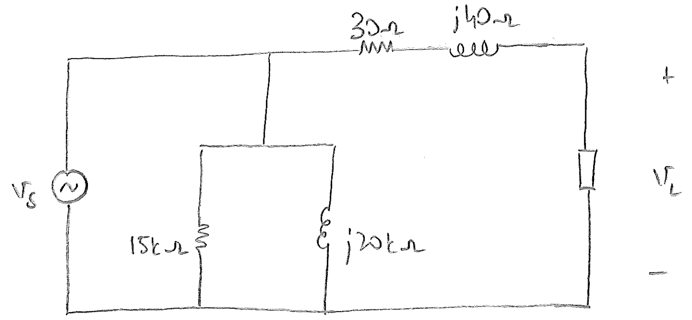
Complex power supplied by the source $S_s = S_R + S_L + S_C = 500 - j500\ VA$

$\Rightarrow$



$$\Rightarrow pf = \cos\phi_s = \boxed{\frac{1}{\sqrt{2}}\ ,\ \text{leading}}$$

Example



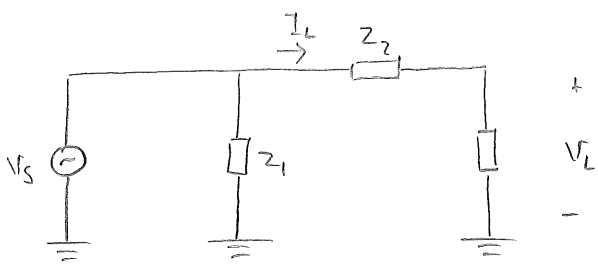
Given :

→ $V_{s,rms} = 3000V$

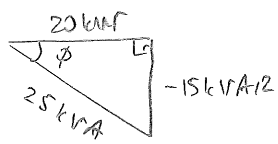
→ 20kW power is drawn from the source at pf = 0.8 , leading

Find $V_{L,rms}$ & pf at load.

Sol'n



Known: $V_{s,rms}$, S_s, Z_1, Z_2



$S_s = 20 - j15\ kVA$

$$Z_1 = 15 \,//\, j20 = 9.6 + j7.2\ k\Omega$$

$$Z_2 = 30 + j40\ \Omega$$

$$S_1 = \frac{V_{S,rms}^2}{Z_1^*} = 600 + j450 \text{ VA} \qquad \left(\text{or} \quad S_1 = \frac{V_{S,rms}^2}{(15k)^*} + \frac{V_{S,rms}^2}{(j20k)^*} = 8000^2 \left[\frac{1}{15{,}000} + j\frac{1}{20{,}000} \right] \right)$$

$$S_2 + S_L = S_S - S_1 = 19400 - j15450 \text{ VA}$$

$$I_{L,rms} = \frac{|S_2 + S_L|}{V_{S,rms}} = 3.27 \text{ A}$$

$$S_2 = Z_2 I_{L,rms}^2 = 2050 + j2730 \text{ VA}$$

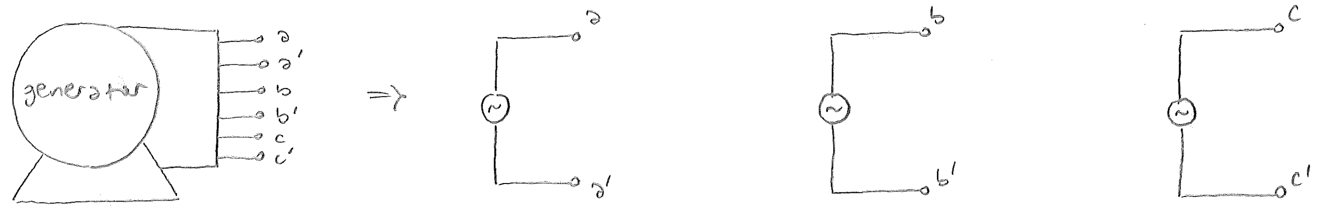
$$S_L = S_2 + S_L - S_2 = 17350 - j18180 \text{ VA} \quad \Rightarrow \quad \boxed{pf_L = 0.69 \text{ leading}}$$

$$V_{L,rms} = \frac{|S_L|}{I_{L,rms}} = \boxed{3040 \text{ V}}$$

Ch. VI Balanced Three-Phase Circuits

Due to economic & operational advantages the electric energy is generated and distributed in three phase form rather than single phase.

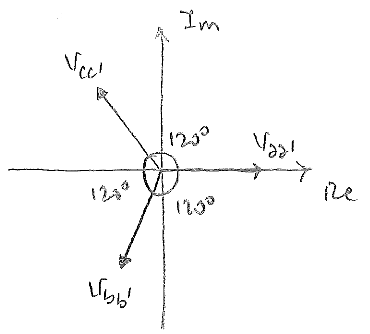
Consider a six-terminal AC generator:



$v_{aa'}(t) = |V_s| \cos \omega t$, $v_{bb'}(t) = |V_s| \cos(\omega t - 120°)$, $v_{cc'}(t) = |V_s| \cos(\omega t - 240°)$

$= |V_s| \cos(\omega t + 120°)$

Phasor diagram:

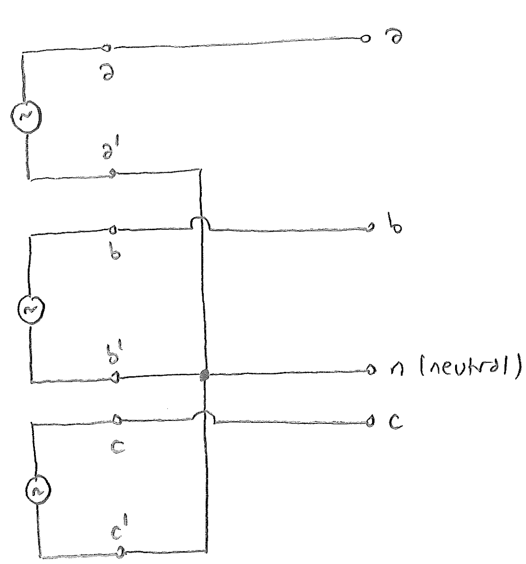


Therefore $\boxed{V_{aa'} + V_{bb'} + V_{cc'} = 0}$

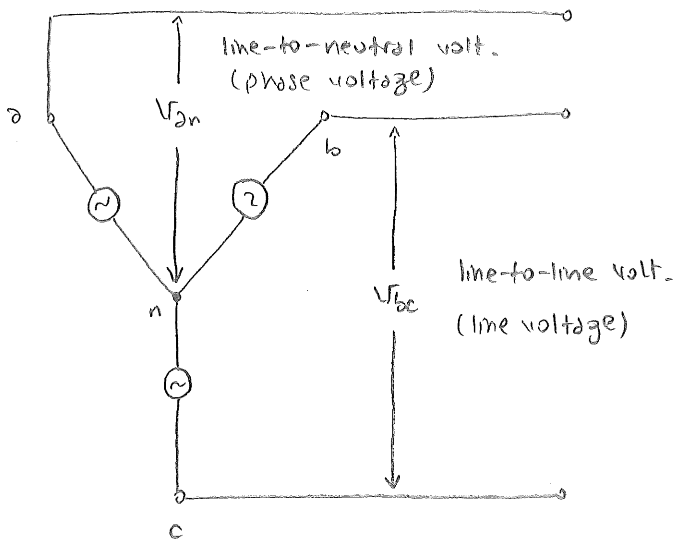
↳ "balanced set" $\{V_{aa'}, V_{bb'}, V_{cc'}\}$

Y-connection

The six-terminal generator can be connected as



The connection is usually drawn as



3-phase Y-connected generator

When the generator is balanced:

$$\left.\begin{aligned} V_{an} &= |V_s| \angle 0^\circ \\ V_{bn} &= |V_s| \angle -120^\circ \\ V_{cn} &= |V_s| \angle 120^\circ \end{aligned}\right\} \quad V_{ab} = V_{an} - V_{bn} = |V_s| \angle 0^\circ - |V_s| \angle -120^\circ$$

$$= |V_s| - |V_s| \left\{ -\frac{1}{2} - j\frac{\sqrt{3}}{2} \right\}$$

$$= |V_s| \left\{ \frac{3}{2} + j\frac{\sqrt{3}}{2} \right\} = \sqrt{3}\,|V_s| \left\{ \frac{\sqrt{3}}{2} + j\frac{1}{2} \right\}$$

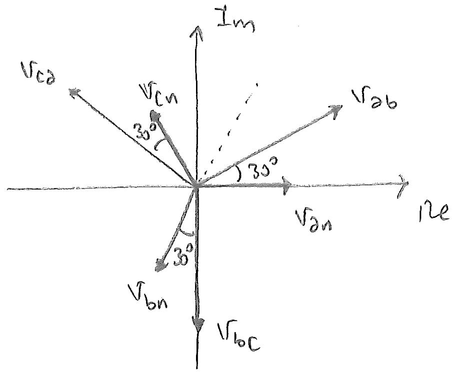
$$= \sqrt{3}\,|V_s| \angle 30^\circ$$

$$V_{bc} = V_{bn} - V_{cn} = |V_s| \angle -120^\circ - |V_s| \angle -240^\circ$$

$$= 1\angle -120^\circ \times \underbrace{\{ |V_s| \angle 0^\circ - |V_s| \angle -120^\circ \}}_{\sqrt{3}|V_s| \angle 30^\circ} = \sqrt{3}\,|V_s| \angle -90^\circ$$

$$V_{ca} = V_{cn} - V_{an} = \sqrt{3}\,|V_s| \angle 150^\circ$$

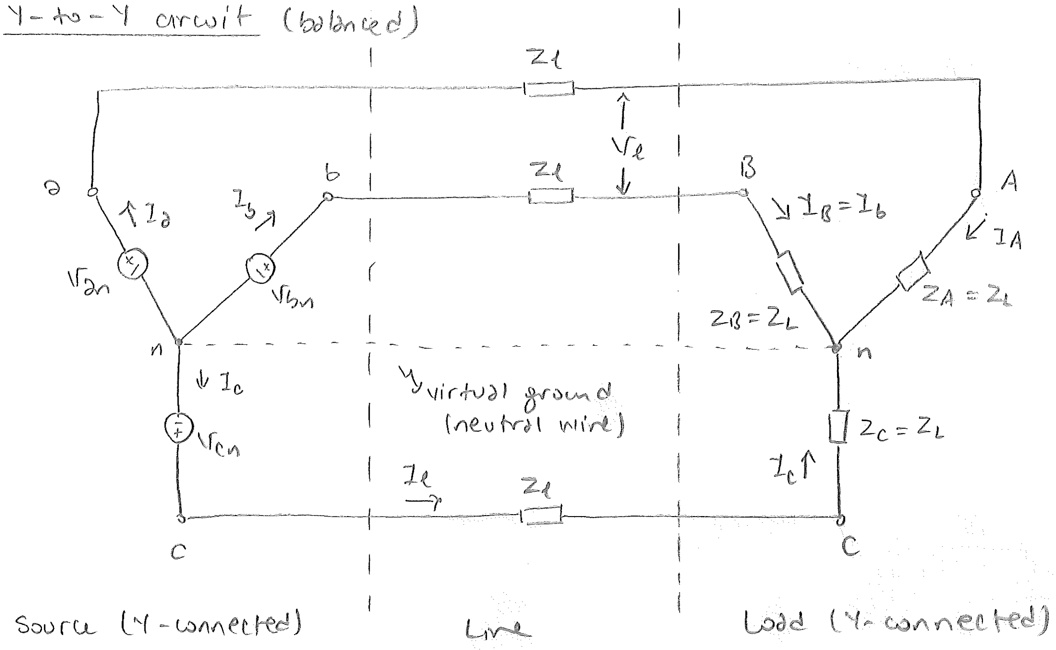
Phasor diagram:



Hence, for Y-connection:

$$|V_{line}| = \sqrt{3}\,|V_{phase}|$$

Y-to-Y circuit (balanced)



Definitions

V_{an}, V_{bn}, V_{cn} : phase voltages on the source (Y) side

V_{An}, V_{Bn}, V_{Cn} : phase voltages on the load (Y) side

I_a, I_b, I_c : phase currents (source)

I_A, I_B, I_C : phase currents (load)

I_ℓ : line current

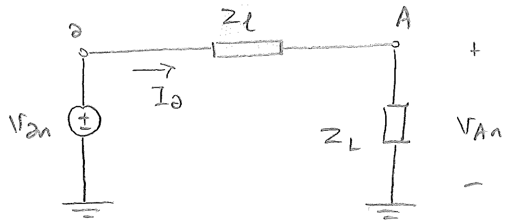
V_ℓ : line voltage (at the load side)

Balanced :
1) Equal magnitude phase voltages with $120°$ phase difference,
2) Same line impedance Z_ℓ,
3) Same load impedance Z_L.

Remark When $Z_A = Z_B = Z_C = Z_L$ the load is said to be balanced. (We will always assume this.) Then the neutral wire (if it exists) carries no current (why?) Therefore in a balanced circuit even when the neutral line does not exist, the neutral points at the load side and the generator side have equal potentials. This allows us to work with:

#4 7 Nisan 2019

per-phase circuit



[Recall: $\{V_{an}, V_{bn}, V_{cn}\}$ is a balanced set.]

$$I_a = \frac{V_{an}}{Z_\ell + Z_L} = |I_L| \angle \theta \quad \& \quad V_{An} = I_a Z_L = |V_L| \angle \alpha$$

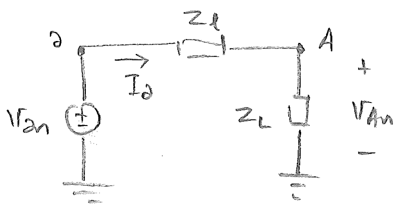
$$I_b = \frac{V_{bn}}{Z_\ell + Z_L} = |I_L| \angle \theta - 120° \quad \& \quad V_{Bn} = I_b Z_L = |V_L| \angle \alpha - 120°$$

$$I_c = \frac{V_{cn}}{Z_\ell + Z_L} = |I_L| \angle \theta + 120° \quad \& \quad V_{Cn} = I_c Z_L = |V_L| \angle \alpha + 120°$$

Note: $I_a + I_b + I_c = 0$ & $V_{An} + V_{Bn} + V_{Cn} = 0$ (balanced)

Complex power (delivered to the load)

Consider the per-phase circuit



$S_A = I_{a,rms}^2 Z_L$ (per phase complex power)

since $|I_a| = |I_b| = |I_c|$ we have $S_A = S_B = S_C$

Let $I_{P,rms} := I_{a,rms}$ (rms phase current)

$V_{P,rms} := V_{An,rms}$ (rms phase voltage)

Then

	per-phase	total
complex power	$S_P = I_{P,rms}^2 Z_L$	$S_{3\phi} = 3S_P$
real power	$P_P = I_{P,rms}^2 \mathrm{Re}\{Z_L\}$	$P_{3\phi} = 3P_P$
reactive power	$Q_P = I_{P,rms}^2 \mathrm{Im}\{Z_L\}$	$Q_{3\phi} = 3Q_P$
apparent power	$\lvert S_P\rvert = I_{P,rms}^2 \lvert Z_L\rvert = I_{P,rms} V_{P,rms}$	$\lvert S_{3\phi}\rvert = 3\lvert S_P\rvert$

#5 7 Nisan 2019

Example Consider Y-Y connected balanced 3ϕ circuit where

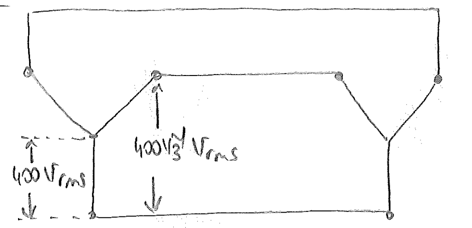
→ generator's line-to-line voltage $= 400\sqrt{3}\ V_{rms}$

→ load's per-phase impedance $= 4+j3\ \Omega$

→ transmission line impedance $= 1+j2\ \Omega$

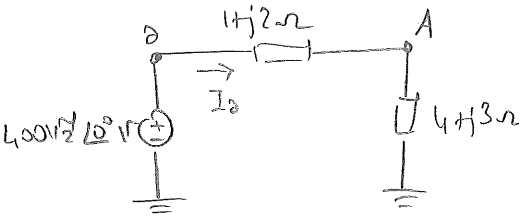
Take V_{an} as reference. Find the phase currents, the average & reactive power delivered to the load.

Sol'n



V_{an} reference ⇒ $V_{an} = 400\sqrt{2}\angle 0°\ V$

per-phase circuit:



$$I_a = \frac{400\sqrt{2}}{5+j5} = \frac{400\sqrt{2}}{5\sqrt{2}\angle 45°} = 80\angle -45°\ A$$

Then

$$I_b = 80\angle -45°-120° = 80\angle -165°\ A$$

$$I_c = 80\angle -45°+120° = 80\angle 75°\ A$$

$$P_{3\phi} = 3\, I_{a,rms}^2\, Re\{Z_L\} = 3\left(\frac{80}{\sqrt{2}}\right)^2 Re\{4+j3\} = 38400\ W$$

& $Q_{3\phi} = 3\, I_{a,rms}^2\, Im\{Z_L\} = 28800\ VAR$

Instantaneous Power (delivered to the load)

Let phase voltages be

$$\begin{cases} v_{An}(t) = |V_p| \cos \omega t \\ v_{Bn}(t) = |V_p| \cos(\omega t - 120°) \\ v_{Cn}(t) = |V_p| \cos(\omega t + 120°) \end{cases}$$

& the phase currents be

$$\begin{cases} i_a(t) = |I_p| \cos(\omega t - \phi) \\ i_b(t) = |I_p| \cos(\omega t - 120° - \phi) \\ i_c(t) = |I_p| \cos(\omega t + 120° + \phi) \end{cases}$$

Then $\quad p_A(t) = i_a(t)\, v_{An}(t) = |I_p| \cdot |V_p| \cos\omega t \cdot \cos(\omega t - \phi)$

$$= \underbrace{\tfrac{1}{2} |I_p| \cdot |V_p|}_{I_{P,rms} \cdot V_{P,rms}} \left\{ \cos(2\omega t - \phi) + \cos\phi \right\}$$

& $\quad p_B(t) = I_{P,rms} V_{P,rms} \left\{ \cos(2\omega t - 240^\circ - \phi) + \cos\phi \right\}$

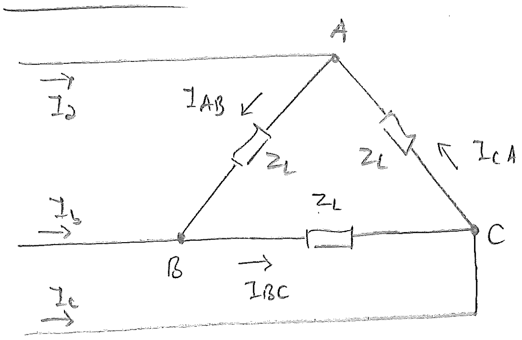
$\quad\;\; p_C(t) = I_{P,rms} V_{P,rms} \left\{ \cos(2\omega t + 240^\circ + \phi) + \cos\phi \right\}$

$\Rightarrow \quad p_A(t) + p_B(t) + p_C(t) = 3 I_{P,rms} V_{P,rms} \cos\phi = P_{3\phi,av}$

Hence the instantaneous power is constant and equal to the total average power. Constant power demand allows generators to operate smoothly (no vibration) and they last longer. ($P_{mech} = \tau \cdot \omega$)

— o —

Δ-connection



For Δ-connection phase currents & phase voltages are defined as:

phase currents : I_{AB}, I_{BC}, I_{CA}

phase voltages : V_{AB}, V_{BC}, V_{CA}

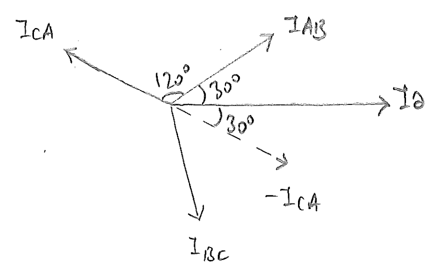
Under balanced conditions the phase currents are equal in magnitude and have 120° phase separation.

$(I_{AB}, I_{BC}, I_{CA}) \sim (I_a, I_b, I_c)$ relation ?

$I_{AB} = |I_p| \angle \alpha$

$I_{BC} = |I_p| \angle \alpha - 120^\circ$

$I_{CA} = |I_p| \angle \alpha + 120^\circ$



By KCL:

$I_a = I_{AB} - I_{CA} = \sqrt{3}\, |I_p| \angle \alpha - 30^\circ$

Likewise:

$I_b = I_{BC} - I_{AB} = \sqrt{3}\, |I_p| \angle \alpha - 150^\circ$

$I_c = I_{CA} - I_{BC} = \sqrt{3}\, |I_p| \angle \alpha + 90^\circ$

Note that for Y-connection : $|V_\ell| = \sqrt{3}\,|V_p|$ & $|I_\ell| = |I_p|$

for Δ-connection : $|I_\ell| = \sqrt{3}\,|I_p|$ & $|V_\ell| = |V_p|$

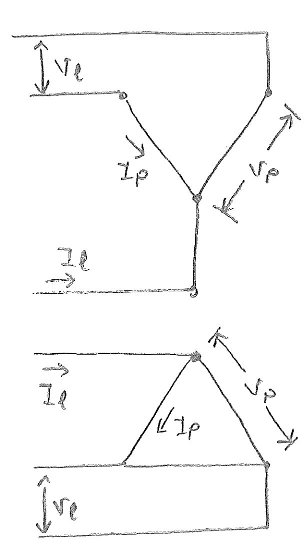
Hence for both (Δ & Y) connections we have :

$$|S_{3\phi}| = 3\,V_{p,rms}\,I_{p,rms} = \sqrt{3}\,V_{\ell,rms}\,I_{\ell,rms}$$

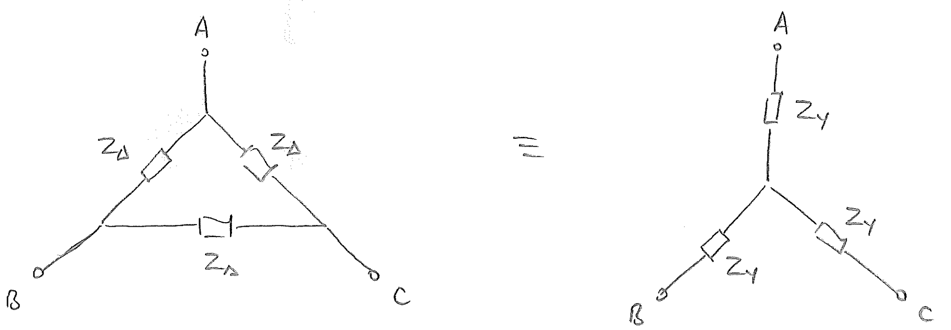
$$P_{3\phi} = 3\,V_{p,rms}\,I_{p,rms}\cos\phi = \sqrt{3}\,V_{\ell,rms}\,I_{\ell,rms}\cos\phi$$

$\hookrightarrow$ powerfactor (of the load)

$$Q_{3\phi} = 3\,V_{p,rms}\,I_{p,rms}\sin\phi = \sqrt{3}\,V_{\ell,rms}\,I_{\ell,rms}\sin\phi$$

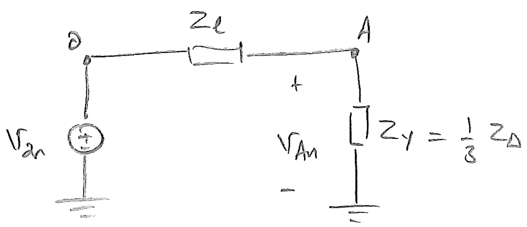


Remark To make use of per-phase circuit we apply Δ-Y transformation to the load.



where $\boxed{Z_Y = \frac{1}{3} Z_\Delta}$

Equivalent per-phase circuit:



Example A Δ-connected load with per-phase impedance $Z_\Delta = 75 + j225\ \Omega$ is supplied with power from a Y-connected generator whose phase voltage is 110 V_{rms}. The transmission line impedance is $Z_\ell = 10 + j5\ \Omega$. Take V_{an} on the generator side as reference. Find :

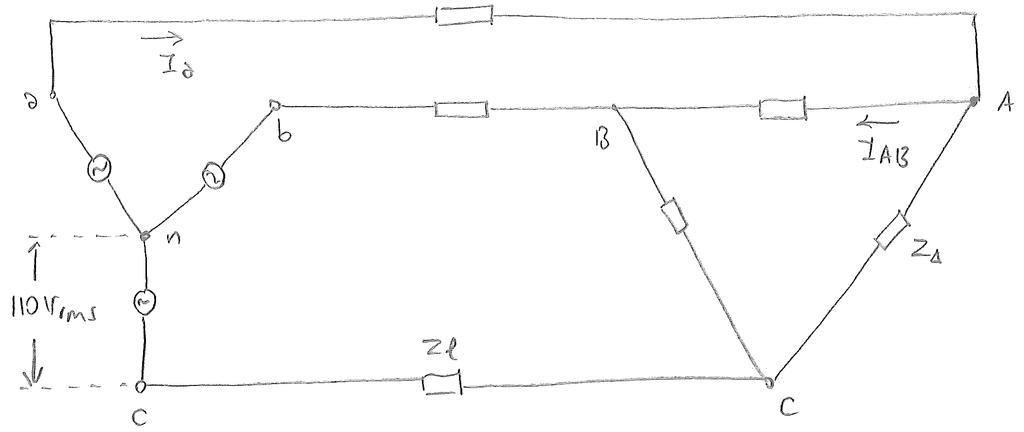
a) Phase currents of the load,

b) Line-to-line voltages on the load side.

