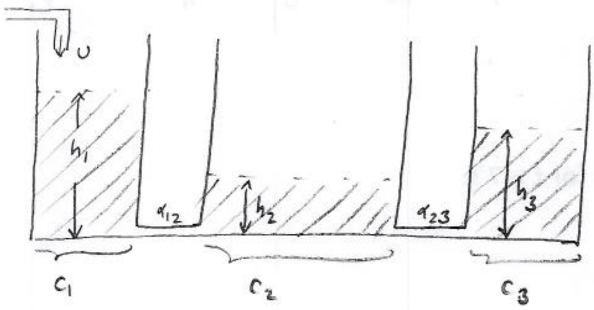
EE 502 LINEAR SYSTEMS II

Example: Water reservoir system: rate of water flow through links is proportional to the relative pressure.



c_1, c_2, c_3: areas of tanks (m^2)

h_1, h_2, h_3: height of water (m)

u: rate of water inflow (m^3/s) input

y: total volume of water (m^3) output

Model ?

v_1, v_2, v_3 : volume of water in tanks $(v_i = c_i h_i)$

$$\left.\begin{aligned} \dot{v}_1 &= \alpha_{12}(h_2 - h_1) + u \\ \dot{v}_2 &= \alpha_{12}(h_1 - h_2) + \alpha_{23}(h_3 - h_2) \\ \dot{v}_3 &= \alpha_{23}(h_2 - h_3) \end{aligned}\right\} \quad \dot{v}_i = c_i \dot{h}_i \quad \left\} \begin{aligned} \dot{h}_1 &= -\frac{\alpha_{12}}{c_1} h_1 + \frac{\alpha_{12}}{c_1} h_2 + \frac{1}{c_1} u \\ \dot{h}_2 &= \frac{\alpha_{12}}{c_2} h_1 - \frac{\alpha_{12}+\alpha_{23}}{c_2} h_2 + \frac{\alpha_{23}}{c_2} h_3 \\ \dot{h}_3 &= \frac{\alpha_{23}}{c_3} h_2 - \frac{\alpha_{23}}{c_3} h_3 \end{aligned}\right.$$

Let $x = [h_1 \ h_2 \ h_3]^T$ be the state vector. Then

$$\left.\begin{aligned} \dot{x} &= \overbrace{\begin{bmatrix} -\frac{\alpha_{12}}{c_1} & \frac{\alpha_{12}}{c_1} & 0 \\ \frac{\alpha_{12}}{c_2} & -\frac{\alpha_{12}+\alpha_{23}}{c_2} & \frac{\alpha_{23}}{c_2} \\ 0 & \frac{\alpha_{23}}{c_3} & -\frac{\alpha_{23}}{c_3} \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} \frac{1}{c_1} \\ 0 \\ 0 \end{bmatrix}}^{B} u \\ y &= \underbrace{\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}}_{C} x \end{aligned}\right\} \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

A linear system in state-space form.

Questions

Q1: Suppose input u is constant. Will the output $y(t)$ stay bounded? (STABILITY)

Q2: Suppose initially $h_1 = 5m$, $h_2 = 10m$, $h_3 = 5m$. Can we find an input $u(t)$ so that $h_1 = 10m$, $h_2 = 5m$, $h_3 = 10m$ an hour later? (CONTROLLABILITY)

Q3: Suppose $u(t)$ is known and $y(t)$ is measured throughout a week. Can we determine h_1, h_2, h_3 at the beginning of the week? (OBSERVABILITY)

In this course we will mainly be interested in STABILITY, CONTROLLABILITY, and OBSERVABILITY of linear systems.

Following is what we call a continuous-time linear system

$$\boxed{\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t) \\ y(t) &= C(t)\,x(t) + D(t)\,u(t) \end{aligned}}$$

$A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times k}$ for all $t \geq 0$

$C(t) \in \mathbb{R}^{m\times n}$, $D(t) \in \mathbb{R}^{m\times k}$

The signals are called

$x : [0, \infty) \to \mathbb{R}^n$ state

$u : [0, \infty) \to \mathbb{R}^k$ input

$y : [0, \infty) \to \mathbb{R}^m$ output

Discrete-time linear system:

$$\begin{aligned} x(t+1) &= A(t)\,x(t) + B(t)\,u(t) \\ y(t) &= C(t)\,x(t) + D(t)\,u(t) \end{aligned}$$

where, this time, $t \in \{0, 1, 2, \ldots\} =: \mathbb{N}$

Compact notation for DT linear system:

$$\boxed{\begin{aligned} x^+ &= A(t)\,x + B(t)\,u \\ y &= C(t)\,x + D(t)\,u \end{aligned}}$$

Whenever the matrices $A(t), B(t), C(t), D(t)$ are all constant, then the system is called linear time-invariant (LTI). The general case, where the matrices are allowed to be time-dependent, is called linear time-varying. (LTV)

Remark: Note that there are different ways to represent the same system.