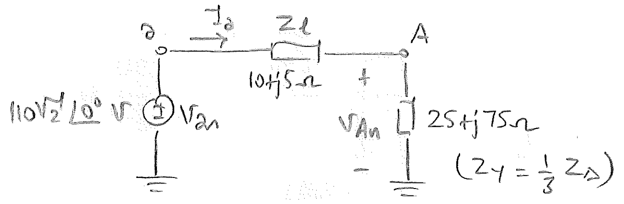
c) Real power consumed by the load.

#8 7 Nisan 2019

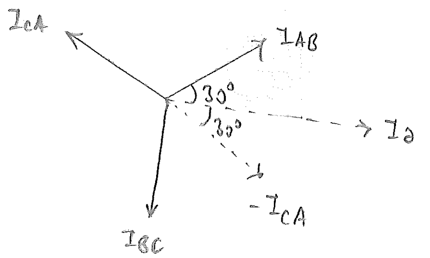
Sol'n



per-phase circuit:



$$I_a = \frac{110\sqrt{2}}{35+j80} = 1.78\angle -66° \text{ A}$$



$$I_{AB} = \left(\frac{1}{\sqrt{3}}\angle 30°\right) \times I_a = 1.03\angle -36° \text{ A}$$

Then $I_{BC} = 1.03\angle -156°$ A

& $I_{CA} = 1.03\angle 84°$ A

$$V_{AB} = Z_\Delta I_{AB} = (75+j225)\, 1.03\angle -36° = 244\angle 35° \text{ V}$$

Then $V_{BC} = 244\angle -85°$ V & $V_{CA} = 244\angle 155°$ V

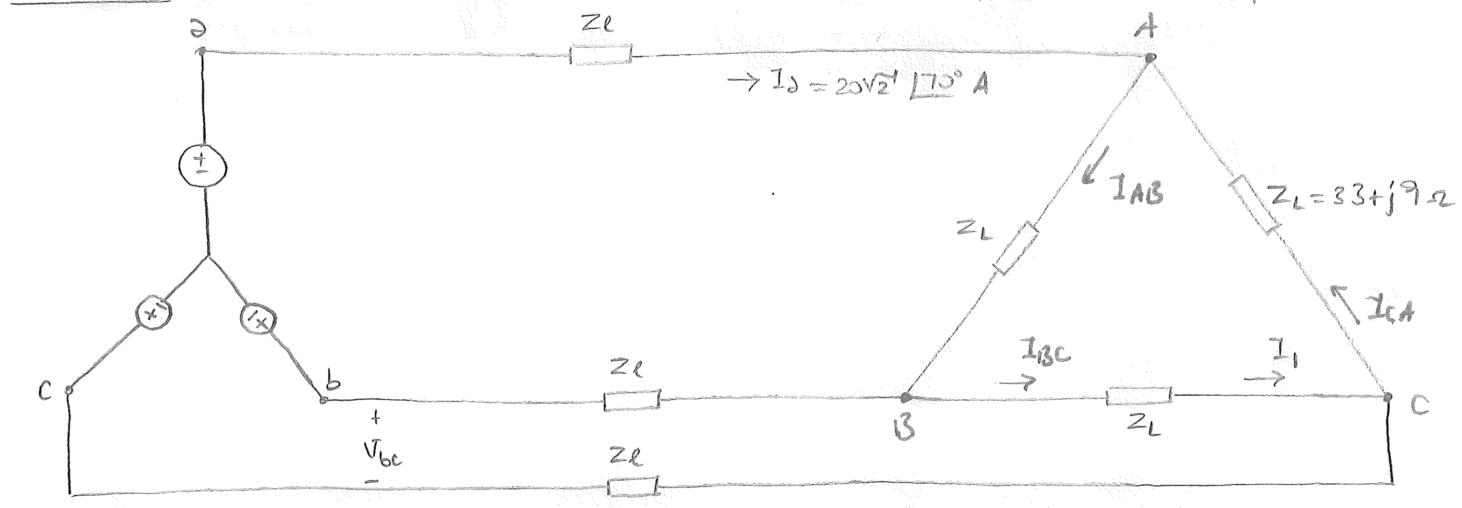
per-phase real power $P_A = I_{AB,rms}^2 \cdot \mathrm{Re}\{Z_\Delta\} = I_{a,rms}^2 \cdot \mathrm{Re}\{Z_Y\}$

$$= \left(\frac{1.03}{\sqrt{2}}\right)^2 \cdot 75$$

$$= 39.8 \text{ W}$$

Then $P_{3\phi} = 3P_A = \boxed{119 \text{ W}}$

Example: Consider the balanced 3ϕ circuit with positive phase sequence (a-b-c)



Given

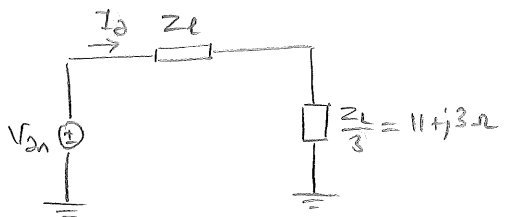
→ The total real power lost on the trans. lines : $P_{\ell,3\phi} = 1200\,W$

→ The power factor of the source side $pf_s = 12/13$ lag.

→ The frequency of operation $f = 50\,Hz$.

Find $S_{s,3\phi}$, η (efficiency), Z_ℓ, I_1, $v_{bc}(t)$.

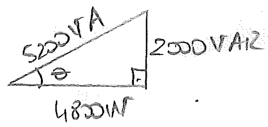
Sol'n per-phase circuit:



$I_{a,eff} = 20\,A$, $P_{\ell,\phi} = 1200/3 = 400\,W$

$P_{s,\phi} = P_{\ell,\phi} + P_{L,\phi} = 400 + 11\cdot 20^2 = 4800\,W$

$pf_s = \frac{12}{13}$ lag. $\Rightarrow$

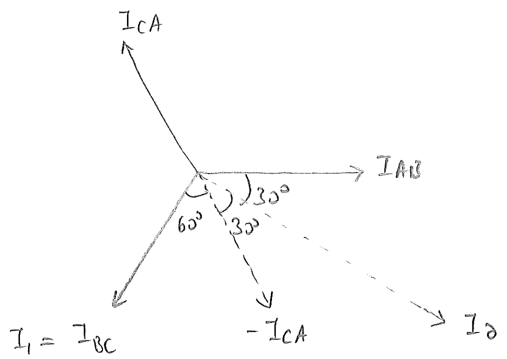


$\Rightarrow S_{s,\phi} = 4800 + j2000$ VA $\Rightarrow S_{s,3\phi} = 3S_{s,\phi} = \boxed{14,400 + j6000 \text{ VA}}$

$$\eta = \frac{P_{L,3\phi}}{P_{s,3\phi}} = \frac{P_{L,\phi}}{P_{s,\phi}} = \frac{4800-400}{4800} = \frac{11}{12} \cong \boxed{91\,\%}$$

$$S_{\ell,\phi} = S_{s,\phi} - S_{L,\phi} = 4800 + j2000 - (11+j3)\cdot 20^2 = 400 + j800 \text{ VA}$$

$$S_{\ell,\phi} = I_{a,eff}^2\, Z_\ell \Rightarrow Z_\ell = \frac{400+j800}{400} = \boxed{1+j2\,\Omega}$$

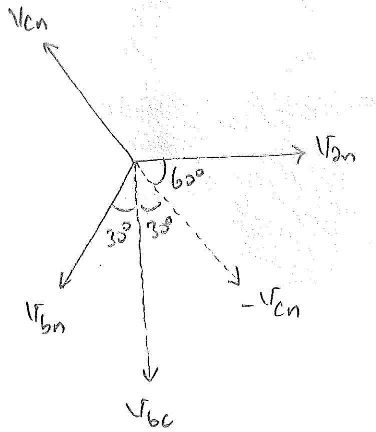


$$I_a = I_{AB} - I_{CA}$$

$$\Rightarrow I_1 = I_a \frac{1}{\sqrt{3}} \angle -90° = \boxed{\frac{20\sqrt{2}}{\sqrt{3}} \angle -20° \text{ A}}$$

$$V_{an} = I_a \left(Z_\ell + \frac{Z_L}{3}\right) = 20\sqrt{2} \angle 70° \,(12+j5) = 20\sqrt{2} \angle 70° \;\; 13 \angle \arctan 5/12$$

$$= 260\sqrt{2} \angle 70° + \arctan 5/12 \quad V$$



$$V_{bc} = V_{bn} - V_{cn}$$

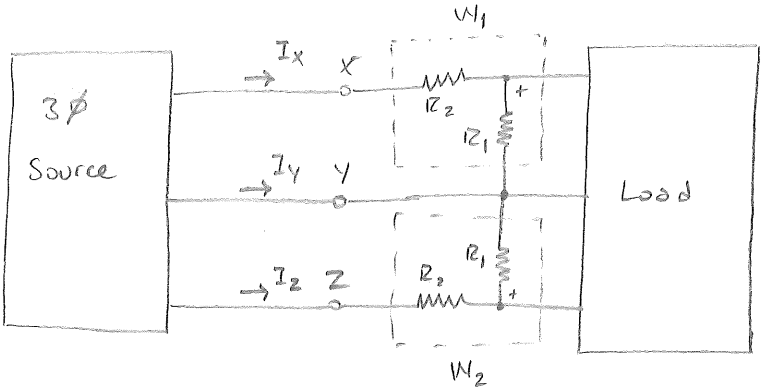
$$\Rightarrow V_{bc} = V_{an} \sqrt{3} \angle -90° = 260\sqrt{6} \angle -20° + \arctan 5/12 \quad V$$

$$f = 50 \text{Hz} \Rightarrow \omega = 100\pi \text{ rad/sec}$$

$$\Rightarrow \boxed{v_{bc}(t) = 260\sqrt{6} \cos\left(100\pi t - 20° + \arctan \frac{5}{12}\right) \; V}$$

#1 3 Nisan 2011

TWO WATTMETER POWER MEASUREMENT



W_1, W_2 wattmeters

$R_1 \approx \infty \Omega$, $R_2 \approx 0 \Omega$ } Here the measurement does not disturb the system.

$\left(S_L = P_L + jQ_L, \quad pf_L = \cos\theta\right)$

W_1 reads: $Re\left\{\frac{1}{2} V_{xy} I_x^*\right\} =: P_1$

W_2 reads: $Re\left\{\frac{1}{2} V_{zy} I_z^*\right\} =: P_2$

We know the circuit is balanced, but we don't know the sequencing. (That's why we labeled the nodes X, Y, Z instead of A, B, C.)

— o —

Let us take V_{xn} as reference. That is, $V_{xn} = \frac{|V_\ell|}{\sqrt{3}} \angle 0^\circ$.

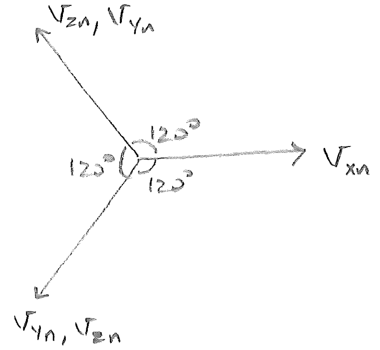
($|V_\ell|$: magnitude of line-to-line voltage.)

$$V_{xn} = \frac{|V_\ell|}{\sqrt{3}} \angle 0^\circ$$

$$V_{yn} = \frac{|V_\ell|}{\sqrt{3}} \angle \mp 120^\circ$$

$$V_{zn} = \frac{|V_\ell|}{\sqrt{3}} \angle \pm 120^\circ$$

Note that $\mp$ & $\pm$ are not the same!



$$\Rightarrow V_{xy} = V_{xn} - V_{yn} = \frac{|V_\ell|}{\sqrt{3}}\left(\angle 0^\circ - \angle \mp 120^\circ\right) = |V_\ell| \angle \pm 30^\circ$$

$$\Rightarrow V_{zy} = V_{zn} - V_{yn} = \frac{|V_\ell|}{\sqrt{3}}\left(\angle \pm 120^\circ - \angle \mp 120^\circ\right) = |V_\ell| \angle \pm 90^\circ$$

How about currents?

$$I_x = |I_\ell| \angle -\theta \qquad P_1 = \frac{1}{2} Re\left\{ |V_\ell| \angle \pm 30^\circ \; |I_\ell| \angle \theta \right\}$$

$$= \frac{1}{2} |V_\ell||I_\ell| \cos(\theta \pm 30^\circ)$$

$$I_y = |I_\ell| \angle \mp 120^\circ - \theta$$

$$I_z = |I_\ell| \angle \pm 120^\circ - \theta \qquad P_2 = \frac{1}{2} Re\left\{ |V_\ell| \angle \pm 90^\circ \; |I_\ell| \angle \theta \mp 120^\circ \right\}$$

$$= \frac{1}{2} |V_\ell||I_\ell| \cos(\theta \mp 30^\circ)$$

$$\Rightarrow P_1 + P_2 = V_{\ell,rms} I_{\ell,rms} \left\{ \cos(\theta \pm 30^\circ) + \cos(\theta \mp 30^\circ) \right\}$$

$$= V_{\ell,rms} I_{\ell,rms} \left\{ \cos\theta \cos 30^\circ \cancel{\mp \sin\theta \sin 30^\circ} + \cos\theta \cos 30^\circ \cancel{\pm \sin\theta \sin 30^\circ} \right\}$$

$$= \sqrt{3}\, V_{\ell,rms} I_{\ell,rms} \cos\theta = \text{power of load}$$

$$= P_L$$

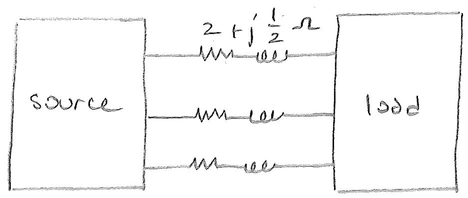
$$\Rightarrow P_1 - P_2 = V_{\ell,rms} I_{\ell,rms} \left\{ \cos(\theta \pm 30^\circ) - \cos(\theta \mp 30^\circ) \right\}$$

$$= \mp V_{\ell,rms} I_{\ell,rms} \sin\theta = \mp \frac{Q_L}{\sqrt{3}} \Rightarrow |Q_L| = \sqrt{3}\, |P_1 - P_2|$$

Note that two wattmeter method does not tell us whether the load is capacitive or inductive.

#1 25 Nisan 2016

Example: A 3ϕ load is powered by the following circuit.



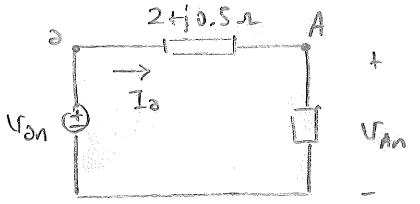
→ line voltage at the load side = 380 V_{rms}

→ $P_{load} = 5kW$

→ $pf_{load} = 0.8$ lag

Find the rms value of the line voltage at the source side.

Sol'n: per-phase circuit



$$V_{An,rms} = \frac{380}{\sqrt{3}} = 220V$$

per-phase complex power consumed by the load:

$$S_A = \frac{5000}{0.8}(0.8+j0.6) \times \frac{1}{3}$$

$$= 1667 + j1250 \text{ VA}$$

$$\Rightarrow I_{a,rms} = \frac{|S_A|}{V_{An,rms}} = 9.47 \text{ A}$$

per-phase complex power delivered by the source:

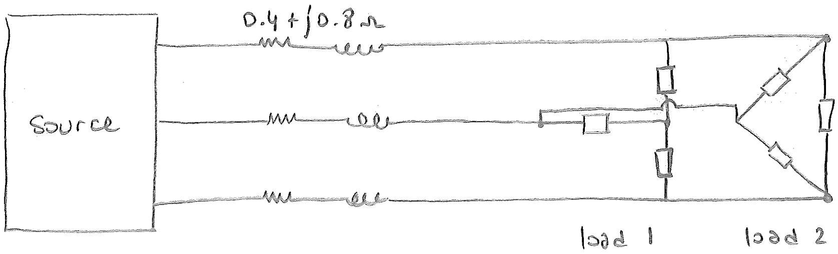
$$S_a = I_{a,rms}^2 (2+j0.5) + S_A$$

$$= 1846 + j1295 \text{ VA}$$

$$\Rightarrow V_{an,rms} = \frac{|S_a|}{I_{a,rms}} = 238V$$

$$\Rightarrow V_{\ell,source,rms} = \sqrt{3}\, V_{an,rms} = \boxed{413V}$$

Example



Given

- → $V_{\ell,rms} = 4.16$ kV at load side
- → load 1 : 1.5 MVA at pf 0.75 lag
- → load 2 : 2 MW at pf 0.8 lag

Find

a) line-to-line source voltage (rms)

b) real power supplied by the source

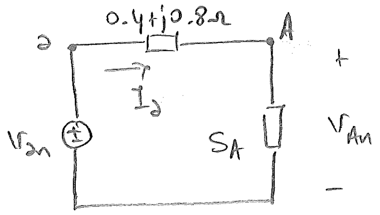
Sol'n

$$S_1 = 1.5\left(\frac{3}{4} + j\frac{\sqrt{7}}{4}\right) = 1.125 + j0.992 \text{ MVA}$$

$$S_2 = \frac{2}{0.8}(0.8 + j0.6) = 2 + j1.5 \text{ MVA}$$

$$S_L = S_1 + S_2 = 3.125 + j2.492 \text{ MVA}$$

per-phase circuit :



$$S_A = \frac{1}{3} S_L = 1.04 + j0.83 \text{ MVA}$$

$$V_{An,rms} = \frac{4160}{\sqrt{3}} = 2400 \text{ V}$$

$$I_{a,rms} V_{An,rms} = |S_A| = 1.33 \times 10^6 \text{ VA}$$

$$\Rightarrow I_{a,rms} = 555 \text{ A}$$

$$S_a = (0.4 + j0.8) I_{a,rms}^2 + S_A = 1.16 + j1.08 \text{ MVA}$$

$$I_{a,rms} V_{an,rms} = |S_a| = 1.58 \times 10^6 \text{ VA}$$

$$\Rightarrow V_{an,rms} = 2855 \text{ V} \quad \Rightarrow \quad V_{\ell,source,rms} = 2855\sqrt{3} = \boxed{4946 \text{ V}}$$

$$P_s = 3\,\mathrm{Re}\{S_a\} = \boxed{3.48 \text{ MW}}$$

#1 27 Nisan 2016

S-Domain Circuit Analysis

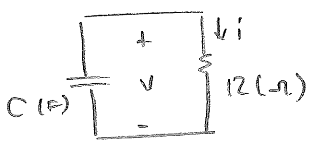
Idea : Given an LTI circuit follow the steps below:

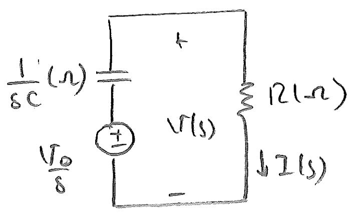
1) Replace each component by its s-domain equivalent (?) & obtain the "s-domain circuit".

2) Treat the s-domain circuit as an LTI resistive circuit & solve for the Laplace transforms of the asked variables.

3) Obtain the time domain signals of the asked variables through inverse Laplace transform.

Ex : time-domain circuit $\xrightarrow{(?)}$ s-domain circuit





$v(0^-) = V_o$

$i(t) = ?$

$$I(s) = \frac{V_o/s}{R + \frac{1}{sC}} = \frac{V_o}{R} \cdot \frac{1}{s + \frac{1}{RC}}$$

$$\Rightarrow \quad i(t) = \mathcal{L}^{-1}\{I(s)\} = \frac{V_o}{R} e^{-t/RC} \quad \text{for } t \geq 0.$$

s-Domain Circuit

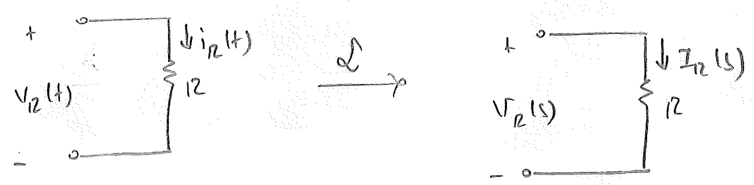
→ KCL / KVL hold in s-domain

$$i_1(t) + i_2(t) + \cdots + i_n(t) = 0 \Rightarrow I_1(s) + I_2(s) + \cdots + I_n(s) = 0$$

→ Terminal equations

1) Resistor

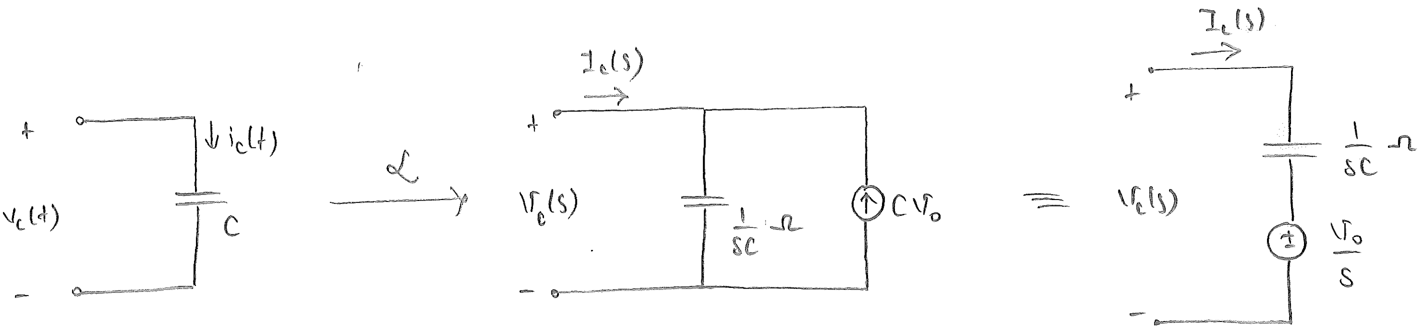
$$v_R(t) = R\, i_R(t) \xrightarrow{\mathcal{L}} \boxed{V_R(s) = R\, I_R(s)}$$



2) Capacitor

$$i_c(t) = C\,Dv_c(t) \xrightarrow{\mathcal{L}} I_c(s) = C\{sV_c(s) - v_c(0^-)\} \Rightarrow \boxed{I_c(s) = sC\,V_c(s) - CV_0} \quad (1)$$

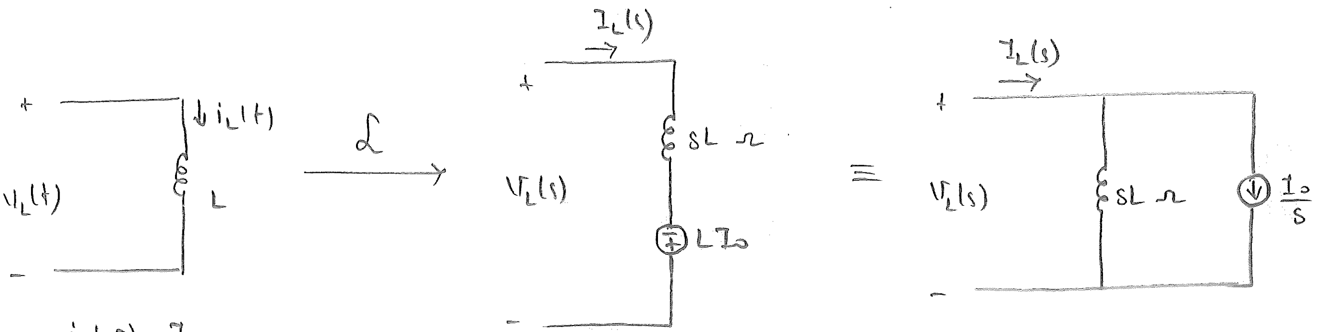
$$v_c(t) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i_c(\tau)\,d\tau \xrightarrow{\mathcal{L}} \boxed{V_c(s) = \frac{V_0}{s} + \frac{1}{sC} I_c(s)} \quad (2)$$



$v_c(0^-) = V_0$ (1) (2)

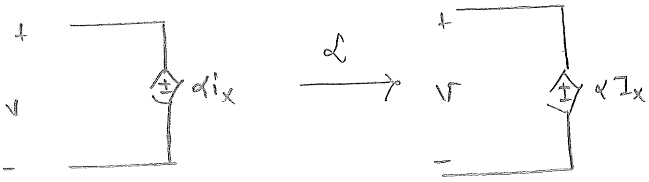
3) Inductor

$$\boxed{V_L(s) = sL\, I_L(s) - LI_0} \qquad \boxed{I_L(s) = \frac{I_0}{s} + \frac{1}{sL} V_L(s)}$$



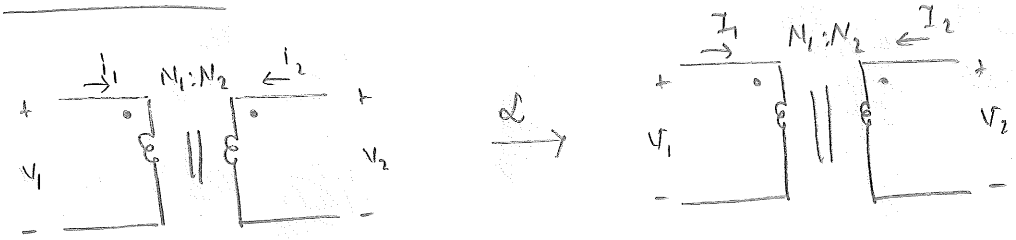
$i_L(0^-) = I_0$

Dependent source



$v(t) = \alpha\, i_x(t)$ $\qquad\qquad$ $V(s) = \alpha\, I_x(s)$

Ideal Transformer (IT)



$$\frac{v_1(t)}{N_1} = \frac{v_2(t)}{N_2} \qquad\qquad \frac{V_1(s)}{N_1} = \frac{V_2(s)}{N_2}$$

$$N_1 i_1(t) + N_2 i_2(t) = 0 \qquad\qquad N_1 I_1(s) + N_2 I_2(s) = 0$$

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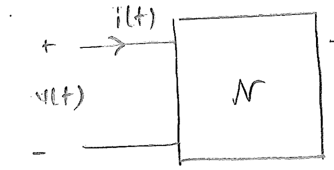
Definition Consider one-port



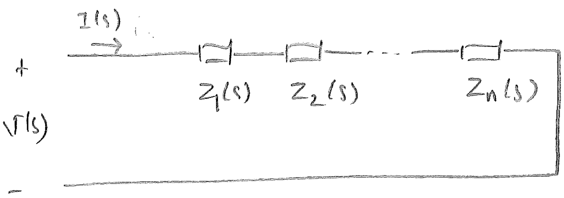
⇒ No ind. source & zero init. cond.

impedance of N is $\boxed{Z(s) = \dfrac{V(s)}{I(s)}}$ Z is measured in ohms (Ω)

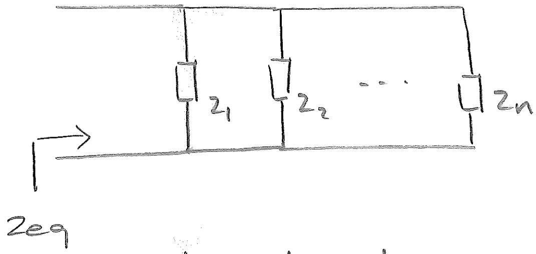
admittance $Y(s) = \dfrac{1}{Z(s)}$, measured in mhos ($\mho$)

R	$Z_R(s) = R$	$Y_R(s) = \frac{1}{R}$
L	$Z_L(s) = sL$	$Y_L(s) = \frac{1}{sL}$
C	$Z_C(s) = \frac{1}{sC}$	$Y_C(s) = sC$

Series / parallel connection

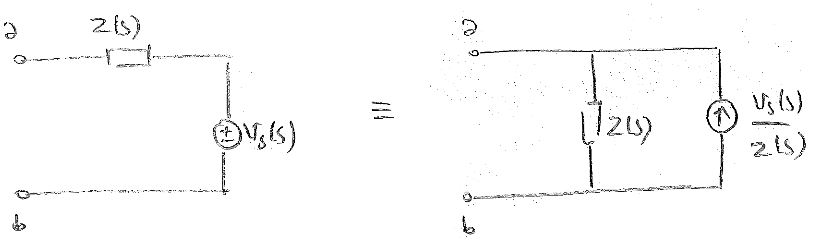


$$\frac{V(s)}{I(s)} = Z_{eq}(s) = Z_1(s) + Z_2(s) + \cdots + Z_n(s)$$

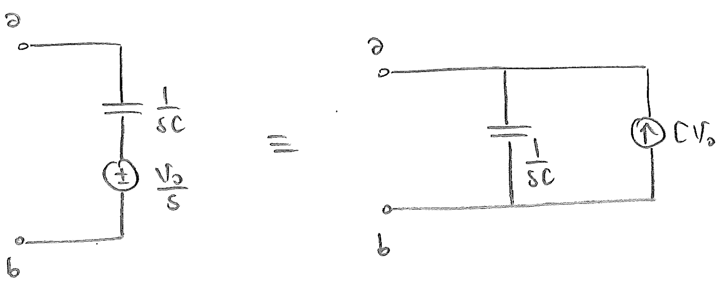


$$\frac{1}{Z_{eq}(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \cdots + \frac{1}{Z_n(s)}$$

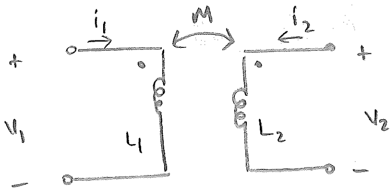
Source transformation



Ex :



Coupled Inductors

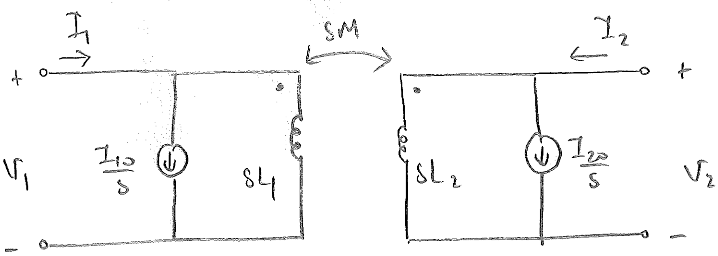


$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} Di_1 \\ Di_2 \end{bmatrix} \quad (1)$$

$i_1(0^-) = I_{10}$, $i_2(0^-) = I_{20}$

$$(1) \Rightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} sI_1 - I_{10} \\ sI_2 - I_{20} \end{bmatrix} = \begin{bmatrix} sL_1 & sM \\ sM & sL_2 \end{bmatrix} \begin{bmatrix} I_1 - \frac{I_{10}}{s} \\ I_2 - \frac{I_{20}}{s} \end{bmatrix}$$

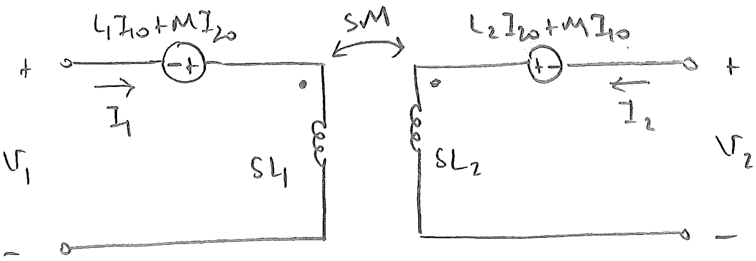
Schematics:



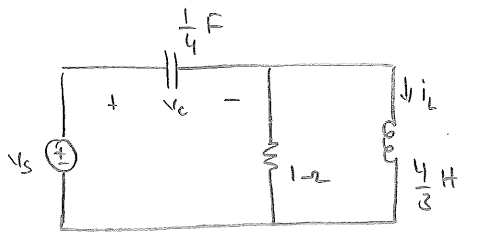
or,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} sI_1 - I_{10} \\ sI_2 - I_{20} \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} I_{10} \\ I_{20} \end{bmatrix}}_{\begin{bmatrix} V_1 + (L_1 I_{10} + M I_{20}) \\ V_2 + (L_2 I_{20} + M I_{10}) \end{bmatrix}} = \begin{bmatrix} sL_1 & sM \\ sM & sL_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Schematics;



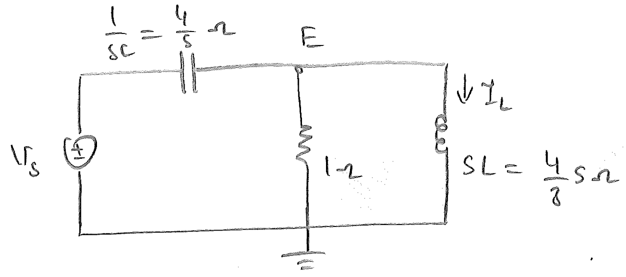
Example



$v_s(t) = \frac{4}{3} e^{-2t} u(t)$ V , $v_c(0^-) = 0$, $i_L(0^-) = 0$

Find $i_L(t)$ for $t \geq 0$.

Sol'n Transform the circuit to its s-domain equivalent.



Node eqn: $\frac{E - V_s}{4/s} + \frac{E}{1} + \frac{E}{\frac{4}{3}s} = 0$

$$\Rightarrow \left(\frac{s}{4} + 1 + \frac{3}{4s}\right) E = \frac{s}{4} V_s$$

$$\Rightarrow \frac{s^2 + 4s + 3}{4s} E = \frac{s}{4} V_s \quad \Rightarrow \quad E = \frac{s^2}{s^2+4s+3} V_s$$

$$\Rightarrow I_L = \frac{E}{\frac{4}{3}s} = \frac{3s/4}{s^2+4s+3} V_s \qquad (1)$$

Now, $V_s = \mathcal{L}\left\{\frac{4}{3} e^{-2t} u(t)\right\} = \frac{4}{3} \cdot \frac{1}{s+2}$

$$(1) \Rightarrow I_L = \frac{3s/4}{s^2+4s+3} \cdot \frac{4}{3} \cdot \frac{1}{s+2} = \frac{s}{(s+2)(s^2+4s+3)} = \frac{s}{(s+1)(s+2)(s+3)}$$

$$\Rightarrow I_L(s) = \frac{-1/2}{s+1} + \frac{2}{s+2} + \frac{-3/2}{s+3}$$

$$\Rightarrow \boxed{i_L(t) = -\frac{1}{2} e^{-t} + 2e^{-2t} - \frac{3}{2} e^{-3t} \text{ A}}$$ (for $t \geq 0$)

#4 29 Nisan 2012
[#7 27 Nisan 2016]

Superposition: For linear resistive circuits, we can write by superposition

$$y = k_1 w_1 + k_2 w_2 + \dots + k_n w_n$$

y: any output of the circuit

w_i: inputs of the circuit (ind. voltage / current sources)

k_i: weighing factors that depend on circuit (constants)

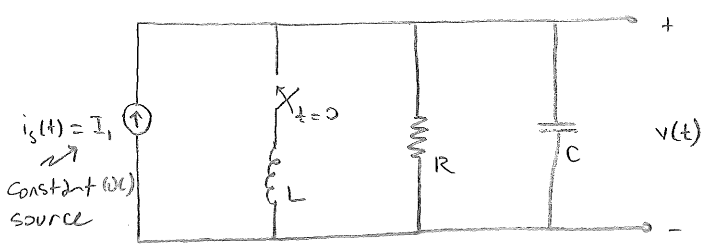
The superposition principle applies to linear dynamic circuits in s-domain with the difference that weighing factors are rational functions of s rather than constants. Also, in the s-domain there are two types of independent sources:

1) voltage / current sources representing external driving forces.

2) voltage / current sources representing the energy stored at $t=0$ (due to nonzero initial cond.)

The overall response is due both to 1 & 2. If we kill the sources that belong to group 2, the component we compute is zero-state response. When group 1 is killed, we obtain zero-input response. The sum of these components gives us the overall response.

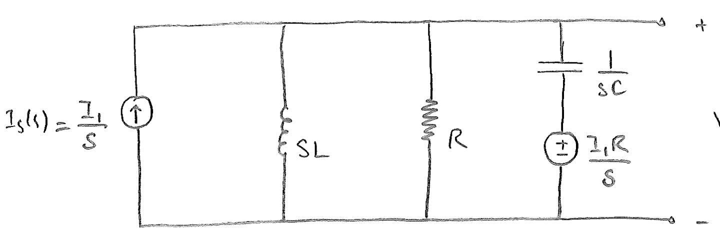
Example: [The circuit is in steady state at $t=0^-$.]



a) Find the zero-state & zero-input components of $V(s)$.

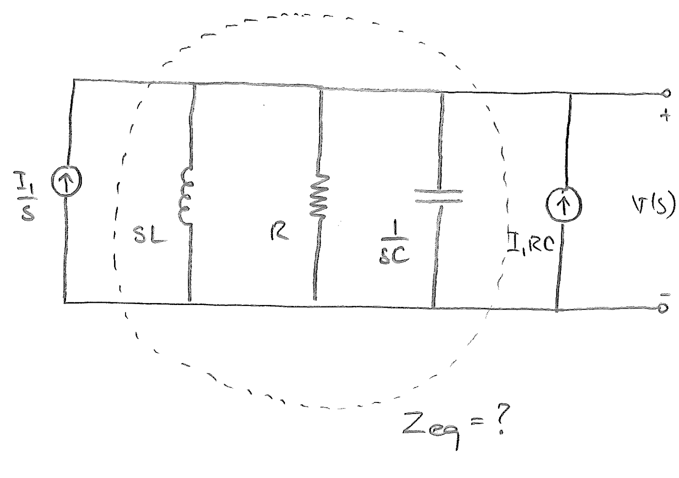
b) Find $v(t)$ for $I_1 = 1mA$, $L = 2H$, $R = \frac{3}{2}k\Omega$, $C = \frac{1}{6}\mu F$, for $t \geq 0$

a) s-domain



$(v_c(0^-) = I_1 R$ Volts$)$

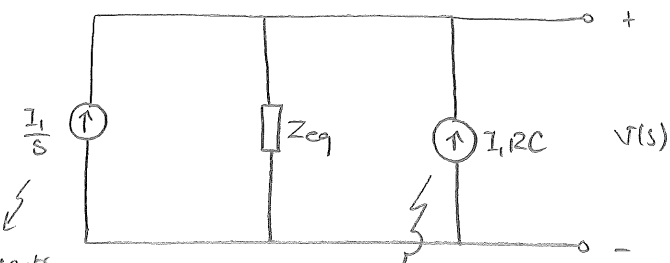
$\equiv$



#5 29 Nisan 2012
[#8 27 Nisan 2016]

$$Z_{eq}(s) = \frac{1}{\frac{1}{sL} + \frac{1}{R} + sC} = \frac{RLs}{RLCs^2 + Ls + R} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$\Rightarrow$



represents external inputs

represents init. cond.

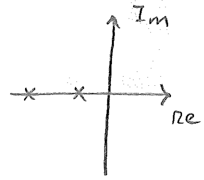
$$\Rightarrow V_{\substack{zero\\state}}(s) = Z_{eq}(s)\frac{I_1}{s} = \frac{I_1/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$V_{\substack{zero\\input}}(s) = Z_{eq}(s)\, I_1RC = \frac{I_1Rs}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

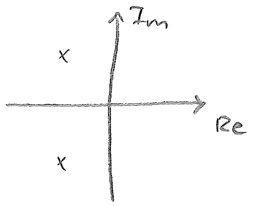
$$\Rightarrow V(s) = V_{zs}(s) + V_{zi}(s) = \frac{I_1Rs + I_1/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

What can be said about $v(t)$? Since the poles of $V(s)$ $\left\{ \text{i.e. roots of } s^2 + \frac{1}{RC}s + \frac{1}{LC} \right\}$

are either



or



i.e. $Re\{x\} < 0$

$\Rightarrow v(t) \to 0$ as $t \to \infty$

b) $$V_{zs}(s) = \frac{6000}{s^2 + 4000s + 3\cdot 10^6} = \frac{6000}{(s+3000)(s+1000)} = \frac{3}{s+1000} - \frac{3}{s+3000}$$

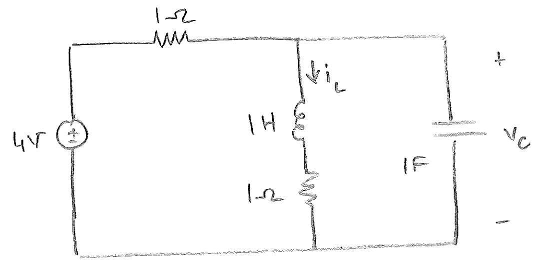
$$\Rightarrow \boxed{v_{zs}(t) = 3e^{-1000t} - 3e^{-3000t} \text{ Volts}} \quad (t \geq 0)$$

$$V_{zi}(s) = \frac{1.5s}{(s+3000)(s+1000)} = \frac{-3/4}{s+1000} + \frac{9/4}{s+3000}$$

$$\Rightarrow \boxed{v_{zi}(t) = -\frac{3}{4}e^{-1000t} + \frac{9}{4}e^{-3000t} \text{ Volts}} \quad (t \geq 0)$$

$\Rightarrow v(t) = v_{zs}(t) + v_{zi}(t)$.

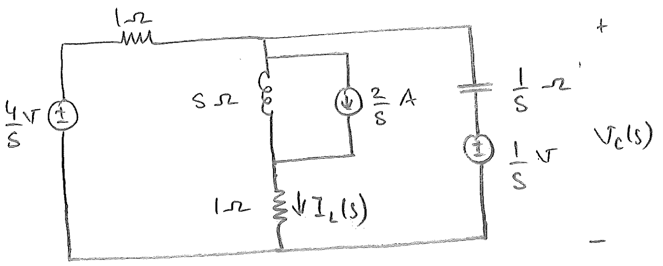
Example :



$i_L(0) = 2A$, $v_c(0) = 1V$

Find $v_c(t)$.

Sol'n :



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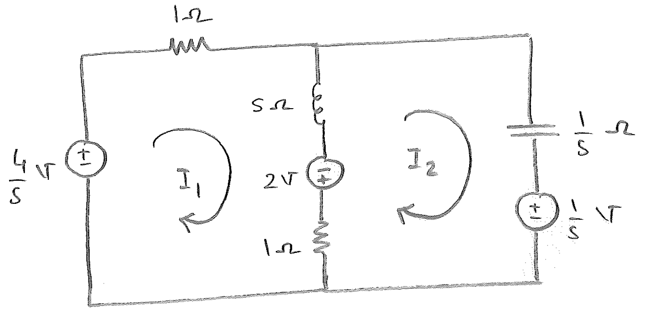

Loop 1: $-\frac{4}{s} + I_1 + s(I_1 - I_2) - 2 + I_1 - I_2 = 0 \Rightarrow (2+s)I_1 - (s+1)I_2 = 2 + \frac{4}{s}$ (1)

Loop 2: $I_2 - I_1 + 2 + s(I_2 - I_1) + \frac{1}{s}I_2 + \frac{1}{s} = 0 \Rightarrow -(s+1)I_1 + (s+1+\frac{1}{s})I_2 = -2 - \frac{1}{s}$ (2)

(1)&(2) $\Rightarrow$ $\underbrace{\begin{bmatrix} s+2 & -(s+1) \\ -(s+1) & \frac{s^2+s+1}{s} \end{bmatrix}}_{\text{let det of this matrix be } \Delta(s)} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 2+\frac{4}{s} \\ -2-\frac{1}{s} \end{bmatrix}$

Remark : Nat freq. of circuit are the roots of $\Delta(s)$!

$$\Delta(s) = (s+2)\frac{s^2+s+1}{s} - (s+1)^2 = \frac{s^2+2s+2}{s} \Rightarrow \text{Nat. freq. } s_{1,2} = -1 \mp j$$

$$\Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{s}{s^2+2s+2} \begin{bmatrix} \frac{s^2+s+1}{s} & s+1 \\ s+1 & s+2 \end{bmatrix} \begin{bmatrix} 2+\frac{4}{s} \\ -2-\frac{1}{s} \end{bmatrix}$$

$$\Rightarrow I_2(s) = \frac{s}{s^2+2s+2} \begin{bmatrix} s+1 & s+2 \end{bmatrix} \begin{bmatrix} 2+\frac{4}{s} \\ -2-\frac{1}{s} \end{bmatrix} = \frac{s}{s^2+2s+2} \cdot \left\{ \frac{(s+1)(2s+4)}{s} - \frac{(s+2)(2s+1)}{s} \right\}$$

$$\Rightarrow I_2(s) = \frac{s+2}{s^2+2s+2} \Rightarrow V_c(s) = \frac{1}{s}I_2(s) + \frac{1}{s} = \frac{s^2+3s+4}{s(s^2+2s+2)} = \frac{s^2+3s+4}{\underset{\downarrow\ \text{constant term}}{s}\underbrace{(s+1+j)(s+1-j)}_{\downarrow\ e^{-t}\cos t \ \& \ e^{-t}\sin t \text{ terms}}}$$

$$\Rightarrow V_c(s) = \frac{k_1}{s} + A\frac{s+1}{(s+1)^2+1} + B\frac{1}{(s+1)^2+1}$$

$$k_1 = sV_c(s)\Big|_{s=0} = 2$$

$$\Rightarrow \frac{2}{s} + A\frac{s+1}{(s+1)^2+1} + B\frac{1}{(s+1)^2+1} = \frac{2(s^2+2s+2) + As(s+1) + Bs}{s(s^2+2s+2)} = \frac{s^2+3s+4}{s(s^2+2s+2)}$$

$$\Rightarrow (A+2)s^2 + (4+A+B)s + 4 = s^2+3s+4 \quad \Rightarrow \quad A=-1 \ \& \ B=0$$

$$\Rightarrow V_c(s) = \frac{2}{s} - \frac{s+1}{(s+1)^2+1} \quad \Rightarrow \quad \boxed{v_c(t) = 2 - e^{-t}\cos t \ \text{V}} \quad \text{for } t>0$$

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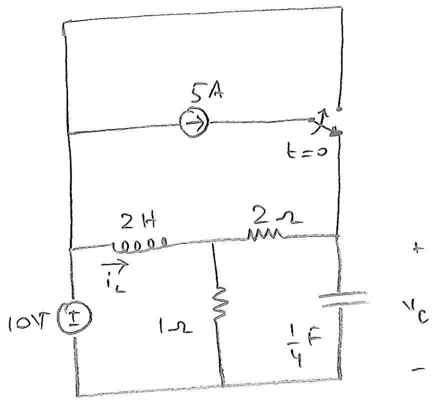
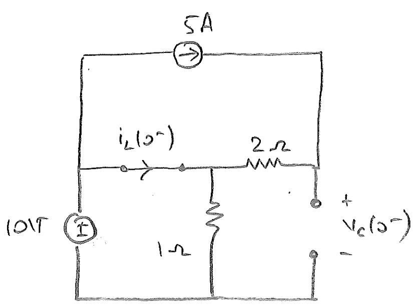
Example [The circuit is in steady state at $t=0^-$]



Find $v_c(t)$ for $t>0$.

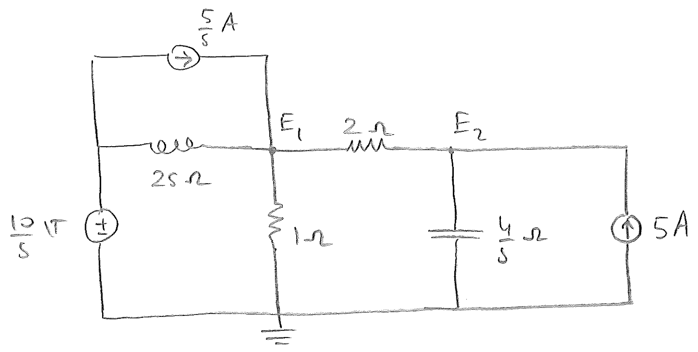
Sol'n: at $t=0^-$



$i_L(0^-) = 5A$

$v_c(0^-) = 20V$

S-domain circuit (after the switching)



Node 1: $\dfrac{E_1 - 10/s}{2s} - \dfrac{5}{s} + \dfrac{E_1}{1} + \dfrac{E_1 - E_2}{2} = 0 \Rightarrow \left\{\dfrac{1}{2s} + 1 + \dfrac{1}{2}\right\}E_1 + \left\{-\dfrac{1}{2}\right\}E_2 = \dfrac{5}{s^2} + \dfrac{5}{s}$

$\Rightarrow \quad \dfrac{3s+1}{2s}E_1 - \dfrac{1}{2}E_2 = 5\,\dfrac{s+1}{s^2} \qquad (1)$

Node 2: $\dfrac{E_2 - E_1}{2} + \dfrac{E_2}{4/s} - 5 = 0 \Rightarrow -\dfrac{1}{2}E_1 + \dfrac{s+2}{4}E_2 = 5 \qquad (2)$

$(2) \Rightarrow \quad E_1 = \dfrac{s+2}{2}E_2 - 10 \qquad (3)$

$(1)\,\&\,(3) \Rightarrow \quad \dfrac{3s+1}{2s}\left\{\dfrac{s+2}{2}E_2 - 10\right\} - \dfrac{1}{2}E_2 = 5\,\dfrac{s+1}{s^2}$

$\Rightarrow \quad \left\{\dfrac{(3s+1)(s+2)}{4s} - \dfrac{1}{2}\right\}E_2 = 5\,\dfrac{s+1}{s^2} + 5\,\dfrac{3s+1}{s} = 5\,\dfrac{3s^2+2s+1}{s^2}$

$\Rightarrow \quad \dfrac{3s^2+7s+2-2s}{4s}E_2 = 5\,\dfrac{3s^2+2s+1}{s^2} \Rightarrow \dfrac{3s^2+5s+2}{4}E_2 = 5\,\dfrac{3s^2+2s+1}{s}$

$\Rightarrow E_2 = 20\,\dfrac{3s^2+2s+1}{s(3s^2+5s+2)} = \dfrac{20s^2 + \frac{40}{3}s + \frac{20}{3}}{s\left(s^2+\frac{5}{3}s+\frac{2}{3}\right)} = \dfrac{20s^2+\frac{40}{3}s+\frac{20}{3}}{s\left(s+\frac{2}{3}\right)(s+1)}$

$\Rightarrow E_2(s) = \dfrac{k_1}{s} + \dfrac{k_2}{s+\frac{2}{3}} + \dfrac{k_3}{s+1}$

$\Rightarrow k_1 = 10$

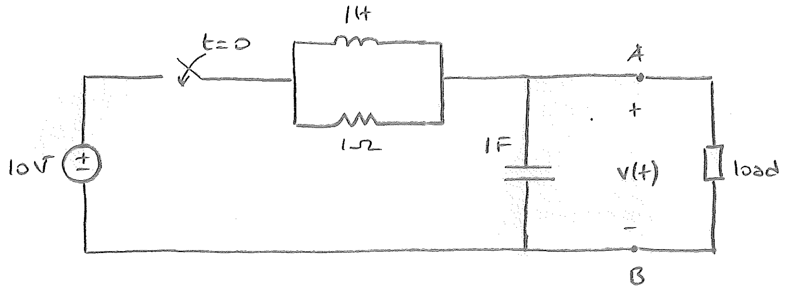
$k_2 = \left.\dfrac{20s^2+\frac{40}{3}s+\frac{20}{3}}{s(s+1)}\right|_{s=-2/3} = -30$

$k_3 = \dfrac{40/3}{1/3} = 40$

$\boxed{v_c(t) = e_2(t) = 10 - 30e^{-2t/3} + 40e^{-t}\ \text{V}}$

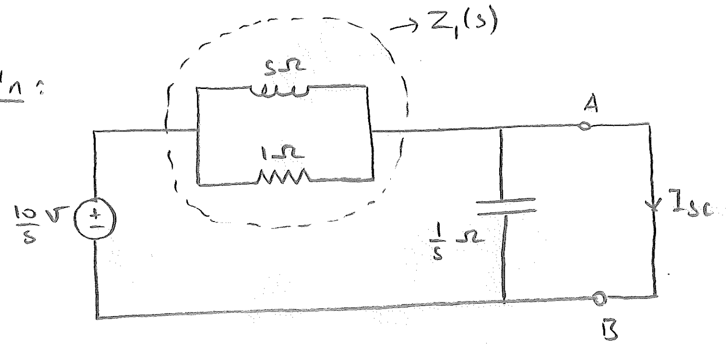
for $t > 0$

Example :



The circuit is at zero-state before the switch is shut at $t=0$. Obtain the Norton Equiv. to the left of the terminals AB in s-domain to solve for $v(t)$ when the load is a) 1Ω resistor
b) 1F capacitor

Sol'n :



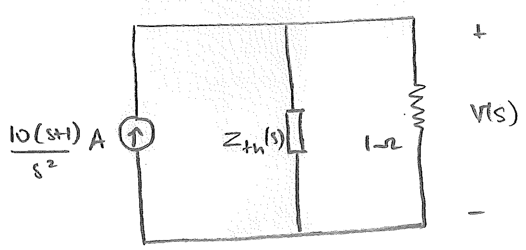
$$Z_1(s) = \frac{s}{s+1}$$

$$I_{sc}(s) = \frac{10/s}{Z_1(s)} = \frac{10(s+1)}{s^2} \text{ A}$$

$$Z_{th}(s) = Z_1(s) \mathbin{/\!/} \frac{1}{s}$$

$$= \frac{\frac{1}{s}\cdot\frac{s}{s+1}}{\frac{1}{s}+\frac{s}{s+1}} = \frac{s}{s^2+s+1}$$

a)

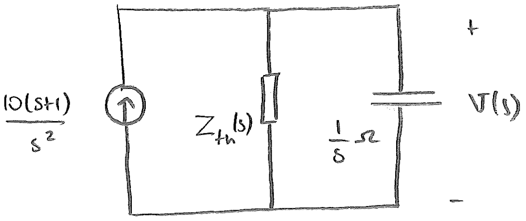


$$V(s) = \frac{10(s+1)}{s^2}\cdot\left(Z_{th} \mathbin{/\!/} 1\right) = \frac{10(s+1)}{s^2}\cdot\frac{\frac{s}{s^2+s+1}}{1+\frac{s}{s^2+s+1}} = \frac{10(s+1)}{s^{\cancel{2}}}\cdot\frac{\cancel{s}}{s^2+2s+1} = \frac{10}{s(s+1)}$$

$$= \frac{10}{s} - \frac{10}{s+1}$$

$$\Rightarrow \boxed{v(t) = 10 - 10e^{-t} \text{ V}} \quad \text{for } t>0$$

b)



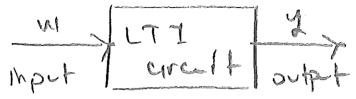
$$V(s) = \frac{10(s+1)}{s^2}\left(Z_{th} \mathbin{/\!/} \frac{1}{s}\right) = \frac{5(s+1)}{s\left(s^2+\frac{1}{2}s+\frac{1}{2}\right)}$$

$$\Rightarrow \boxed{v(t) = 10 - 10e^{-t/4}\cos\frac{\sqrt{7}}{4}t + \frac{10}{\sqrt{7}}e^{-t/4}\sin\frac{\sqrt{7}}{4}t \text{ V}}$$

for $t>0$

TRANSFER FUNCTION

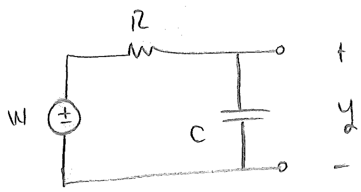
Definition For a LTI circuit the transfer function $H(s)$ from input w to output y is the ratio of a zero-state response transform $Y(s)$ to the excitation transform $W(s)$.



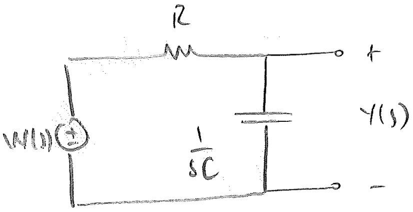
$$\boxed{H(s) = \frac{Y(s)}{W(s)}} \Rightarrow Y(s) = H(s)W(s)$$

single input,
zero init. cond.

Ex



$\Rightarrow$



$$\Rightarrow H(s) = \frac{Y(s)}{W(s)} = \frac{1/sC}{R + 1/sC} = \frac{1/RC}{s + 1/RC}$$

The poles of $Y(s)$ come from either the TF $H(s)$ or the input signal $W(s)$

When there are no repeated poles,

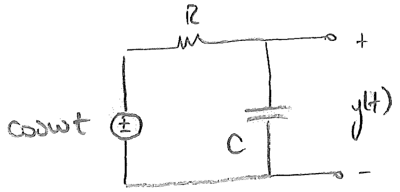
$$Y(s) = \sum_i \frac{k_j}{s-p_j} + \sum_i \frac{l_i}{s-q_i}$$

p_j : poles of $H(s)$ = nat. frequencies

q_i : poles of $W(s)$

Then $$y(t) = \underbrace{\sum_i k_j e^{p_j t}}_{\text{homogeneous sol'n}} + \underbrace{\sum_i l_i e^{q_i t}}_{\text{particular sol'n}}$$

Ex

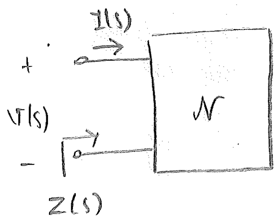


$$\Rightarrow Y(s) = \underbrace{\frac{1/RC}{s + 1/RC}}_{H(s)} \cdot \underbrace{\frac{s}{s^2+w^2}}_{W(s)}$$

$$\Rightarrow y(t) = \underbrace{k_1 e^{-t/RC}}_{\text{hmg. sol'n}} + \underbrace{l_1 e^{jwt} + l_2 e^{-jwt}}_{\text{part. sol'n}} \qquad (l_2 = l_1^*)$$

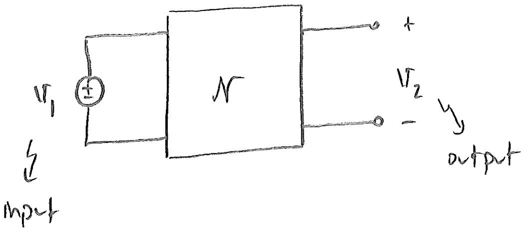
In a stable circuit (i.e. all the natural frequencies λ_i satisfy $Re\{\lambda_i\}<0$) the homogeneous solution decays to zero. The poles of the input $W(s)$ lead to the particular solution. (In a stable circuit) those elements in the particular solution that do not decay to zero and stay bounded form the steady-state response (or SS solution).

Transfer functions of one-port & two-port circuits

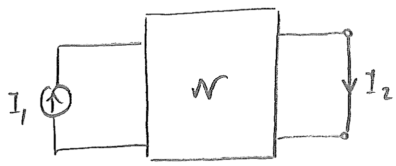


Driving point impedance (or simply impedance) is a transfer function $Z(s) = \dfrac{V(s)}{I(s)}$

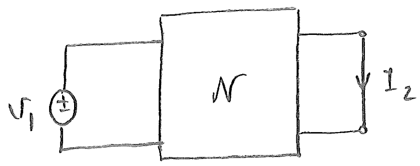
For two ports the possibilities are diverse:



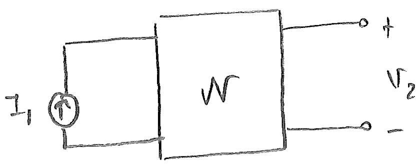
$H_V(s) = \dfrac{V_2(s)}{V_1(s)}$: voltage trans. func.



$H_I(s) = \dfrac{I_2(s)}{I_1(s)}$: current TF



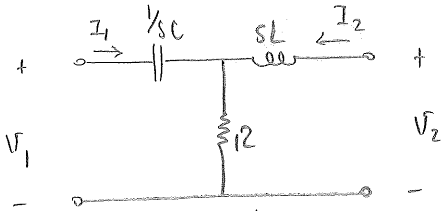
$H_Y(s) = \dfrac{I_2(s)}{V_1(s)}$: transfer admittance



$H_Z(s) = \dfrac{V_2(s)}{I_1(s)}$: transfer impedance

#3 3 Mayıs 2016

Example [Impedance Parameters]



obtain the impedance param.

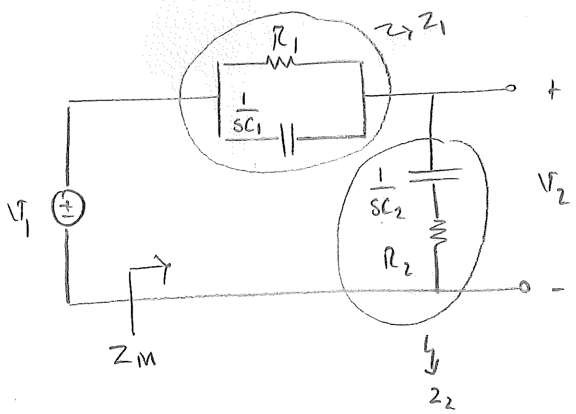
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix}}_{Z(s)} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Sol'n

$$\left.\begin{aligned} V_1 &= \frac{1}{sC} I_1 + R(I_1+I_2) = \left(R+\frac{1}{sC}\right) I_1 + R I_2 \\ V_2 &= sL I_2 + R(I_1+I_2) = R I_1 + (R+sL) I_2 \end{aligned}\right\} \quad Z(s) = \begin{bmatrix} \frac{RCs+1}{Cs} & R \\ R & Ls+R \end{bmatrix}$$

Exercise obtain the admittance, hybrid, and chain parameters.

Example



a) Find the driving point impedance Z_{in}.

b) Find the TF $H_v(s) = \frac{V_2(s)}{V_1(s)}$.

c) Locate the poles & zeros of $H_v(s)$

for $R_1 = 10k\Omega$, $R_2 = 20k\Omega$, $C_1 = 100nF$, $C_2 = 50nF$

Sol'n a) $Z_1 = \dfrac{R_1 \cdot \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \dfrac{R_1}{R_1C_1s+1}$, $Z_2 = R_2 + \dfrac{1}{sC_2} = \dfrac{R_2C_2s+1}{C_2s}$

$$\Rightarrow Z_{in} = Z_1 + Z_2 = \frac{R_1C_1R_2C_2s^2 + (R_1C_2 + R_1C_1 + R_2C_2)s + 1}{C_2s(R_1C_1s+1)}$$

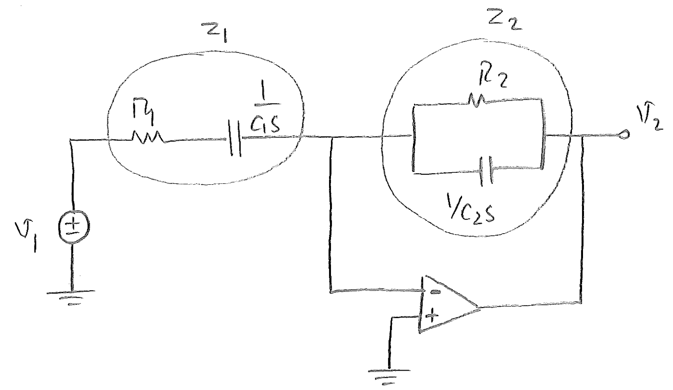
b) $$H_v(s) = \frac{Z_2(s)}{Z_{in}(s)} = \frac{(R_2C_2s+1)(R_1C_1s+1)}{R_1C_1R_2C_2s^2 + (R_1C_2+R_1C_1+R_2C_2)s+1}$$

c) $$H_v(s) = \frac{(10^{-3}s+1)(10^{-3}s+1)}{10^{-6}s^2 + 2.5\times10^{-3}s+1} = \frac{(s+1000)^2}{s^2+2500s+10^6} = \frac{(s+1000)^2}{(s+500)(s+2000)}$$

$\Rightarrow$ poles : $-500, -2000$ & zeros : $-1000, -1000$

#4 3 Mayıs 2016

Example



a) Find $H_v(s) = \dfrac{V_2(s)}{V_1(s)}$

b) poles are located at

$P_1 = -1000$ rad/sec &

$P_2 = -5000$ rad/sec for $R_1 = R_2 = 20k\Omega$

Find C_1 & C_2.

Sol'n a) $Z_2 = \dfrac{R_2 \cdot \frac{1}{C_2 s}}{R_2 + \frac{1}{C_2 s}} = \dfrac{R_2}{R_2 C_2 s + 1}$ &

$Z_1 = R_1 + \dfrac{1}{C_1 s} = \dfrac{R_1 C_1 s + 1}{C_1 s}$

We have $\dfrac{V_1}{Z_1} + \dfrac{V_2}{Z_2} = 0 \Rightarrow H_v(s) = -\dfrac{Z_2}{Z_1} = -\dfrac{\dfrac{R_2}{R_2 C_2 s + 1}}{\dfrac{R_1 C_1 s + 1}{C_1 s}} = -\dfrac{R_2 C_1 s}{(R_2 C_2 s + 1)(R_1 C_1 s + 1)}$

b) $P_1 = -\dfrac{1}{R_1 C_1} \Rightarrow -1000 = -\dfrac{1}{20\times10^3 \times C_1} \Rightarrow \boxed{C_1 = 50nF}$

(or $C_1 = 10nF$ & $C_2 = 50nF$)

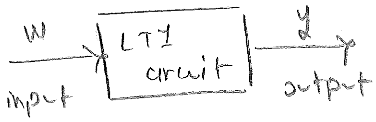
$P_2 = -\dfrac{1}{R_2 C_2} \Rightarrow -5000 = -\dfrac{1}{20\times10^3 \times C_2} \Rightarrow \boxed{C_2 = 10nF}$

#5 8 Mayıs 2016

Transfer function & Impulse response

Recall: impulse response is the zero-state response of a circuit when the driving force is a unit impulse at $t=0$.

For $w(t) = \delta(t) \Rightarrow W(s) = \mathcal{L}\{\delta(t)\} = 1$

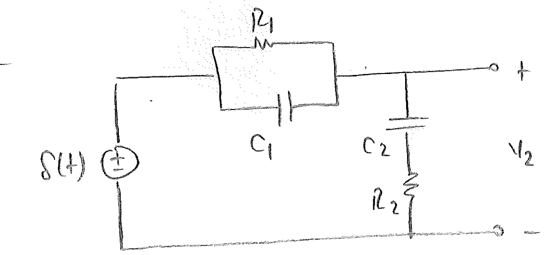


$$\Rightarrow Y(s) = H(s)\cdot 1$$

$$Y(s) = H(s)\,W(s) \qquad \Rightarrow y(t) = \mathcal{L}^{-1}\{H(s)\}$$

Therefore $\boxed{h(t) = \mathcal{L}^{-1}\{H(s)\}}$ is the impulse response.

Example



Find $v_2(t)$ (i.e., impulse response) for

$R_1 = 10k\Omega$, $C_1 = 1\mu F$

$R_2 = 12.5k\Omega$, $C_2 = 2\mu F$

Sol'n Earlier we've obtained $H_v(s) = \dfrac{(R_1C_1s+1)(R_2C_2s+1)}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_1C_2 + R_2C_2)s + 1}$

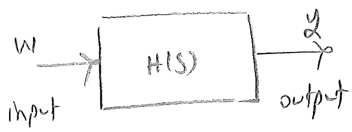
$$\Rightarrow H_v(s) = \frac{(10^{-2}s+1)(2.5\times10^{-2}s+1)}{2.5\times10^{-4}s^2 + 5.5\times10^{-2}s+1} = \frac{(s+100)(s+40)}{s^2+220s+4000} = \frac{s^2+140s+4000}{s^2+220s+4000}$$

$$\Rightarrow H_v(s) = 1 - \frac{80s}{s^2+220s+4000} = 1 - \frac{80s}{(s+20)(s+200)} = 1 + \frac{80/9}{s+20} - \frac{800/9}{s+200}$$

$$\Rightarrow \boxed{h(t) = \delta(t) + \frac{80}{9}e^{-20t}u(t) - \frac{800}{9}e^{-200t}u(t)} \quad V$$

Step Response

$w(t) = u(t)$ (unit step) $\Rightarrow W(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$



$\Rightarrow Y(s) = H(s) \cdot \frac{1}{s}$

Therefore $\boxed{y_{step}(t) = \mathcal{L}^{-1}\left\{ \frac{H(s)}{s} \right\}}$ is the step response

Equivalently, $y_{step}(t) = \int_{0^-}^{t} h(\tau)\, d\tau$, where $h(t)$ is the impulse response.

Impulse response & convolution

Given: impulse response $h(t)$ & input $w(t)$

Want: zero-state response $y(t)$ for $w(t)$

$H(s) = \mathcal{L}\{h(t)\}$ & $W(s) = \mathcal{L}\{w(t)\}$ $\Rightarrow$ $Y(s) = H(s)\,W(s)$

↳ transfer function

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{H(s)W(s)\} = \underbrace{\int_{0^-}^{t} h(t-\tau)\, w(\tau)\, d\tau}_{\text{convolution integral}} =: h * w$$

Example Let $h(t) = 200 e^{-100t} u(t)$. Find the ramp response $y(t)\Big|_{w(t) = t\,u(t)}$

$$H(s) = \mathcal{L}\{200 e^{-100t}\} = \frac{200}{s+100}$$

$$W(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$Y(s) = \frac{200}{s+100} \cdot \frac{1}{s^2} = \frac{k_{10}}{s^2} + \frac{k_{11}}{s} + \frac{k_2}{s+100}$$

$$= \frac{2}{s^2} + \frac{-1/50}{s} + \frac{1/50}{s+100} .$$

$$\Rightarrow \boxed{y(t) = \left(2t - \frac{1}{50} + \frac{1}{50} e^{-100t}\right) u(t)}$$

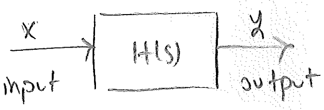
012

$$y(t) = \int_0^t h(t-\tau)\, w(\tau)\, d\tau = \int_0^t 200\, e^{-100(t-\tau)}\, \tau\, d\tau = 200\, e^{-100t} \underbrace{\int_0^t e^{100\tau}\, \tau\, d\tau}_{?}$$

$$\int_0^t \underbrace{\tau}_{v}\, \underbrace{e^{100\tau} d\tau}_{du} = \underbrace{\tau}_{v} \cdot \underbrace{\frac{1}{100} e^{100\tau}}_{u} \Big|_0^t - \int_0^t \underbrace{\frac{1}{100} e^{100\tau}}_{u}\, \underbrace{d\tau}_{dv} = \frac{t}{100} e^{100t} - \left\{ \frac{1}{10^4} e^{100t} - \frac{1}{10^4} \right\}$$

$$\Rightarrow y(t) = 200\, e^{-100t} \left\{ \frac{t}{100} e^{100t} - \frac{1}{10^4} e^{100t} + \frac{1}{10^4} \right\} = \boxed{2t - \frac{1}{50} + \frac{1}{50} e^{-100t} \quad \text{for } t > 0}$$

<u>Transfer Function & Sinusoidal Steady-State Response</u>



let input $x(t) = A\cos(\omega t + \phi) = A\{\cos\omega t \cos\phi - \sin\omega t \sin\phi\}$

$$\Rightarrow X(s) = A\left\{ \cos\phi \frac{s}{s^2+\omega^2} - \sin\phi \frac{\omega}{s^2+\omega^2} \right\}$$

Then $\quad Y(s) = H(s)X(s) = A \dfrac{\cos\phi \cdot s - \sin\phi \cdot \omega}{s^2+\omega^2} H(s)$

$$= \underbrace{\frac{k}{s-j\omega} + \frac{k^*}{s+j\omega}}_{\substack{\text{terms related} \\ \text{to input poles} \\ \text{(particular sol'n)}}} + \underbrace{\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}}_{\substack{\text{terms related to} \\ \text{natural frequencies} \\ \text{(hmg. sol'n)}}}$$

Circuit stable $\Rightarrow$ hmg. sol'n $\to 0$ as $t \to \infty$

$\Rightarrow y(t) \to y_{ss}(t)$ sinusoidal steady-state response (i.e. the particular sol'n)

<u>$y_{ss}(t) = ?$</u>

#8 3 Mayıs 2016

$$y_{ss}(t) = k e^{j\omega t} + k^* e^{-j\omega t} = 2\,\mathrm{Re}\{k e^{j\omega t}\}$$

$$k = H(s)\,X(s)\,(s-j\omega)\Big|_{s=j\omega} = A\,\frac{\cos\phi \cdot s - \sin\phi \cdot \omega}{s+j\omega}\,H(s)\Big|_{s=j\omega}$$

$$= A\,\frac{\cos\phi \cdot j\omega - \sin\phi \cdot \omega}{j2\omega}\,H(j\omega)$$

$$= \frac{1}{2}A\,\underbrace{(\cos\phi + j\sin\phi)}_{e^{j\phi}}\,\underbrace{H(j\omega)}_{|H(j\omega)|\cdot e^{j\theta(\omega)}} \quad \text{where } \theta(\omega) = \underline{/H(j\omega)}$$

$$= \frac{1}{2}A\,|H(j\omega)|\,e^{j(\phi+\theta(\omega))}$$

Then
$$y_{ss}(t) = 2\,\mathrm{Re}\Big\{\underbrace{\frac{1}{2}A\,|H(j\omega)|\,e^{j[\phi+\theta(\omega)]}}_{k}\,e^{j\omega t}\Big\}$$

$$= A\,|H(j\omega)|\,\mathrm{Re}\{e^{j[\omega t+\phi+\theta(\omega)]}\}$$

$$= \boxed{A\,|H(j\omega)|\cos(\omega t + \phi + /H(j\omega))}$$

Conclusion:

$$A\cos(\omega t+\phi) \xrightarrow[\text{input}]{} \boxed{\begin{array}{c}\text{stable circuit}\\ TF: H(s)\end{array}} \xrightarrow[\text{output}]{} y_{ss}(t) = A\,|H(j\omega)|\cos(\omega t + \phi + /H(j\omega))$$

Example The impulse response of the circuit is

$$h(t) = 5000\left[2e^{-1000t}\cos 2000t - e^{-1000t}\sin 2000t\right]u(t)$$

Find the SSS response $y_{ss}(t)$ for

a) input $= 5\cos 1000t$

b) input $= 5\cos\left(3000t + \frac{\pi}{4}\right)$

Sol'n

$$H(s) = 5000\left\{2\,\frac{s+1000}{(s+1000)^2+2000^2} - \frac{2000}{(s+1000)^2+2000^2}\right\} = \frac{10^4 s}{s^2+2000s+5\times10^6}$$

a) $\omega = 1000\,\text{rad/sec} \Rightarrow$

$$H(s)\Big|_{s=j1000} = \frac{j10^7}{-10^6+j2\times10^6+5\times10^6} = \frac{j10}{4+j2} = 1+j2$$

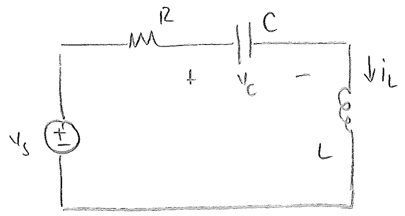
$$\Rightarrow |H(j1000)| = |1+j2| = \sqrt{5} \quad \& \quad \angle H(j1000) = \angle 1+j2 = \arctan 2$$

$$\Rightarrow \boxed{y_{ss}(t) = 5\sqrt{5}\cos(1000t + \arctan 2)}$$

b) $H(j3000) = \frac{45}{13} - j\frac{30}{13} \Rightarrow |H(j3000)| = \frac{15}{\sqrt{13}}$ & $\angle H(j3000) = -\arctan\frac{2}{3}$

$$\Rightarrow \boxed{y_{ss}(t) = \frac{75}{\sqrt{13}}\cos\left(3000t + \frac{\pi}{4} - \arctan\frac{2}{3}\right)}$$

Example [Transfer function & Diff. Eqn.]



input: v_s , output: i_L

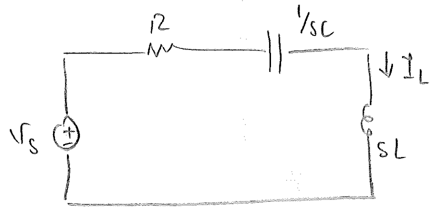
a) obtain the diff. eqn.

b) obtain the TF.

a) $$v_s = Ri_L + v_C(0^-) + \frac{1}{C}\int_{0^-}^{t} i_L(\tau)d\tau + LDi_L$$

$$\Rightarrow LD^2 i_L + RDi_L + \frac{1}{C} i_L = Dv_s \quad \Rightarrow \quad D^2 i_L + \frac{R}{L} Di_L + \frac{1}{LC} i_L = \frac{1}{L} D v_s \quad (1)$$

b)



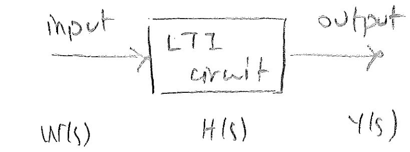
$$V_s = \left(R + \frac{1}{Cs} + Ls\right) I_L = \frac{LCs^2 + RCs + 1}{Cs} I_L$$

$$\Rightarrow \frac{I_L}{V_s} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \quad (2)$$

Note that (2) $\Rightarrow$ $s^2 I_L + \frac{R}{L} s I_L + \frac{1}{LC} I_L = \frac{1}{L} s V_s \quad (3)$

Compare (1) & (3) !

FREQUENCY RESPONSE



$H(s)$: transfer function

$Y(s) = H(s)\,W(s)$

If the circuit is stable then

$$A\cos(\omega t+\phi) \xrightarrow{\text{input}} \boxed{H(s)} \xrightarrow{\text{output}} A\,|H(j\omega)|\cos(\omega t+\phi+\angle H(j\omega)) \quad \text{(in steady state)}$$

That is,

→ output magnitude = input magnitude × $|H(j\omega)|$

→ output phase = input phase + $\angle H(j\omega)$

(→ output freq. = input freq.)

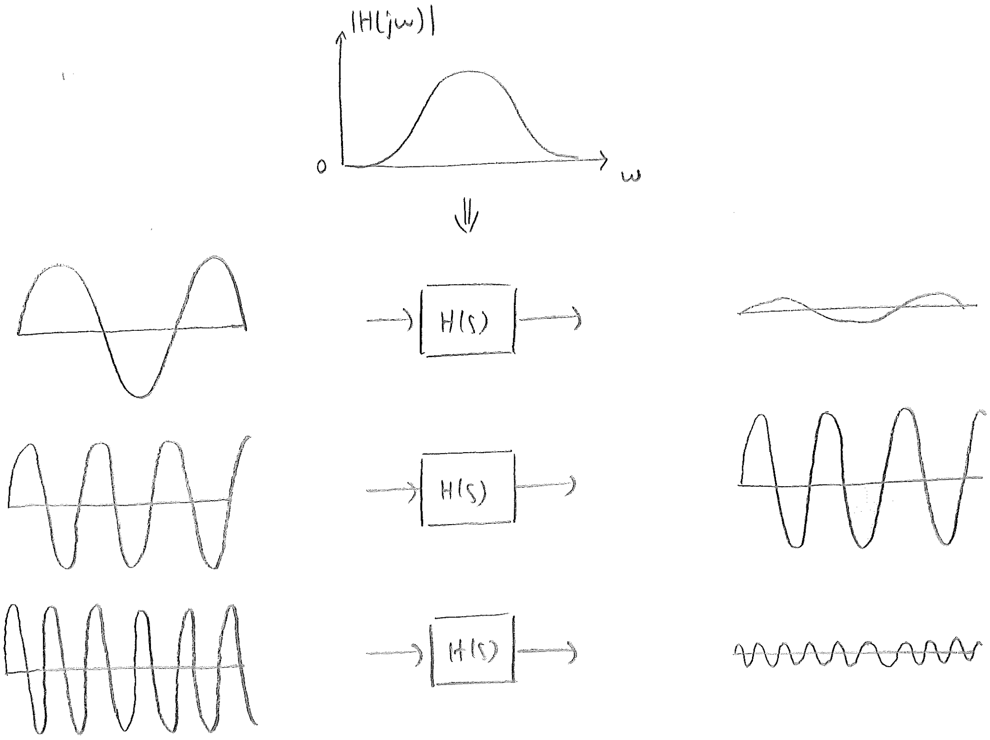
Gain function := $|H(j\omega)|$

Phase function := $\angle H(j\omega) = \theta(\omega)$

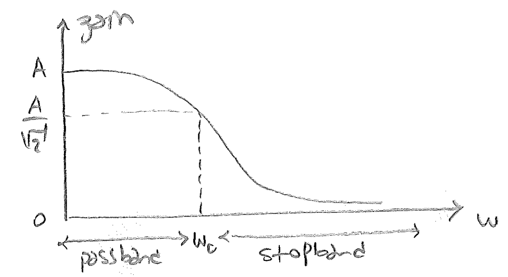
These two functions define the <u>frequency</u> <u>response</u> of the circuit.

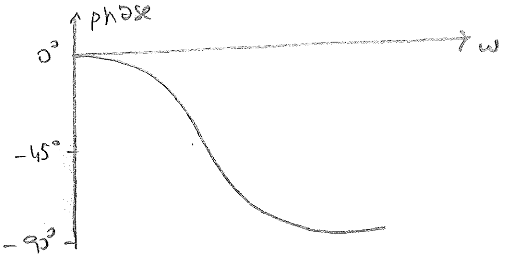
Taken together, the gain & phase functions show how the circuit modifies the input amplitude and phase angle to produce the output sinusoid in steady state.

<u>Ex</u>



Frequency Response Plots





Definitions

Stopband: The range of frequencies over which the output is significantly attenuated.

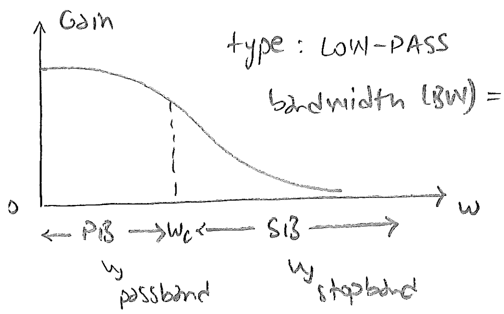
Passband: The frequency range over which there is little attenuation

Cutoff freq.: The frequency associated with the boundary between a passband and an adjacent stopband. Usually cutoff freq. w_c is the frequency satisfying

$$|H(jw_c)| = \frac{1}{\sqrt{2}} \max_{w} |H(jw)| \quad (\text{Half-power})$$

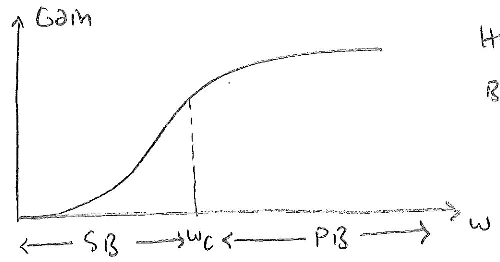
Bandwidth: The frequency range spanned by its passband.

Prototype gain characteristics



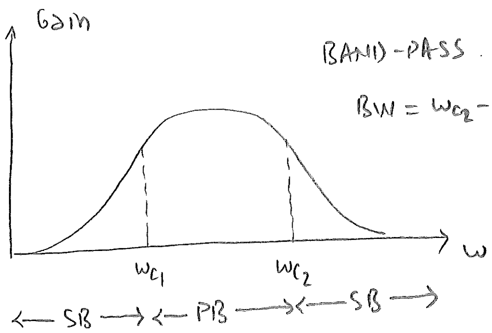
type: LOW-PASS

bandwidth (BW) = w_c



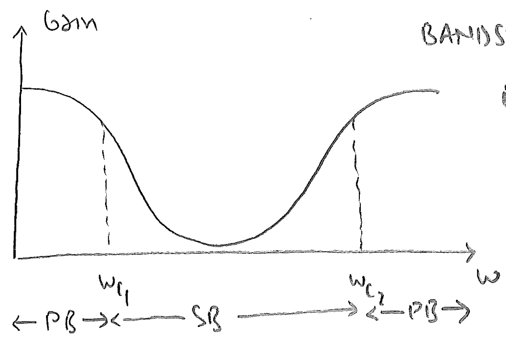
HIGH-PASS

BW = ∞



BAND-PASS

BW = $w_{c_2} - w_{c_1}$



BANDSTOP

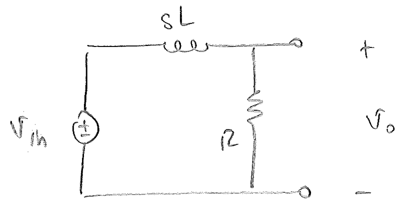
BW = ∞

#1 10 Mayıs 2016

First-order Circuit Frequency Response

Low-pass

Ex:



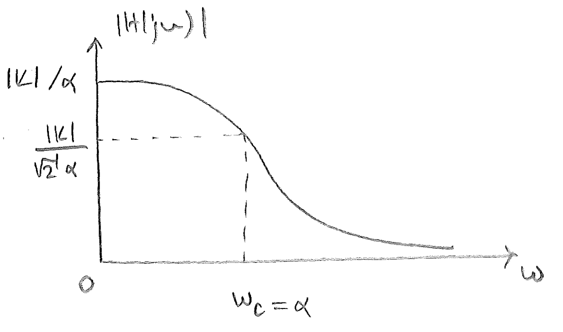
$$H(s) = \frac{R}{R+sL}$$

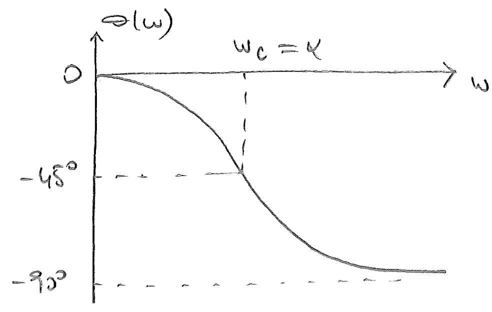
$$\Rightarrow H(j\omega) = \frac{R/L}{R/L + j\omega}$$

In general : $\boxed{H(j\omega) = \frac{K}{\alpha + j\omega}}$ $(\alpha > 0)$

Gain function: $$|H(j\omega)| = \frac{|K|}{\sqrt{\alpha^2+\omega^2}} = \frac{|K|/\alpha}{\sqrt{1+\left(\frac{\omega}{\alpha}\right)^2}}$$

Phase function: $$\theta(\omega) = \begin{cases} -\arctan(\omega/\alpha) & \text{for } K>0 \\ 180^\circ - \arctan(\omega/\alpha) & \text{for } K<0 \end{cases}$$





$(K>0)$

$$\max_{\omega} |H(j\omega)| = \frac{|K|}{\alpha}$$

$\omega_c = ?$ (Def: $|H(j\omega_c)| = \max_{\omega} |H(j\omega)| \cdot \frac{1}{\sqrt{2}}$)

$\Rightarrow \omega_c = \alpha$ (cutoff freq.)

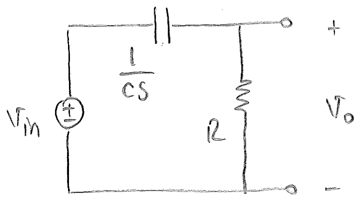
passband : $0 < \omega < \alpha$

stopband : $\omega > \alpha$

bandwidth : $B = \omega_c = \alpha$

Highpass

Ex :

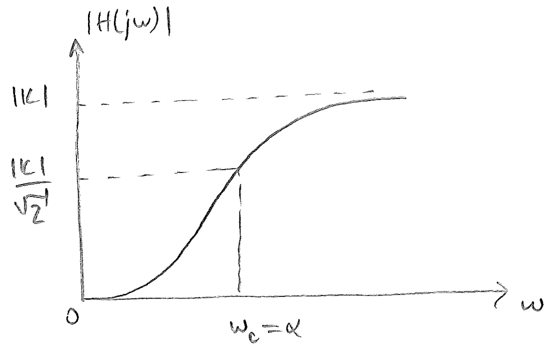


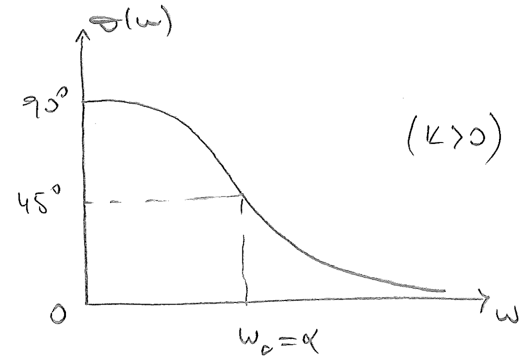
$$H(s) = \frac{R}{R + \frac{1}{Cs}} = \frac{RCs}{RCs + 1} = \frac{s}{s + \frac{1}{RC}}$$

In general :
$$\boxed{H(j\omega) = \frac{Kj\omega}{\alpha + j\omega}} \qquad (\alpha > 0)$$

Gain function :
$$|H(j\omega)| = \frac{|K|\,\omega}{\sqrt{\alpha^2 + \omega^2}} = \frac{|K|}{\sqrt{1 + \left(\frac{\alpha}{\omega}\right)^2}}$$

Phase function :
$$\theta(\omega) = \begin{cases} 90^\circ - \arctan(\omega/\alpha) & \text{for } K > 0 \\ 270^\circ - \arctan(\omega/\alpha) & \text{for } K < 0 \end{cases}$$





$$\max_{\omega} |H(j\omega)| = |K|$$

$$\frac{|K|}{\sqrt{1 + \left(\frac{\alpha}{\omega_c}\right)^2}} = \frac{|K|}{\sqrt{2}}$$

$\Rightarrow \omega_c = \alpha$

passband : $\omega > \alpha$

stopband : $0 < \omega < \alpha$

bandwidth: $B = \infty$

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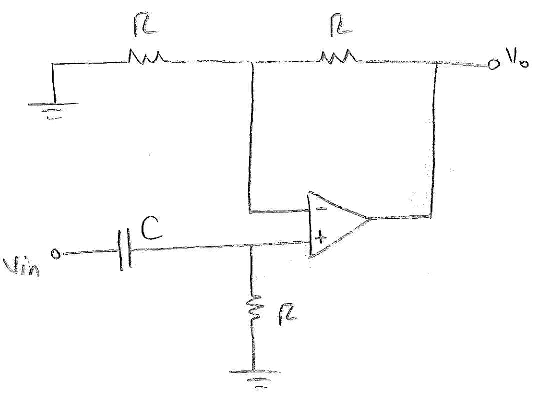
Example



a) Find TF $H(s) = \frac{V_o(s)}{V_{in}(s)}$

b) Type of filter?

c) Passband gain?

d) For $R = 10k\Omega$, choose C such that the cutoff freq. is $f_c = 500Hz$.

e) Phase shift at cutoff freq.?

Sol'n

$$V_+ = \frac{R}{R + \frac{1}{Cs}} V_{in} = \frac{RCs}{RCs+1} V_{in}$$

$$V_- = \frac{V_o}{2}$$

$$V_+ = V_- \Rightarrow \frac{RCs}{RCs+1} V_{in} = \frac{V_o}{2}$$

$$\Rightarrow \boxed{H(s) = \frac{2s}{s + \frac{1}{RC}}}$$

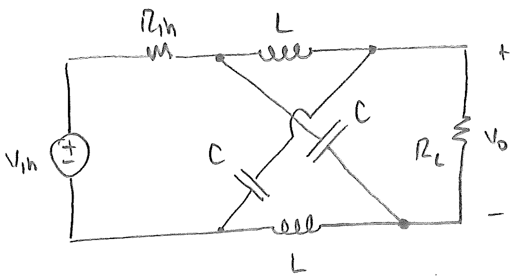
b) $H(j\omega) = \frac{2j\omega}{\frac{1}{RC} + j\omega} \Rightarrow$ highpass

c) passband gain $= \max_{\omega} |H(j\omega)| = 2$

d) $\omega_c = \frac{1}{RC} \Rightarrow 2\pi f_c = \frac{1}{RC} \Rightarrow 2\pi(500) = \frac{1}{10^4 C} \Rightarrow C = \frac{1}{10^7 \pi} F \cong \boxed{32nF}$

e) $\angle H(j\omega) \Big|_{\omega=\omega_c} = \angle \frac{2j\, ^1/_{RC}}{\frac{1}{RC} + j\frac{1}{RC}} = \angle \frac{j2}{1+j} = \angle j2 - \angle 1+j = \boxed{45^\circ}$

First-order All pass filter



condition: $R_L = \sqrt{L/C}$

Exercise Show that $H(j\omega) = A \frac{1 - j\,\omega/\omega_o}{1 + j\,\omega/\omega_o}$

with $A = \frac{R_L}{R_{in} + R_L}$ and $\omega_o = \frac{1}{\sqrt{LC}}$

Also, sketch $|H(j\omega)|$ vs. ω & $\angle H(j\omega)$ vs. ω characteristics.

Assignment Search about possible uses of all-pass filters.

Second-order bandpass filter

General form: $\boxed{H(s) = \dfrac{Ks}{s^2 + 2\alpha\omega_0 s + \omega_0^2}}$ $\omega_0 > 0$, $\alpha > 0$

(ω_0 is called the "center frequency" & α the "damping factor".)

$$H(j\omega) = \frac{Kj\omega}{-\omega^2 + 2\alpha\omega_0 j\omega + \omega_0^2} = \frac{K\cancel{j\omega}}{\cancel{j\omega}\omega_0 \left\{ 2\alpha + \frac{\omega_0}{j\omega} - \frac{\omega}{j\omega_0} \right\}}$$

$$= \frac{K/\omega_0}{2\alpha + j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

Gain function: $$|H(j\omega)| = \frac{|K|/\omega_0}{\left[(2\alpha)^2 + \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 \right]^{1/2}}$$

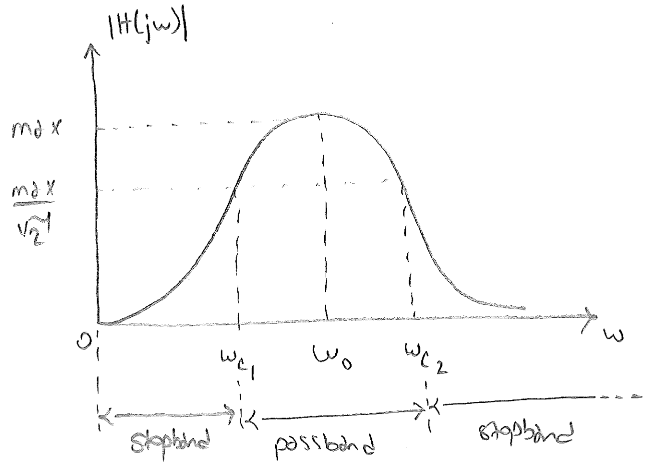
$$\Rightarrow |H(j\omega)|_{max} = |H(j\omega_0)| = \frac{|K|}{2\alpha\omega_0}$$

at low frequencies $\omega \ll \omega_0$ $|H(j\omega)| \to 0$

at high frequencies $\omega \gg \omega_0$ $|H(j\omega)| \to 0$

Hence,

$\frac{|K|}{2\alpha\omega_0}$ = max



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Cutoff frequencies ?

$$|H(j\omega_c)| = \frac{|H(j\omega)|_{max}}{\sqrt{2}} \Rightarrow \frac{|K|/\omega_0}{\left[4\alpha^2 + \left(\frac{\omega_c}{\omega_0} - \frac{\omega_0}{\omega_c}\right)^2\right]^{1/2}} = \frac{|K|}{2\alpha\omega_0} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow 8\alpha^2 = 4\alpha^2 + \left(\frac{\omega_c}{\omega_0} - \frac{\omega_0}{\omega_c}\right)^2 \Rightarrow (2\alpha)^2 = \left(\frac{\omega_c}{\omega_0} - \frac{\omega_0}{\omega_c}\right)^2 \quad (1)$$

$$(1) \Rightarrow \boxed{\begin{aligned} \omega_{c_1} &= \omega_0\left[\sqrt{1+\alpha^2} - \alpha\right] \\ \omega_{c_2} &= \omega_0\left[\sqrt{1+\alpha^2} + \alpha\right] \end{aligned}} \Rightarrow \omega_{c_1}\omega_{c_2} = \omega_0^2$$

i.e. the center freq. ω_0 is the geometric mean of cutoff freq. $\omega_{c_1}, \omega_{c_2}$

Bandwidth $\boxed{B = \omega_{c_2} - \omega_{c_1} = 2\alpha\omega_0}$

Quality factor $\boxed{Q := \frac{\omega_0}{B} = \frac{1}{2\alpha}}$

Quality factor is an indicator of how sharp the peak of the gain response is.

$Q > 1$: "narrowband" filter & $Q < 1$: "broadband" filter.

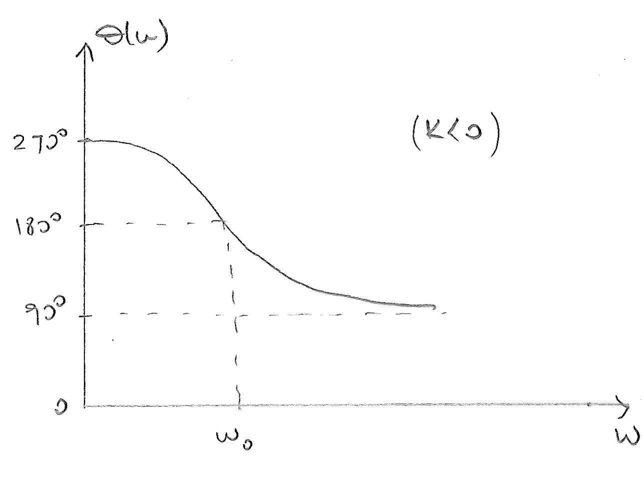
Phase function:

$$H(j\omega) = \frac{K/\omega_0}{2\alpha + j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

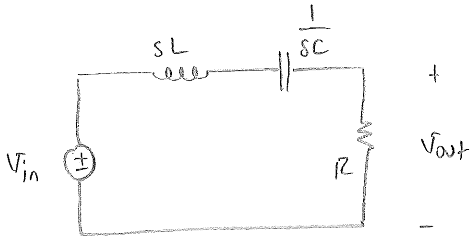
at low freq. $\omega \ll \omega_0$ $\quad \theta(\omega) \to \angle K + 90°$

at high freq $\omega \gg \omega_0$ $\quad \theta(\omega) \to \angle K - 90°$

$$\left[\begin{aligned} \angle K &= 0 \quad \text{for} \quad K > 0 \\ \angle K &= 180° \quad \text{for} \quad K < 0 \end{aligned}\right]$$



Example



$$H(s) = \frac{R}{sL + \frac{1}{sC} + R} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\left(\text{prototype} \quad H(s) = \frac{Ks}{s^2 + 2\alpha\omega_0 s + \omega_0^2} \right)$$

$$\Rightarrow K = \frac{R}{L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

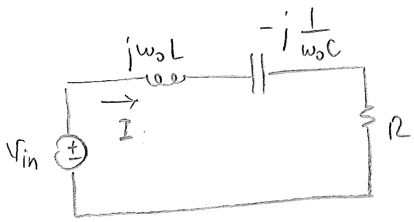
$$2\alpha\omega_0 = \frac{R}{L} \Rightarrow \alpha = \frac{R}{2\omega_0 L} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$|H(j\omega)|_{max} = \frac{|K|}{2\alpha\omega_0} = 1, \quad B = 2\alpha\omega_0 = \frac{R}{L}, \quad Q = \frac{\omega_0}{B} = \frac{1}{2\alpha} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

[Exercise : cutoff frequencies in terms of R, L, C ?]

Physical meaning of Q ?

Consider the circuit at $\omega = \omega_0$



total average stored energy $E_s = E_L + E_C$

$$E_L = \frac{1}{2}L I_{eff}^2, \quad E_C = \frac{1}{2}C V_{C,eff}^2 = \frac{1}{2}C\left[I_{eff}\frac{1}{\omega_0 C}\right]^2$$

$v_{in}(t) = A\cos(\omega_0 t + \phi), \quad \omega_0 = \frac{1}{\sqrt{LC}}$

$$\Rightarrow E_s = \frac{1}{2}L I_{eff}^2 + \frac{1}{2}C I_{eff}^2 \left(\frac{\sqrt{LC}}{C}\right)^2 = L I_{eff}^2$$

energy dissipated on R during one period: $E_d = P_{av} \cdot T = R I_{eff}^2 \frac{2\pi}{\omega_0} = 2\pi I_{eff}^2 R\sqrt{LC}$

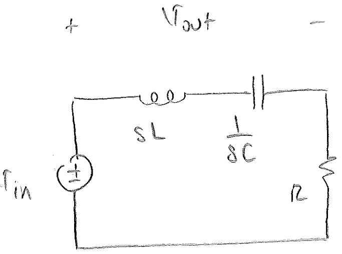
Now, observe that $\frac{E_s}{E_d} = \frac{L \cancel{I_{eff}^2}}{2\pi \cancel{I_{eff}^2} R\sqrt{LC}} = \frac{1}{2\pi} \cdot \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{Q}{2\pi}$

Therefore, $Q = 2\pi \times \frac{\text{Average stored energy}}{\text{Energy dissipated in one cycle}}$ ☒

$$= \omega_0 \times \frac{\text{Average stored energy}}{\text{Average dissipated power}}$$

#7 10 May 15 2016

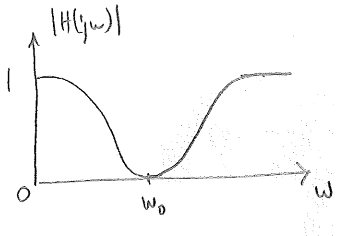
Example



$$H(s) = \frac{sL + \frac{1}{sC}}{sL + \frac{1}{sC} + R} = \frac{LCs^2 + 1}{LCs^2 + RCs + 1}$$

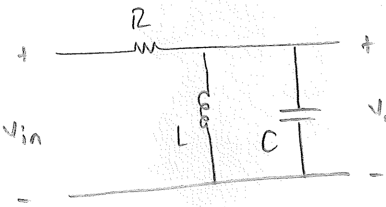
$$\Rightarrow H(s) = \frac{s^2 + 1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Let $w_0 := \frac{1}{\sqrt{LC}}$. Then $H(jw) = \dfrac{w_0^2 - w^2}{(w_0^2 - w^2) + j\frac{R}{L}w}$ $\Rightarrow$ $|H(jw_0)| = 0$, $|H(j0)| = 1$ & $|H(j\infty)| = 1$ } Bandstop filter



Exercise: Sketch the phase plot $\angle H(jw)$

Example



a) Find $H(s)$

b) Sketch the gain & phase plots

c) Compute bandwidth, quality factor, & cut-off freq. for $R = 1k\Omega$, $C = 1\mu F$, $L = 10mH$

Sol'n a)

$$H(s) = \frac{\dfrac{sL \cdot \frac{1}{sC}}{sL + \frac{1}{sC}}}{R + \dfrac{sL \cdot \frac{1}{sC}}{sL + \frac{1}{sC}}} = \frac{\frac{Ls}{LCs^2+1}}{R + \frac{Ls}{LCs^2+1}} = \frac{\frac{1}{RC}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \quad \text{(bandpass)}$$

$$H(jw) = \frac{j\frac{1}{RC}w}{-w^2 + j\frac{1}{RC}w + \frac{1}{LC}}$$

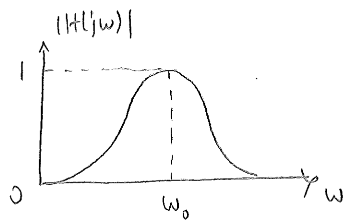
center freq. $w_0 = \frac{1}{\sqrt{LC}}$

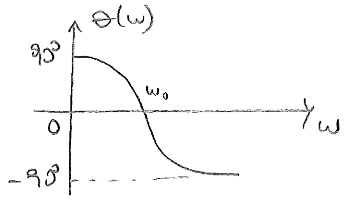
$w \to 0$, $|H(jw)| \to 0$, $\angle H(jw) \to 90°$

$w = w_0$, $|H(jw)| = 1$, $\angle H(jw) = 0$

$w \to \infty$, $|H(jw)| \to 0$, $\angle H(jw) \to -90°$

b)





c) $H(s) = \dfrac{Ks}{s^2 + 2\alpha w_0 s + w_0^2}$

$B = 2\alpha w_0 = \frac{1}{RC} = \frac{1}{10^3 \times 10^{-6}} = 10^3$ rad/sec

$w_0^2 = \frac{1}{LC} = \frac{1}{10^{-2}\, 10^{-6}} = 10^8 \Rightarrow w_0 = 10^4$ rad/sec

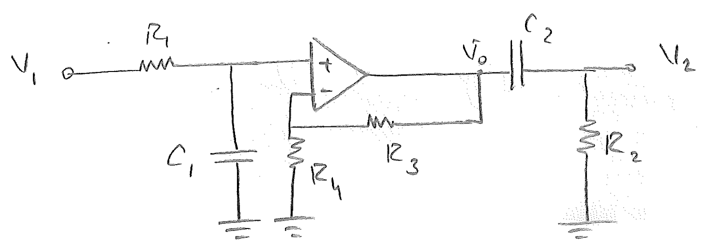
$2\alpha w_0 = 10^3 \Rightarrow \alpha = 0.05$

$Q = \frac{w_0}{B} = \frac{10^4}{10^3} = 10$

$w_{c1,2} = w_0\left(\sqrt{1+\alpha^2} \mp \alpha\right) = 10^4\left(\sqrt{1 + \frac{1}{400}} \mp \frac{1}{20}\right)$

$\Rightarrow w_{c1} \cong 9500$ rad/sec & $w_{c2} \cong 10{,}500$ rad/sec

Example :



a) Find transfer function $\frac{V_2(s)}{V_1(s)} = H(s)$. b) What type ?

c) Choose element values ($R_i > 1k\Omega$, $C_i < 10\mu F$) such that filter has (bandpass filter?) cut-off frequencies at 100 rad/sec & 2500 rad/sec and $|H(j\omega)|_{max} = 10$.

Sol'n : a)

$$\frac{V_+}{V_1} = \frac{\frac{1}{C_1 s}}{\frac{1}{C_1 s} + R_1} = \frac{\frac{1}{R_1C_1}}{s + \frac{1}{R_1C_1}} \quad , \quad \frac{V_-}{V_+} = 1 \quad , \quad \frac{V_o}{V_-} = \frac{R_4 + R_3}{R_4}$$

$$\& \quad \frac{V_2}{V_o} = \frac{R_2}{\frac{1}{C_2 s} + R_2} = \frac{s}{s + \frac{1}{R_2C_2}}$$

Now,
$$\frac{V_2}{V_1} = \frac{V_2}{V_o} \cdot \frac{V_o}{V_-} \cdot \frac{V_-}{V_+} \cdot \frac{V_+}{V_1}$$

$$\Rightarrow \frac{V_2}{V_1} = \underbrace{\left(\frac{R_3 + R_4}{R_4} \cdot \frac{1}{R_1C_1}\right)}_{K} \frac{s}{s^2 + \underbrace{\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right)}_{2\alpha\omega_o} s + \underbrace{\frac{1}{R_1C_1} \cdot \frac{1}{R_2C_2}}_{\omega_o^2}} = \frac{Ks}{s^2 + 2\alpha\omega_o s + \omega_o^2}$$

b) bandpass

c) $\omega_{o_{1,2}} = \sqrt{\omega_{c_1}\omega_{c_2}} \Rightarrow \omega_o = \sqrt{250000} = \boxed{500 \text{ rad/sec}}$

$\omega_{c_{1,2}} = \omega_o \left\{\sqrt{1+\alpha^2} \mp \alpha\right\} \Rightarrow \sqrt{1+\alpha^2} - \alpha = \frac{100}{500} = \frac{1}{5}$

$100 = 500\left\{\sqrt{1+\alpha^2} - \alpha\right\}$

$$\Rightarrow \left(\alpha + \frac{1}{5}\right)^2 = 1 + \alpha^2$$

$$\Rightarrow \alpha^2 + \frac{2}{5}\alpha + \frac{1}{25} = 1 + \alpha^2$$

$$\Rightarrow \boxed{\alpha = \frac{12}{5}}$$

$$s^2 + 2\alpha w_0 s + w_0^2 = 0 \Rightarrow s_{1,2} = -w_0 \left\{ \alpha \mp \sqrt{\alpha^2 - 1} \right\} = -\frac{1}{R_1 C_1}, -\frac{1}{R_2 C_2}$$

$$\Rightarrow s_{1,2} = -500 \left\{ \frac{12}{5} \mp \sqrt{\frac{119}{25}} \right\}$$

$$\Rightarrow s_{1,2} = \{-2290, -110\} \qquad \Rightarrow R_1 C_1 = \frac{1}{2290}$$

$$R_2 C_2 = \frac{1}{110}$$

Let $R_1 = R_2 = 10k\Omega$ $\Rightarrow C_1 = 4.4 \times 10^{-8} F \Rightarrow C_1 = 44nF$

$\Rightarrow C_2 = 9 \times 10^{-7} F \Rightarrow C_2 = 900nF$

$$|H(jw)|_{max} = |H(jw_0)| = \frac{K}{2\alpha w_0} \Rightarrow \frac{K}{2 \cdot \frac{12}{5} \cdot 500} = 10$$

$$\Rightarrow K = 24{,}000$$

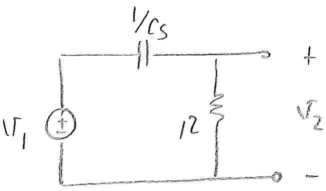
$$24 \times 10^3 = \frac{R_3 + R_4}{R_4} \cdot \frac{1}{R_1 C_1}$$

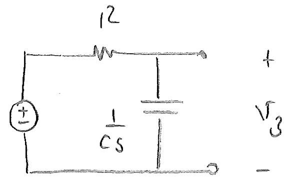
$$= \frac{R_3 + R_4}{R_4} \cdot 2290 \Rightarrow \frac{R_3 + R_4}{R_4} = \frac{24000}{2290} \cong 10.$$

$\Rightarrow R_3 \cong 90k\Omega$, $R_4 \cong 10k\Omega$ (roughly)

5 Mayıs 2018

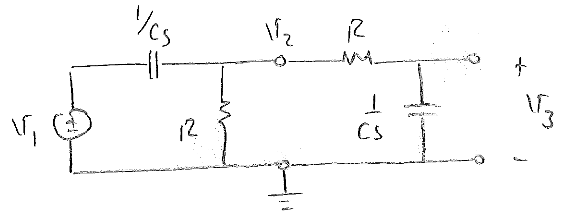
Example [cascade connection] Consider the two filters.





$$H_1(s) = \frac{V_2}{V_1} = \frac{R}{R+\frac{1}{Cs}} = \frac{RCs}{RCs+1} \quad , \quad H_2(s) = \frac{V_3}{V_2} = \frac{\frac{1}{Cs}}{R+\frac{1}{Cs}} = \frac{1}{RCs+1}$$

Now let's check the cascade connection



$$H(s) = \frac{V_3}{V_1} = ?$$

$$\frac{V_2 - V_1}{\frac{1}{Cs}} + \frac{V_2}{R} + \frac{V_2}{R+\frac{1}{Cs}} = 0 \Rightarrow \left\{ Cs + \frac{1}{R} + \frac{Cs}{RCs+1} \right\} V_2 = Cs\, V_1$$

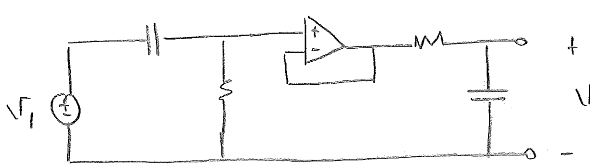
$$\Rightarrow \frac{R(RCs+1)Cs + RCs+1 + RCs}{R(RCs+1)} V_2 = Cs \cdot V_1 \Rightarrow \frac{V_2}{V_1} = \frac{(RCs+1)RCs}{(RCs+1)^2 + RCs} \neq H_1(s)$$

$$\Rightarrow H(s) = \frac{V_2}{V_1} \cdot \frac{V_3}{V_2} = \frac{V_2}{V_1} \cdot H_2(s) = \frac{RCs}{(RCs+1)^2 + RCs} \neq \frac{RCs}{(RCs+1)^2} = H_1(s) \cdot H_2(1)$$

$$\Rightarrow H(s) \neq H_1(s) H_2(s)$$

Question : How to achieve $H(s) = H_1(s) H_2(s)$?

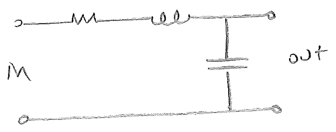
Answer : Use buffer.



#1 11 Mayıs 2015

Second-order low pass filter

ex :



Transfer function $H(s) = \dfrac{K}{s^2 + 2\alpha\omega_0 s + \omega_0^2}$

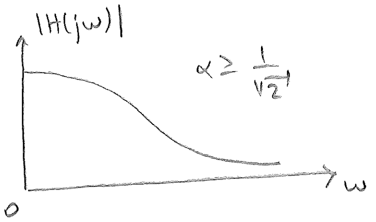
Frequency response $H(j\omega) = \dfrac{K}{\omega_0^2 - \omega^2 + j2\alpha\omega_0\omega}$ $\Rightarrow$ $|H(j\omega)| \to 0$ as $\omega \to \infty$

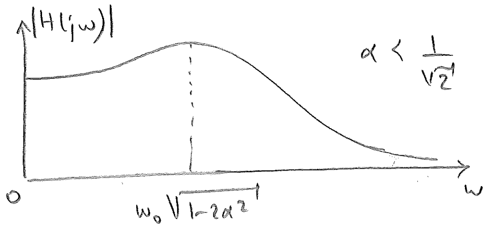
$|H(j\omega)|_{max} = ?$

For $\alpha < \frac{1}{\sqrt{2}}$ $\quad |H(j\omega)|_{max} = \dfrac{|H(0)|}{2\alpha\sqrt{1-\alpha^2}}$ occurs at $\omega_{max} = \omega_0\sqrt{1-2\alpha^2}$

For $\alpha \geq \frac{1}{\sqrt{2}}$ $\quad |H(j\omega)|_{max} = |H(0)|$

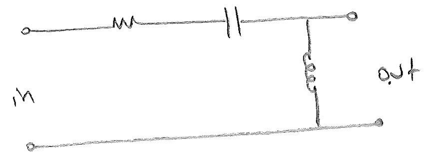
Note when $\alpha = \frac{1}{\sqrt{2}}$, $|H(j\omega_0)| = \dfrac{|H(0)|}{\sqrt{2}}$ $\Rightarrow$ $\omega_c = \omega_0$ for $\alpha = \frac{1}{\sqrt{2}}$.





Second-order high pass filter

ex :

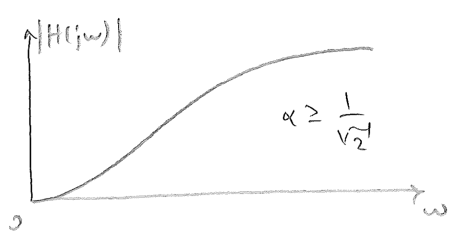


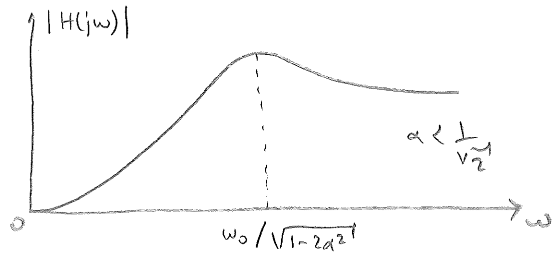
$H(s) = \dfrac{Ks^2}{s^2 + 2\alpha\omega_0 s + \omega_0^2}$

$H(j\omega) = \dfrac{-K\omega^2}{\omega_0^2 - \omega^2 + j2\alpha\omega_0\omega}$

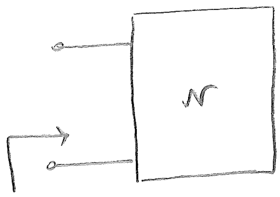
For $\alpha < \frac{1}{\sqrt{2}}$ $\quad |H(j\omega)|_{max} = \dfrac{|H(j\infty)|}{2\alpha\sqrt{1-\alpha^2}}$ occurs at $\omega_{max} = \dfrac{\omega_0}{\sqrt{1-2\alpha^2}}$

For $\alpha \geq \frac{1}{\sqrt{2}}$ $\quad |H(j\omega)|_{max} = |H(j\infty)|$. (when $\alpha = \frac{1}{\sqrt{2}}$, $\omega_c = \omega_0$)





RESONANT FREQUENCY

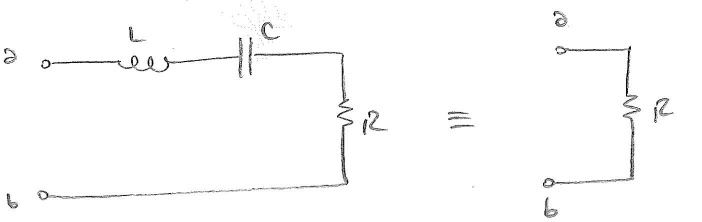


$Z(s)$

$Z(s)$: driving point impedance

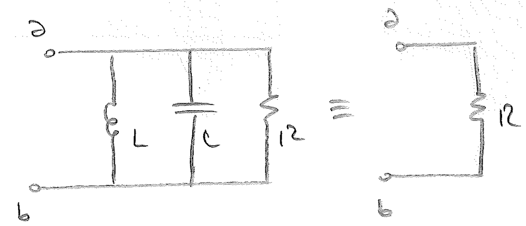
Definition [Resonant frequency] The frequency w_0 for which $Z(jw)$ is purely real $\left(\text{i.e., } Im\{Z(jw)\}=0\right)$ is called the resonant frequency of N.

Exercise : Show that at resonance



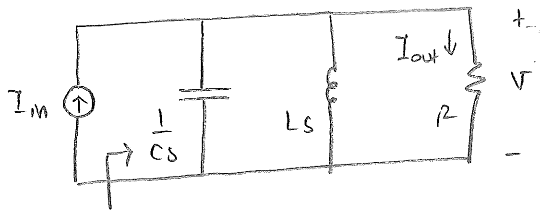
$w_0 = ?$

&



$w_0 = ?$

Example :



$$\left.\begin{aligned} I_{in}(s) &= \frac{V(s)}{Z(s)} \\ I_{out}(s) &= \frac{V(s)}{R} \end{aligned}\right\} \frac{I_{out}}{I_{in}} = H(s) = \frac{Z(s)}{R}$$

$$Z(s) = \frac{1}{Cs + \frac{1}{Ls} + \frac{1}{R}} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \quad \& \quad H(s) = \frac{\frac{1}{RC}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} = \frac{Ks}{s^2 + 2\alpha w_0 s + w_0^2}$$

Recall : $|H(jw)|_{max} = |H(jw_0)|$

$$Z(jw) = \frac{j\frac{w}{C}}{-w^2 + \frac{1}{LC} + j\frac{w}{RC}} \Rightarrow Z(jw)\Big|_{w=\frac{1}{\sqrt{LC}}} = R \quad \text{(purely real)}$$

$$\Rightarrow w_0 = \frac{1}{\sqrt{LC}} \quad \text{is the resonant freq.}$$

Remark : Note that $w_0 = \frac{1}{\sqrt{LC}}$ is both the resonant frequency for $Z(s)$ and the center frequency for $H(s)$. Hence, at resonance, the current through the resistance in a parallel RLC circuit is maximum.

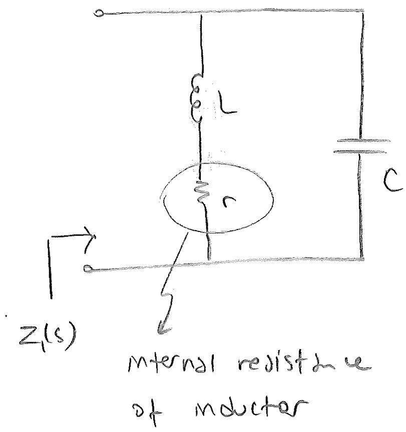
FINITE-Q INDUCTORS / CAPACITORS ("finite-q" means nonideal)

In second (or higher) order RLC filter constructions the LC pair in general appears either in parallel or series connection. An important (from a practical point of view) pair of questions:

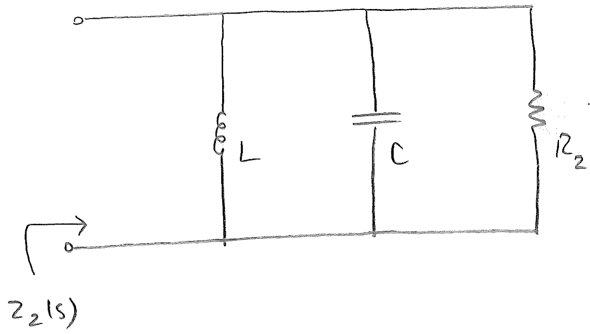
→ What is the effect of the internal resistance of the inductor in the parallel connection?

→ [The dual question] What is the effect of the internal conductance of the capacitor in the series connection?

Nonideal parallel LC:



Approximation:



$R_2 = ?$ To find R_2 let's obtain $Z(s)$ & $Z_2(s)$.

$$Z(s) = \frac{1}{Cs + \frac{1}{Ls+r}} = \frac{1}{C} \cdot \frac{s + r/L}{s^2 + \frac{r}{L}s + \frac{1}{LC}}$$

$$Z_2(s) = \frac{1}{Cs + \frac{1}{Ls} + \frac{1}{R_2}} = \frac{1}{C} \cdot \frac{s}{s^2 + \frac{1}{R_2C}s + \frac{1}{LC}} \quad \Bigg| \quad = \frac{1}{C} \cdot \frac{s}{s^2 + \frac{r}{L}s + \frac{1}{LC}}$$

Choose $R_2 = \frac{L}{rC}$

Remark: If we write $s^2 + \frac{r}{L}s + \frac{1}{LC} = s^2 + 2\alpha\omega_0 s + \omega_0^2$ and define $Q = \frac{1}{2\alpha} = \frac{\sqrt{L}}{r\sqrt{C}}$

Then $R_2 = Q^2 r$.

Now we have

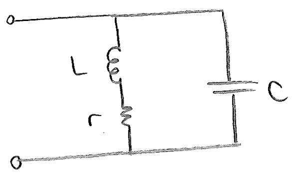
$$Z_1(j\omega) = \frac{1}{C} \cdot \frac{j\omega + r/L}{\frac{1}{LC} - \omega^2 + j\frac{r\omega}{L}} \quad \& \quad Z_2(j\omega) = \frac{1}{C} \cdot \frac{j\omega}{\frac{1}{LC} - \omega^2 + j\frac{r\omega}{L}}$$

when $\omega \cong \frac{1}{\sqrt{LC}}$ we can write

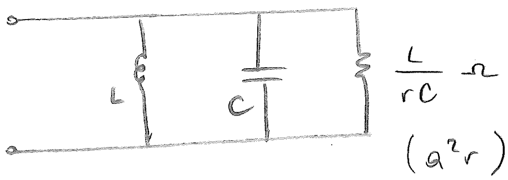
$$Z_1(j\omega) \cong \frac{L}{rC} - j\frac{\sqrt{L}}{\sqrt{C}} \quad \& \quad Z_2(j\omega) \cong \frac{L}{rC}$$

Note that $Z_2(j\omega) \cong Z_1(j\omega)$ if $\frac{L}{rC} \gg \frac{\sqrt{L}}{\sqrt{C}}$

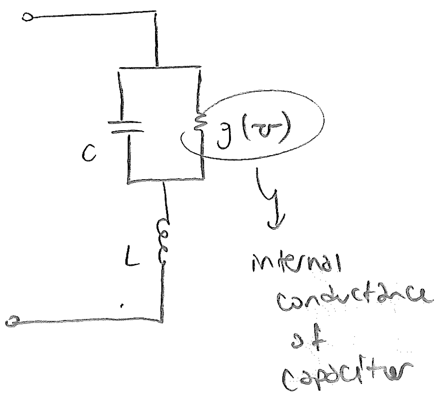
Conclusion : If $\frac{\sqrt{L}}{r\sqrt{C}} \gg 1$ $(a \gg 1)$ then around frequencies $\omega = \frac{1}{\sqrt{LC}}$



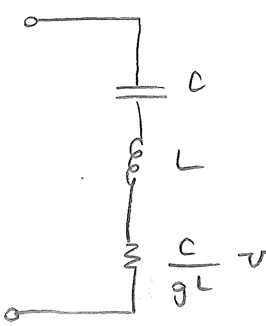
can be approximated as



Dual problem (exercise) : If $\frac{\sqrt{C}}{g\sqrt{L}} \gg 1$ then around frequencies $\omega = \frac{1}{\sqrt{LC}}$



can be approximated as



#5 11 Mayıs 2015

Scaling

a) Magnitude scaling : Purpose is to bring the circuit element values (resistance, inductance, capacitance) to practical ranges without altering the transfer function of the filter.

k_m : magnitude scaling factor

$$R \to k_m R \quad : \quad Z_R(j\omega) = R \Rightarrow \hat{Z}_R(j\omega) = k_m R = k_m Z_R(j\omega)$$

$$L \to k_m L \quad : \quad Z_L(j\omega) = j\omega L \Rightarrow \hat{Z}_L(j\omega) = j\omega k_m L = k_m Z_L(j\omega)$$

$$C \to C/k_m \quad : \quad Z_C(j\omega) = \frac{1}{j\omega C} \Rightarrow \hat{Z}_C(j\omega) = \frac{1}{j\omega C/k_m} = k_m Z_C(j\omega)$$

b) Frequency scaling : Purpose is to modify the frequency characteristics of a filter without altering its type and properties such as $|H(j\omega)|_{max}$, $|H(j\omega)|_{min}$, quality factor.

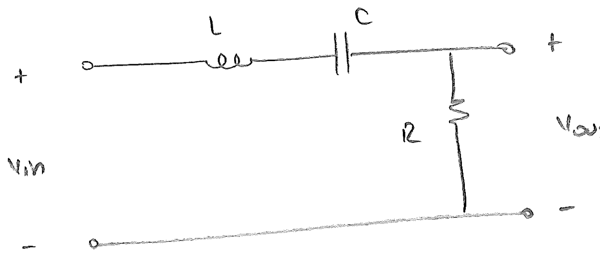
k_f : frequency scaling factor

$$R \to R \quad : \quad Z_R(j\omega) = R \Rightarrow \hat{Z}_R(j\omega) = R = Z_R\left(j\frac{\omega}{k_f}\right)$$

$$L \to L/k_f : Z_L(j\omega) = j\omega L \Rightarrow \hat{Z}_L(j\omega) = j\omega\frac{L}{k_f} = Z_L\left(j\frac{\omega}{k_f}\right)$$

$$C \to C/k_f : Z_C(j\omega) = \frac{1}{j\omega C} \Rightarrow \hat{Z}_C(j\omega) = \frac{1}{j\omega C/k_f} = Z_C\left(j\frac{\omega}{k_f}\right)$$

Example



$$H(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{ks}{s^2 + 2\alpha\omega_0 s + \omega_0^2}$$

$$\underbrace{L = \frac{1}{5}H, \; C = 5F, \; R = 1\Omega}_{\text{initial design parameters}} \Rightarrow H(s) = \frac{5s}{s^2 + 5s + 1} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} = 1\,rad/sec$$

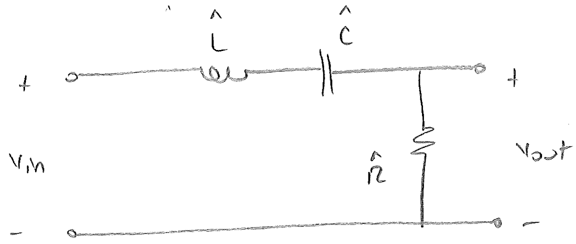
$$B = \frac{R}{L} = 5\,rad/sec$$

$$Q = \frac{\omega_0}{B} = \frac{1}{5}$$

Scale the circuit so that the new resistance value is $\hat{R} = 10k\Omega$ and new peak frequency is $\hat{\omega}_0 = 10^4$ rad/sec

#6 11 Mayıs 2015

Scaled circuit:



$$\hat{L} = \frac{k_m}{k_f} L \ , \quad \hat{C} = \frac{1}{k_m k_f} C \ , \quad \hat{R} = k_m R$$

$\hat{H}(s) = ?$

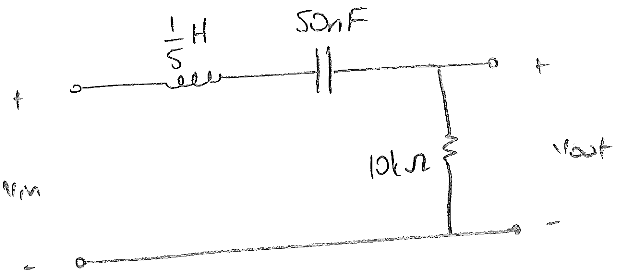
$$\hat{H}(s) = \frac{\frac{\hat{R}}{\hat{L}} s}{s^2 + \frac{\hat{R}}{\hat{L}} s + \frac{1}{\hat{L}\hat{C}}} = \frac{\frac{R}{L} k_f s}{s^2 + \frac{R}{L} k_f s + k_f^2 \frac{1}{LC}} = \frac{\frac{R}{L}\left(\frac{s}{k_f}\right)}{\left(\frac{s}{k_f}\right)^2 + \frac{R}{L}\left(\frac{s}{k_f}\right) + \frac{1}{LC}} = H(s/k_f)$$

$$\Rightarrow \frac{\frac{R}{L} k_f s}{s^2 + \frac{R}{L} k_f s + k_f^2 \underbrace{\frac{1}{LC}}_{w_o^2}} = \frac{\hat{k} s}{s^2 + 2\alpha \hat{w}_o s + \hat{w}_o^2}$$

$$k_f^2 w_o^2 = \hat{w}_o^2 \quad \Rightarrow \quad k_f = \frac{\hat{w}_o}{w_o} = \frac{10^4}{1} = 10^4$$

$$\hat{R} = k_m R \quad \Rightarrow \quad k_m = \frac{\hat{R}}{R} = \frac{10^4}{1} = 10^4$$

Finally, we have



$$\hat{H}(s) = \frac{5\times10^4 s}{s^2 + 5\times10^4 s + 10^8}$$

$$\hat{w}_o = 10^4 \text{ rad/sec}$$

$$\hat{B} = 5\times10^4 \text{ rad/sec}$$

$$\hat{a} = \frac{1}{5} = a$$

YPS III – 1 Given the system function $H(s) = \dfrac{30(s+1)}{s^2+8s+15}$

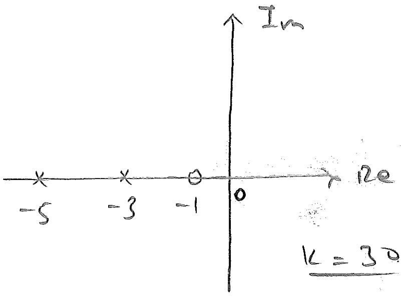
a) Plot the pole / zero diagram.

b) Sketch the magnitude and phase characteristics.

Sol'n:

$$H(s) = 30\,\frac{s+1}{(s+3)(s+5)} \Rightarrow \text{zeros} : \{-1\} \qquad \text{gain} : K = 30$$

$$\text{poles} : \{-3, -5\}$$



b) $H(j\omega) = 30\,\dfrac{1+j\omega}{15-\omega^2+j8\omega} \Rightarrow |H(j\omega)|_{\omega=0} = \dfrac{30}{15} = 2$

$$\lim_{\omega\to\infty} |H(j\omega)| = 0$$

$|H(j\omega)|_{max} = ?$

$$|H(j\omega)|^2 = 30^2\,\frac{1+\omega^2}{(9+\omega^2)(25+\omega^2)}$$

let $\sigma := \omega^2 \Rightarrow |H(j\omega)|^2 = 30^2\,\dfrac{\sigma+1}{\sigma^2+34\sigma+225}$

$$\Rightarrow \frac{d}{d\sigma}\{\cdots\} = 0 \Rightarrow (\sigma^2+34\sigma+225) - (\sigma+1)(2\sigma+34) = 0$$

$$\Rightarrow \sigma^2+34\sigma+225 - (2\sigma^2+36\sigma+34) = 0$$

$$\Rightarrow \sigma^2+2\sigma-191 = 0 \Rightarrow (\sigma+1)^2 = 192$$

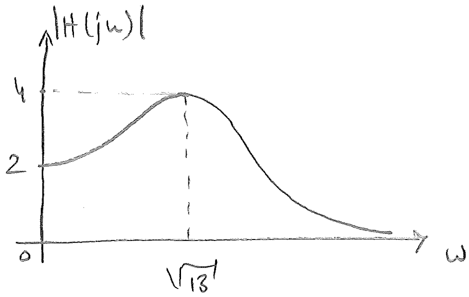
$$\Rightarrow \sigma = \sqrt{192}-1 \cong 13$$

$$\Rightarrow \omega_{max} \cong \sqrt{13} \qquad (\text{actual } \omega_{max} = 3.58)$$

$$\Rightarrow |H(j\omega)|_{max} \cong 30\cdot\left\{\frac{\overbrace{1+13}^{14}}{\underbrace{(9+13)}_{22}\underbrace{(25+13)}_{38}}\right\}^{1/2} \cong 30\cdot\frac{4}{5\cdot 6} = 4$$

$(\text{actual } |\cdot|_{max} = 3.88)$

Hence,



phase ?.

$$\angle H(j\omega) = \angle 1+j\omega - \angle 15-\omega^2+j8\omega$$

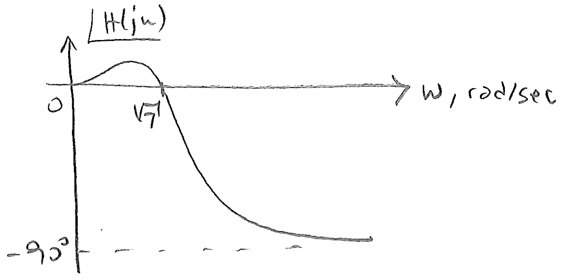
$$H(j\omega)\big|_{\omega=0} = 2 \Rightarrow \angle H(j\omega)\big|_{\omega=0} = 0$$

$$\lim_{\omega\to\infty} \angle H(j\omega) = \lim_{\omega\to\infty} \angle 1+j\omega - \lim_{\omega\to\infty} \angle 15-\omega^2+j8\omega = 90^\circ - 180^\circ = -90^\circ$$

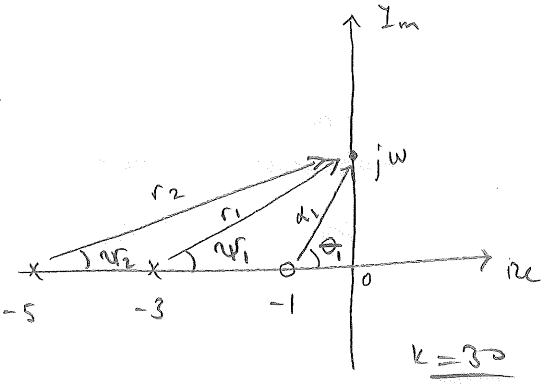
Note that

$$\angle H(j\omega)\big|_{\omega=\sqrt{7}\ \text{rad/sec}} = \angle 1+j\sqrt{7} - \angle 8(1+j\sqrt{7}) = 0$$

Hence,



Exercise: Show that

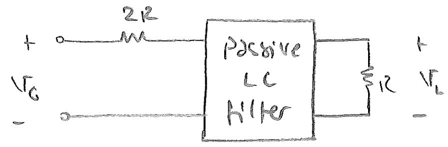


$$\angle H(j\omega) = \theta_1 - \{\psi_1 + \psi_2\}$$

$$|H(j\omega)| = 30 \cdot \frac{\alpha_1}{r_1 r_2}$$

#3 19 Mayıs 2013

YPS VII _5



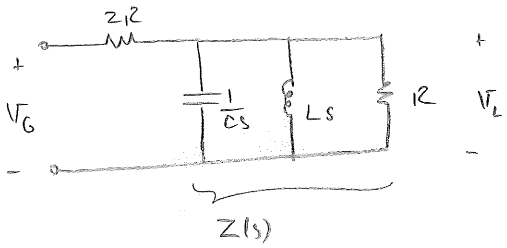
$H(s) = \frac{V_L(s)}{V_G(s)}$ is a 2^{nd} order bandpass TF.

a) Let $R = 1\Omega$. Provide two filter structures & determine the element values so that the peak freq. is 1 rad/sec & the half-power bandwidth is 0.5 rad/sec.

c) Scale the circuits so that the peak freq. is 4 kHz & $R = 2k\Omega$.

—o—

Filter 1:



$$Z(s) = \frac{1}{Cs + \frac{1}{Ls} + \frac{1}{R}} = \frac{LRs}{RLCs^2 + Ls + R} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$\Rightarrow H(s) = \frac{Z(s)}{Z(s)+2R} = \frac{\frac{1}{C}s}{\frac{1}{C}s + 2R\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}$$

$$\Rightarrow H(s) = \frac{\frac{1}{2RC}s}{s^2 + \frac{3}{2}\frac{1}{RC}s + \frac{1}{LC}} = \frac{ks}{s^2 + 2\alpha\omega_o s + \omega_o^2}$$

$$\Rightarrow \left.\begin{array}{l} \frac{1}{LC} = 1 \\ \frac{3}{2}\frac{1}{RC} = \frac{1}{2} \\ R = 1\Omega \end{array}\right\} \begin{array}{l} C = 3F \\ L = \frac{1}{3}H \end{array}$$

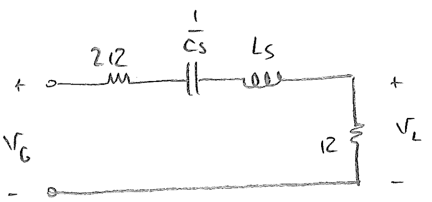
$$\left.\begin{array}{l} \hat{\omega}_o = 4000 \times 2\pi = 8000\pi \text{ rad/sec} \\ \omega_o = 1 \text{ rad/sec} \end{array}\right\} k_f = 8000\pi$$

$$\left.\begin{array}{l} \hat{R} = 2k\Omega \\ R = 1\Omega \end{array}\right\} k_m = 2000$$

$$\hat{L} = \frac{k_m}{k_f}L = \frac{2000}{8000\pi}\cdot\frac{1}{3} = \frac{2}{24\pi}H$$

$$\hat{C} = \frac{C}{k_m k_f} = \frac{3}{2000 \times 8000\pi} = \frac{3}{16\pi}\mu F$$

Filter 2:



$$\Rightarrow H(s) = \frac{R}{3R + \frac{1}{Cs} + Ls} = \frac{\frac{R}{L}s}{s^2 + 3\frac{R}{L}s + \frac{1}{LC}}$$

$$\Rightarrow \left.\begin{array}{l} \frac{1}{LC} = 1 \\ 3\frac{R}{L} = \frac{1}{2} \\ R = 1\Omega \end{array}\right\} \left.\begin{array}{l} L = 6H \\ C = \frac{1}{6}F \end{array}\right\} \begin{array}{l} \hat{L} = \frac{k_m}{k_f}L = \frac{3}{2\pi}H \\ \hat{C} = \frac{C}{k_m k_f} = \frac{1}{96\pi}\mu F \\ \hat{R} = 2k\Omega \end{array}$$

BODE PLOTS

Bode diagrams are separate graphs of the gain & phase (of some transfer function) versus log-frequency scale.

$$H(s) = \frac{Ks(s+\alpha_1)}{(s+\alpha_2)(s+\alpha_3)}$$

$$\Rightarrow H(j\omega) = \frac{Kj\omega(j\omega+\alpha_1)}{(j\omega+\alpha_2)(j\omega+\alpha_3)} = \frac{K\alpha_1}{\alpha_2\alpha_3} \cdot \frac{j\omega\left(1+j\frac{\omega}{\alpha_1}\right)}{\left(1+j\frac{\omega}{\alpha_2}\right)\left(1+j\frac{\omega}{\alpha_3}\right)}$$

$$\Rightarrow |H(j\omega)| = \frac{|K|\alpha_1}{\alpha_2 \cdot \alpha_3} \cdot \frac{\omega\left|1+j\frac{\omega}{\alpha_1}\right|}{\left|1+j\frac{\omega}{\alpha_2}\right| \cdot \left|1+j\frac{\omega}{\alpha_3}\right|}$$

$$\& \quad \underline{/H(j\omega)} = \underbrace{\underline{/K}}_{\substack{0^\circ \text{ or } 180^\circ \\ (K>0) \quad (K<0)}} + \underbrace{90^\circ}_{\substack{\text{due to the} \\ \text{zero at the origin}}} + \arctan\left(\frac{\omega}{\alpha_1}\right) - \arctan\left(\frac{\omega}{\alpha_2}\right) - \arctan\left(\frac{\omega}{\alpha_3}\right)$$

Note that it is easy to sketch the phase $\underline{/H(j\omega)}$: plot each term separately and then add. To make use of some simplicity the gain $|H(j\omega)|$ is expressed in decibels (dB).

$$|H(j\omega)|_{dB} := 20\log|H(j\omega)|$$

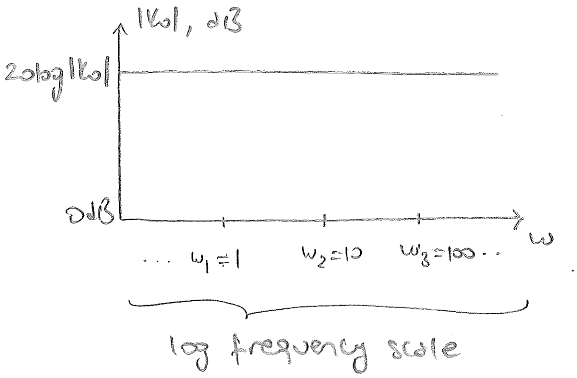
$$\Rightarrow |H(j\omega)|_{dB} = 20\log\left|\frac{K\alpha_1}{\alpha_2\alpha_3}\right| + 20\log\omega + 20\log\left|1+j\frac{\omega}{\alpha_1}\right| - 20\log\left|1+j\frac{\omega}{\alpha_2}\right| - 20\log\left|1+j\frac{\omega}{\alpha_3}\right|$$

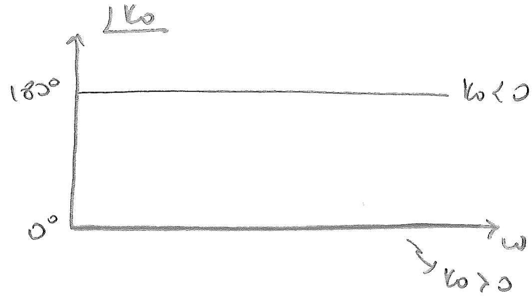
Let us now analyze each term:

1) <u>Constant term</u> $\quad \frac{K\alpha_1}{\alpha_2\alpha_3} =: K_o$

$$\left.\begin{array}{ll} 20\log|K_o| > 0 & \text{if } |K_o| > 1 \\ 20\log|K_o| < 0 & \text{if } |K_o| < 1 \end{array}\right\} \text{the gain contribution is constant (independent of } \omega)$$

$$\left.\begin{array}{ll} \underline{/K_o} = 0^\circ & \text{if } K_o > 0 \\ \underline{/K_o} = 180^\circ & \text{if } K_o < 0 \end{array}\right\} \text{the phase contribution is constant, too.}$$

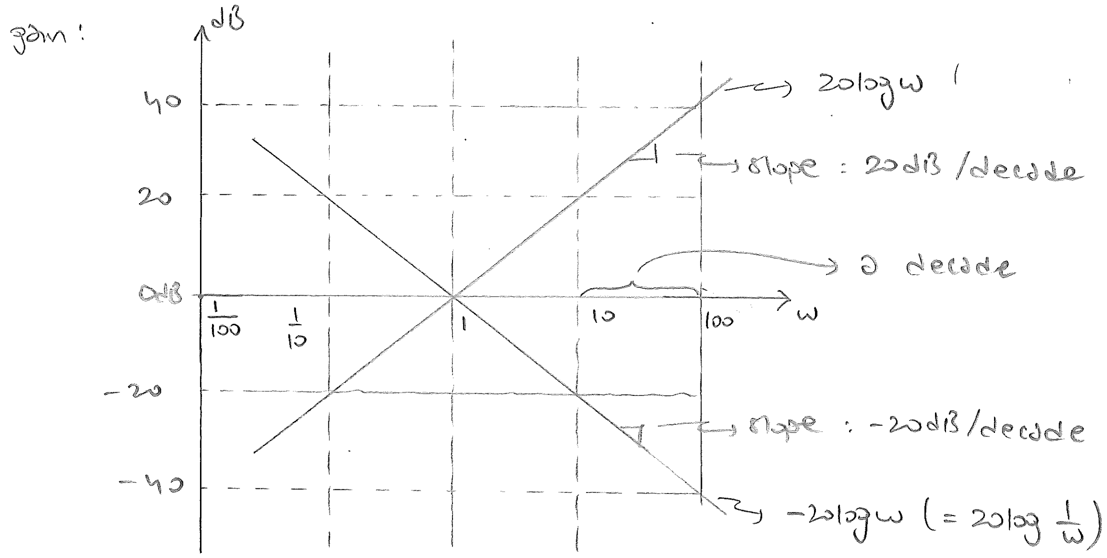




In log frequency scale the "axial" distance between two frequencies ω_1, ω_2 is proportional to $\log\frac{\omega_2}{\omega_1} = \log\omega_2 - \log\omega_1$ (as opposed to $\omega_2-\omega_1$, as in linear scale)

2) <u>jω term</u> : A single pole or zero at the origin contributes $\mp 20\log\omega$ to the gain and $\mp 90°$ to the phase. (+ for zero, − for pole)

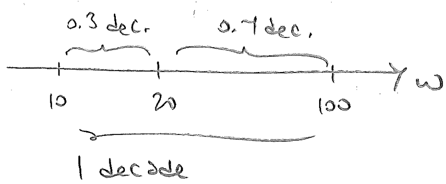
gain :



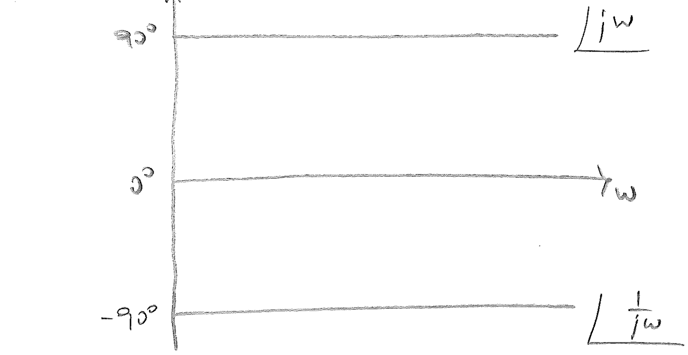
In log scale the distance between two frequencies ω_1, ω_2 is measured in decades:

$\text{dist}(\omega_1, \omega_2) = \left|\log\frac{\omega_1}{\omega_2}\right|$ decades

<u>ex</u>:



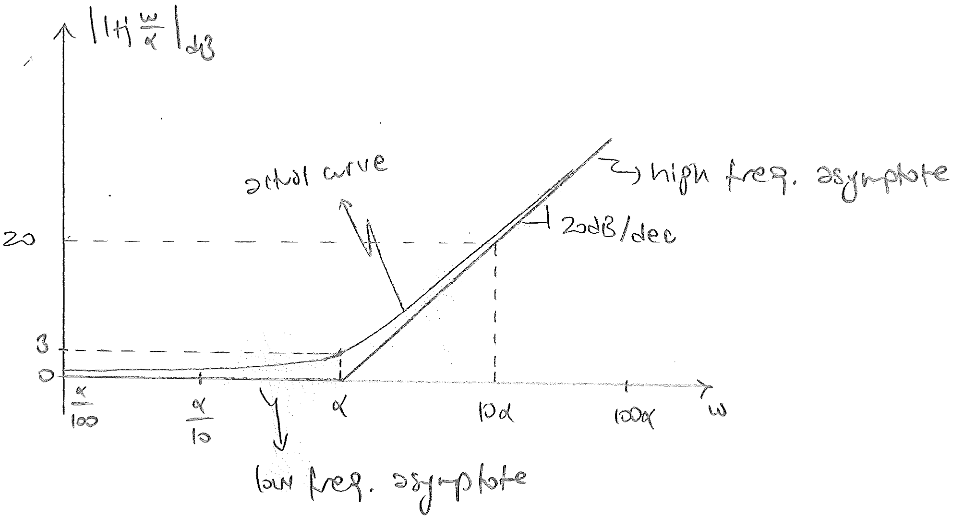
phase :



3) First order term $\left(1+j\frac{w}{\alpha}\right)^{\mp 1}$

Gain : $\left|1+j\frac{w}{\alpha}\right|_{dB}$ at low frequencies $w \ll \alpha$, $20\log\left|1+j\frac{w}{\alpha}\right| \cong 20\log 1 = 0$

at high frequencies $w \gg \alpha$, $20\log\left|1+j\frac{w}{\alpha}\right| \cong 20\log\frac{w}{\alpha} = 20\log w - 20\log\alpha$



Note that $20\log\left|\frac{1}{1+j\frac{w}{\alpha}}\right| = -20\log\left|1+j\frac{w}{\alpha}\right|$ & $20\log\left|1+j\frac{w}{\alpha}\right|_{w=\alpha} = 20\log\sqrt{2} = 10\log 2 = ?$

$\log 2$: $2^{10} = 1024 \cong 10^3 \Rightarrow 2 \cong 10^{3/10} \Rightarrow \log 2 \cong 0.3$

$\log 3$: $3^4 \cong 10^{1/2} \Rightarrow \log 3 \cong 0.5$

$\log 4$: $4 = 2^2 \Rightarrow \log 4 \cong 0.6$

$\log 5$: $5 = \frac{10}{2} \Rightarrow \log 5 = 1 - \log 2 \Rightarrow \log 5 \cong 0.7$

$\log 6$: $6 = 2\cdot 3 \Rightarrow \log 6 = \log 2 + \log 3 \Rightarrow \log 6 \cong 0.8$

$\log 7$: $7 \cong 50^{1/2} \Rightarrow 2\log 7 \cong 1 + \log 5 \Rightarrow \log 7 \cong 0.85$

$\log 8$: $8 = 2^3 \Rightarrow \log 8 \cong 0.9$

$\log 9$: $9 \cong 80^{1/2} \Rightarrow \log 9 \cong 0.95$

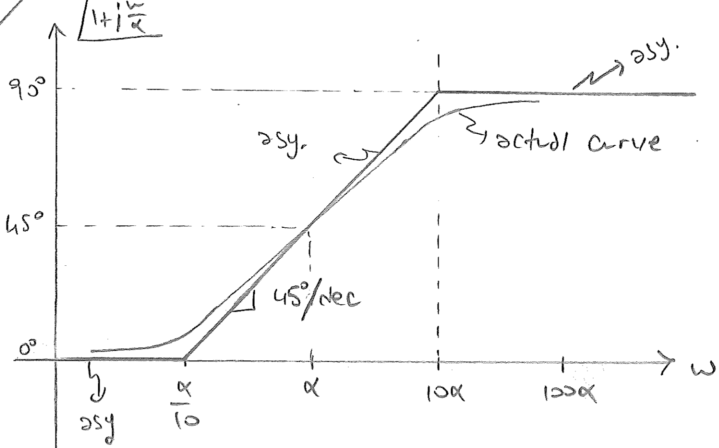
---

phase : $\angle 1+j\frac{w}{\alpha} = \arctan\frac{w}{\alpha}$

at low freq $w \ll \frac{\alpha}{10}$, $\angle 1+j\frac{w}{\alpha} \cong 0°$

at high freq $w \gg 10\alpha$, $\angle 1+j\frac{w}{\alpha} \cong 90°$

for $\frac{\alpha}{10} < w < 10\alpha$, $\angle 1+j\frac{w}{\alpha} \cong$ straight line



Note that $\angle\frac{1}{1+j\frac{w}{\alpha}} = -\angle 1+j\frac{w}{\alpha}$

SUMMARY (Bode Plots)

$$H(jw) = K \frac{jw\left(1+j\frac{w}{\alpha_1}\right)}{\left(1+j\frac{w}{\alpha_2}\right)\left(1+j\frac{w}{\alpha_3}\right)}$$

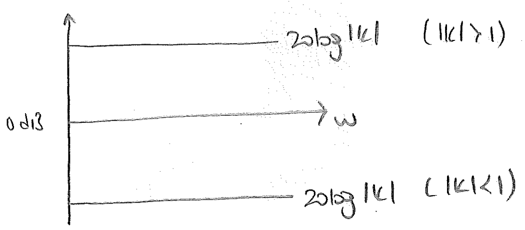
Gain $|H(jw)|_{dB} = 20\log|H(jw)| = 20\log|K| + 20\log w + 20\log\left|1+j\frac{w}{\alpha_1}\right| - 20\log\left|1+j\frac{w}{\alpha_2}\right| - 20\log\left|1+j\frac{w}{\alpha_3}\right|$

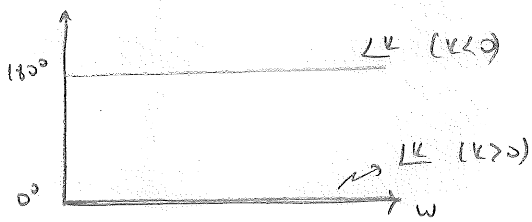
Phase $\angle H(jw) = \angle K + 90^\circ + \angle 1+j\frac{w}{\alpha_1} - \angle 1+j\frac{w}{\alpha_2} - \angle 1+j\frac{w}{\alpha_3}$

$\angle K$: 0° $(K>0)$, 180° $(K<0)$

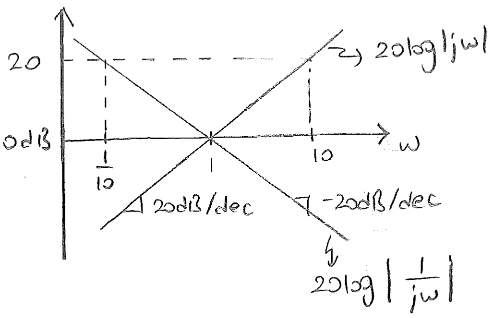
Approach: Sketch each term separately & add them up.

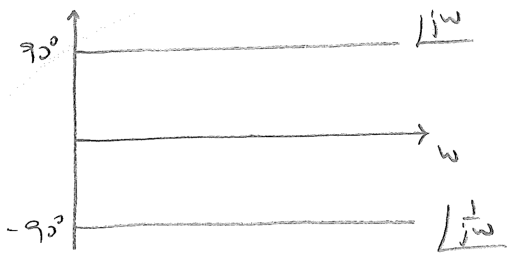
Constant term



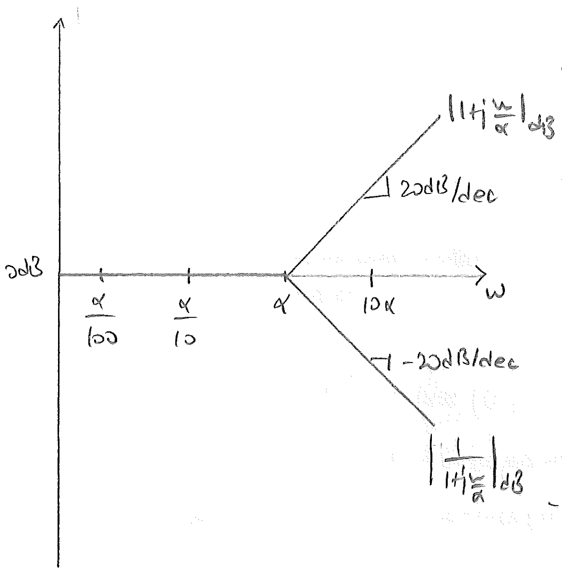


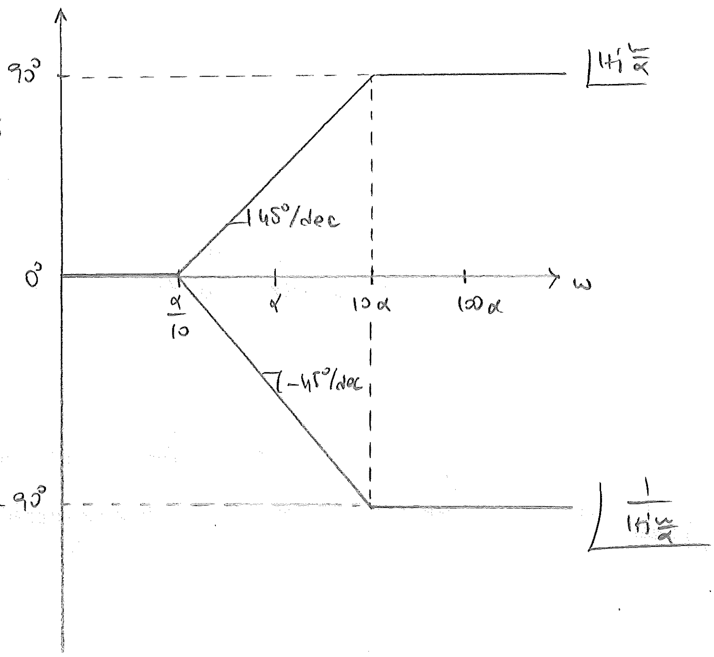
jw term





First-order term $\left(1+j\frac{w}{\alpha}\right)$





Example Consider $H(s) = \dfrac{12500\,(s+10)}{(s+50)\,(s+500)}$

a) Sketch the asymptotic gain plot.

b) Find the freq. ω_1 at which the high-freq gain falls below the DC gain.

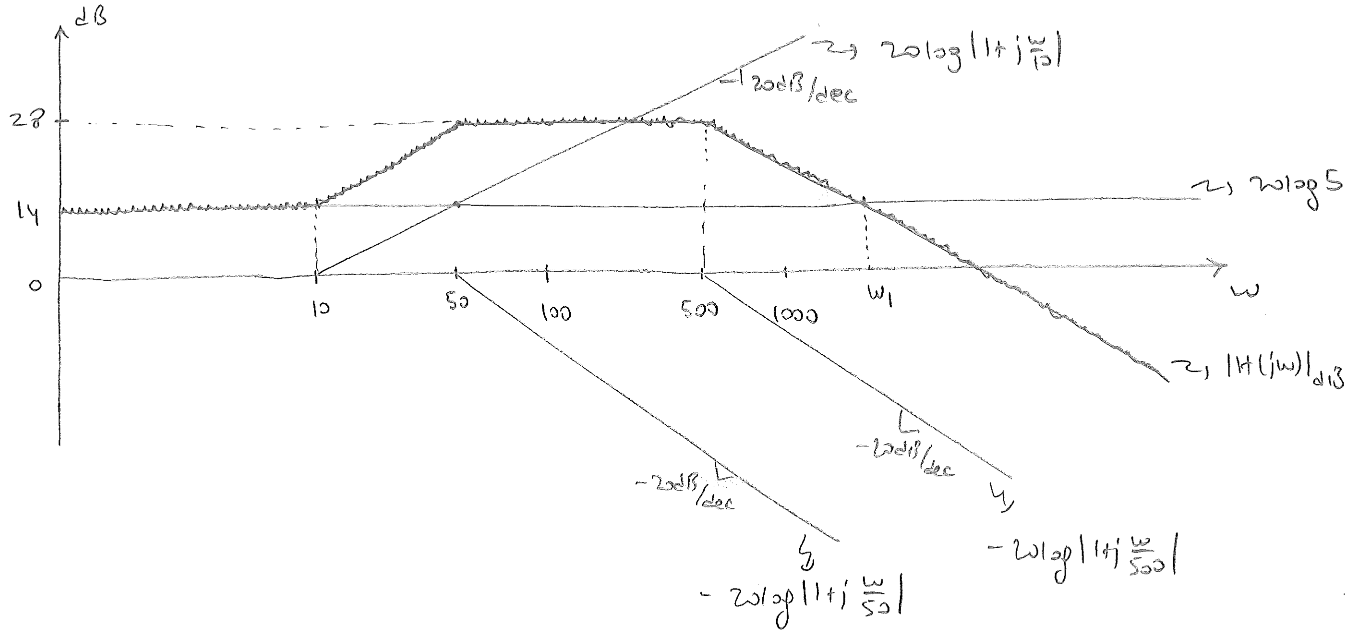
c) Sketch the asymptotic phase plot.

Sol'n $H(s) = \dfrac{12500\,(10)\,(1+s/10)}{(50)(500)\,(1+s/50)\,(1+s/500)}$

$$\Rightarrow \quad H(j\omega) = \frac{5\,\left(1+j\frac{\omega}{10}\right)}{\left(1+j\frac{\omega}{50}\right)\left(1+j\frac{\omega}{500}\right)}$$

$$\Rightarrow \quad |H(j\omega)|_{dB} = \underbrace{20\log 5}_{\cong 14} + 20\log\left|1+j\frac{\omega}{10}\right| - 20\log\left|1+j\frac{\omega}{50}\right| - 20\log\left|1+j\frac{\omega}{500}\right|$$

a) Sketch the individual terms and sum them up.



MATLAB: $\underbrace{\texttt{bode (12500*[1 10]}}_{\text{num. poly. coeff.}}\texttt{,}\ \underbrace{\texttt{[1 550 25000])}}_{\text{den. poly. coeff.}}$

$\left(H(s) = \dfrac{12500\,(s+10)}{s^2+550s+25000} \right.$

b) DC gain : $|H(j0)|_{dB} = 20\log 5 = 14$

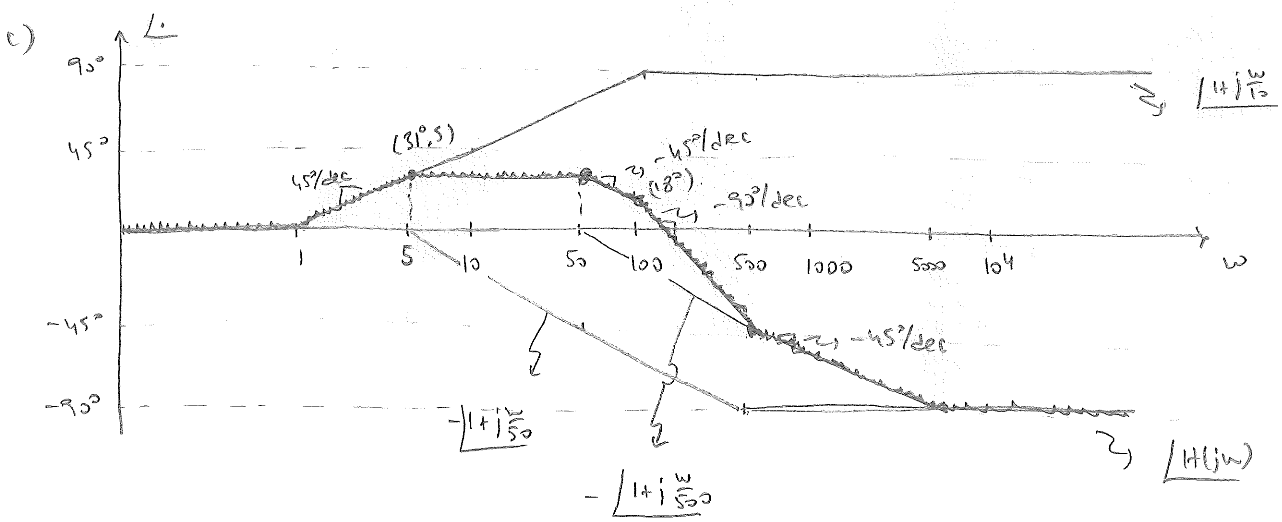
Plot suggests that the frequency we're looking for satisfies $w > 500$ rad/sec.

For $w > 500$ rad/sec $|H(jw)|_{dB} \cong 14 + 20\log\frac{w}{10} - 20\log\frac{w}{50} - 20\log\frac{w}{500}$

$$= 14 + 20\log\left[\frac{2500}{w}\right]$$

Hence $|H(jw)|_{dB} = |H(j0)|_{dB} \Rightarrow 14 + 20\log\frac{2500}{w} = 14 \Rightarrow \boxed{w_1 = 2500 \text{ rad/sec}}$

c)



Example $H(s) = \dfrac{10s^2}{(s+40)(s+200)}$

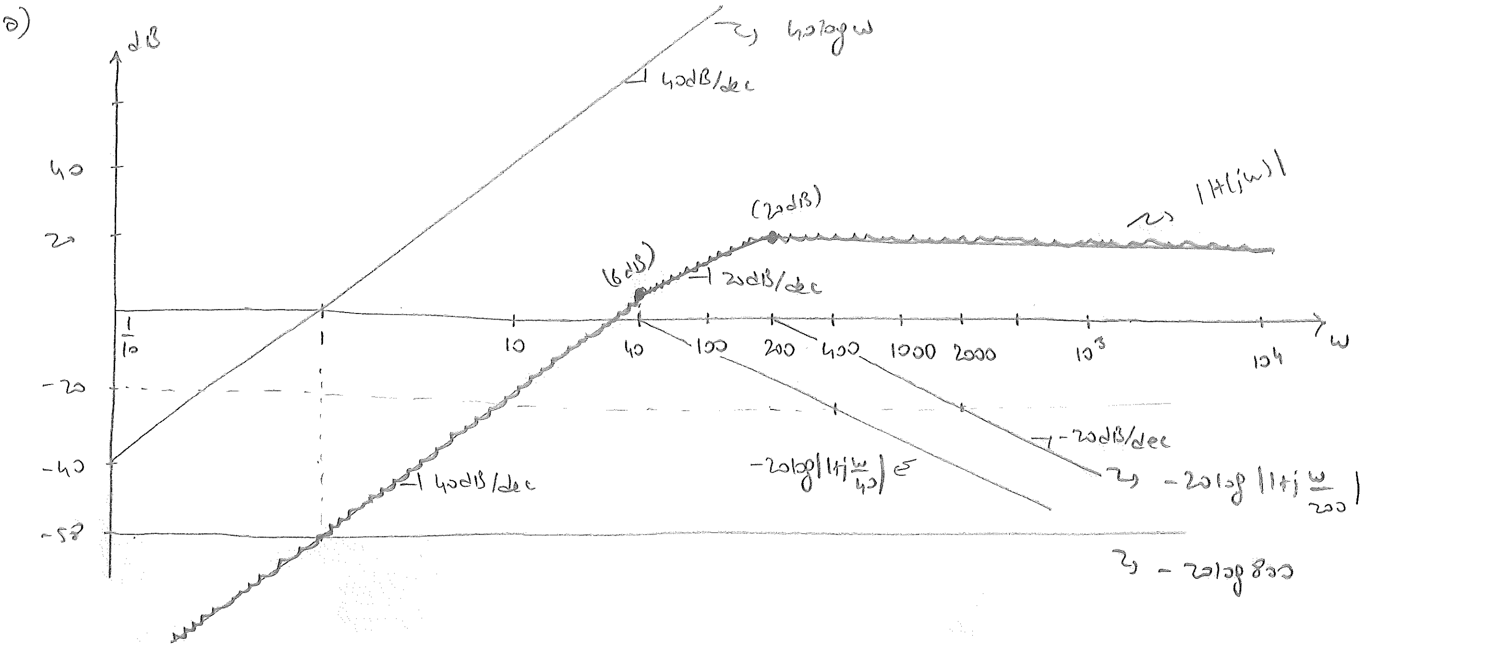
a) Plot the gain diagram

b) Find the freq. w_1 at which the low freq. gain falls 40dB below the passband gain.

Sol'n $H(s) = \dfrac{10s^2}{(40)(200)\left(1+\frac{s}{40}\right)\left(1+\frac{s}{200}\right)} \Rightarrow |H(jw)| = \dfrac{1}{800}\cdot\dfrac{w^2}{\left|1+j\frac{w}{40}\right|\cdot\left|1+j\frac{w}{200}\right|}$

$$\Rightarrow |H(jw)|_{dB} = \underbrace{-20\log 800}_{\cong -58dB} + 40\log w - 20\log\left|1+j\frac{w}{40}\right| - 20\log\left|1+j\frac{w}{200}\right|$$

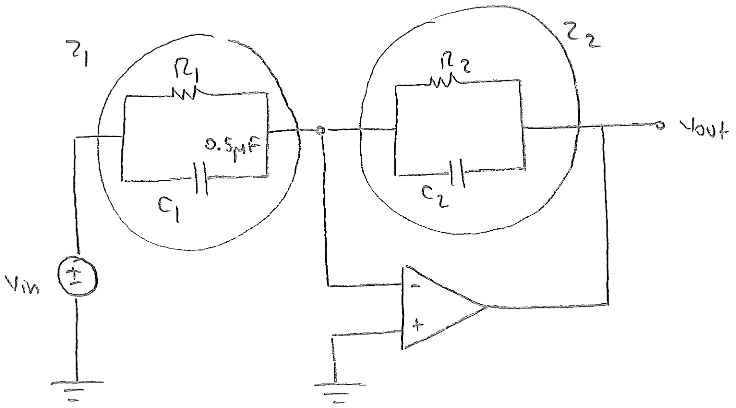
a)

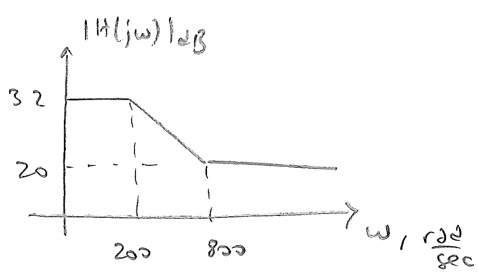


b) Exercise. Passband gain : $|H(j\infty)| = 20dB$.

Then w_1 satisfies $|H(jw_1)| = |H(j\infty)| - 40 = -20dB$. ($w_1 \cong 10$ rad/sec)

<u>Example</u> Design the circuit below to have the following asymptotic Bode plot.





<u>Sol'n</u> $\frac{V_{in}}{Z_1} + \frac{V_{out}}{Z_2} = 0 \Rightarrow H(s) = -\frac{Z_2(s)}{Z_1(s)} \Rightarrow H(s) = -\frac{R_2 \,//\, \frac{1}{C_2 s}}{R_1 \,//\, \frac{1}{C_1 s}}$

$$\Rightarrow H(s) = -\frac{\frac{R_2}{R_2C_2s+1}}{\frac{R_1}{R_1C_1s+1}} = -\frac{R_2}{R_1}\cdot\frac{R_1C_1s+1}{R_2C_2s+1}$$

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$$\Rightarrow |H(j\omega)|_{dB} = \underbrace{20\log\left(\frac{R_2}{R_4}\right)}_{32} + \underbrace{20\log\left|1+j\frac{\omega}{1/R_4C_4}\right|}_{20\log\left|1+j\frac{\omega}{800}\right|} - \underbrace{20\log\left|1+j\frac{\omega}{1/R_2C_2}\right|}_{-20\log\left|1+j\frac{\omega}{200}\right|}$$

$$\Rightarrow \frac{1}{R_4C_4} = 800 \quad \Rightarrow \quad R_4 = \frac{1}{800 \times \frac{1}{2} \times 10^{-6}} = \boxed{2.5\,k\Omega}$$

$$\Rightarrow 20\log\left(\frac{R_2}{R_4}\right) = 32 \quad \Rightarrow \quad \frac{R_2}{R_4} = 10^{1.6} \cong 40 \quad \Rightarrow \quad \boxed{R_2 = 100\,k\Omega}$$

$$\Rightarrow \frac{1}{R_2C_2} = 200 \quad \Rightarrow \quad C_2 = \frac{1}{200 \times 10^5} = \boxed{50\,nF}$$

Second-order (quadratic) terms

Complex poles & zeros occur in conjugate pairs that appear as second-order terms of the form

$$s^2 + 2\alpha \omega_0 s + \omega_0^2 \quad \text{where} \quad \alpha < 1$$

Remark For $\alpha \geq 1$ the roots become real and we can treat them as two separate first-order terms.

Example $H(s) = \dfrac{3(s+3)}{s^2+s+2} = \dfrac{9}{2} \cdot \dfrac{1+\frac{s}{3}}{1+\frac{1}{2}s+\frac{1}{2}s^2}$

↳ second-order term with complex roots

The second-order terms introduce gain & phase terms of the below form:

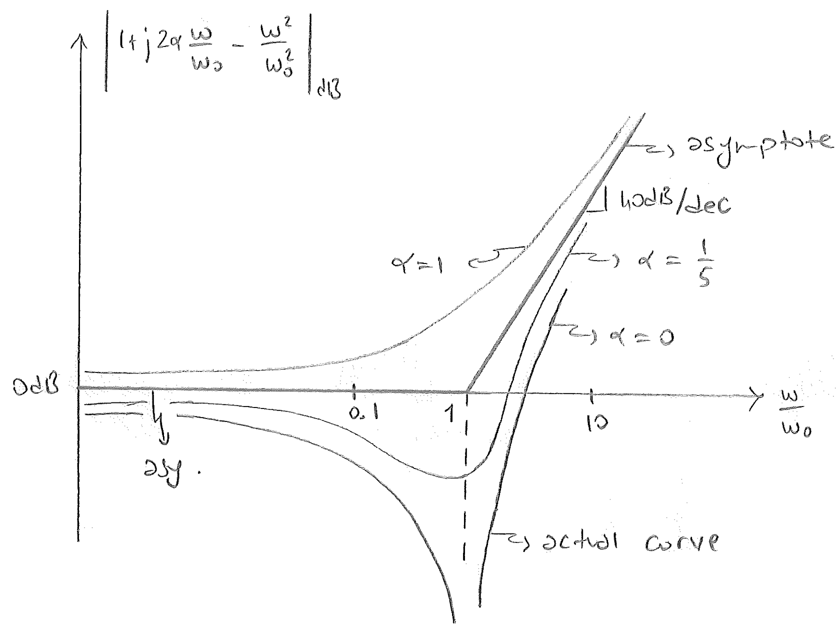
Gain : $$\left| 1 + \frac{2\alpha}{\omega_0} s + \left(\frac{s}{\omega_0}\right)^2 \right|^{\mp 1}_{s=j\omega,\, dB} = \mp 20 \log \left| 1 + j 2\alpha \frac{\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2} \right|$$

Phase : $$\angle \left(1 + \frac{2\alpha}{\omega_0} s + \left(\frac{s}{\omega_0}\right)^2\right)^{\mp 1}_{s=j\omega} = \mp \angle\, 1 + j 2\alpha \frac{\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2}$$

How to approximate these functions ?

Gain at low frequencies ($\omega \ll \omega_0$) $|\cdot|_{dB} \to 0$

Gain at high frequencies ($\omega \gg \omega_0$) $|\cdot|_{dB} \to \mp 20\log\left(\frac{\omega}{\omega_0}\right)^2 = \mp 40\log\left(\frac{\omega}{\omega_0}\right)$



Generally speaking the asymptotic gain is within $\mp 3dB$ of the actual gain for $0.3 \leq \alpha \leq 0.7$.

— o —

Phase at low frequencies ($\omega \ll \frac{\omega_0}{10}$) $\angle \to 0°$

Phase at high frequencies ($\omega \gg 10\omega_0$) $\angle \to \mp 180°$

For $\frac{\omega_0}{10} < \omega < 10\omega_0$ we use the straight-line approximation (slope $= \mp 90°/dec$)