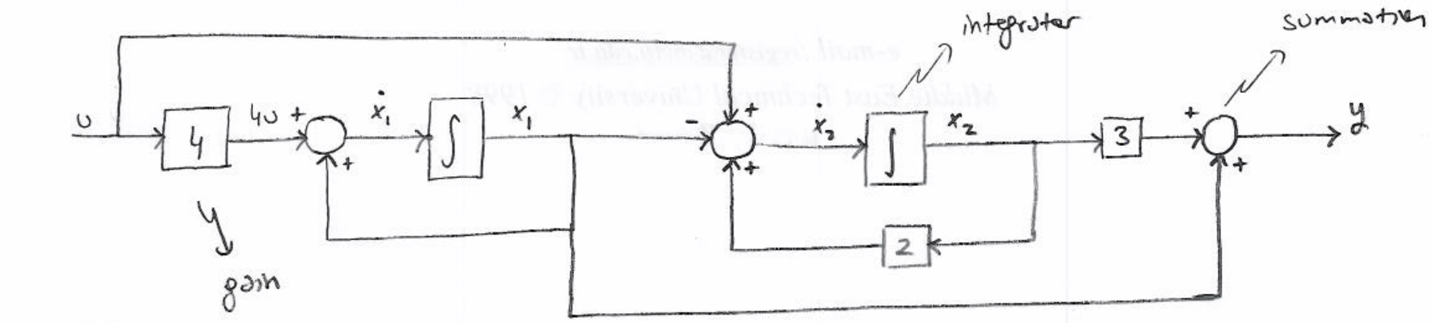
Example: Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Represent this system in terms of a block diagram using only summation, integrator, and gain blocks. (Say, for instance, you want to simulate it in simulink.)

Sol'n :



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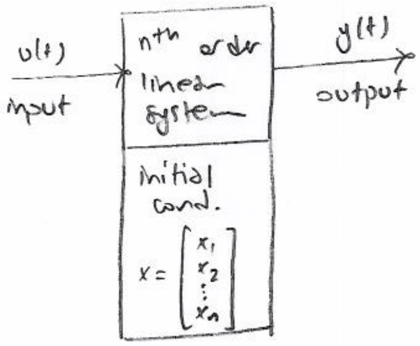
Properties of linear systems



Note that output depends on three things: time, input signal, initial condition vector. Let

$$y = y(t, x, u(\cdot)) \quad , \quad t \geq 0$$

time, init cond. (at $t=0$), input

P1) Superposition : we can write

$$y(t, \alpha x + \beta z, \gamma u(\cdot) + \delta v(\cdot)) = \alpha y(t, x, 0) + \beta y(t, z, 0) + \gamma y(t, 0, u(\cdot)) + \delta y(t, 0, v(\cdot))$$

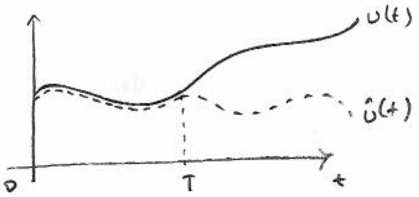
Exercise : Four measurements are made on a second-order linear system :

$u(t)$	x	y
$\sin t$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$y_1(t)$
$\cos t$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$y_2(t)$
0	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$y_3(t)$
0	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$y_4(t)$

Find $y(t)$ in terms of y_1, y_2, y_3, y_4 when $x = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ & $u(t) = A\cos(t+\phi)$

P2) <u>Causality</u>: Given two inputs $u(\cdot)$ & $\hat{u}(\cdot)$ satisfying

$u(t) = \hat{u}(t)$ for all $0 \leq t < T$.



Then $y(t, x, u(\cdot)) = y(t, x, \hat{u}(\cdot))$ for all $0 \leq t < T$.

That is, the present value of the solution depends only on the past values of the input and not on its future behaviour.

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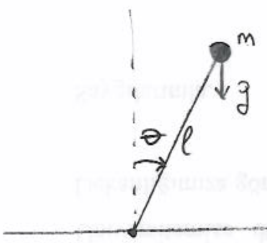
<u>LINEARIZATION</u> (of a nonlinear system around an equilibrium point.)

<u>Example</u> (nonlinear system) Inverted pendulum



$m\ell^2\ddot{\theta} = mg\ell\sin\theta - b\dot{\theta} + \tau$ where τ: torque (input)
b: friction coefficient

State space representation of pendulum

Let $\left.\begin{array}{l} x_1 := \theta \\ x_2 := \dot{\theta} \end{array}\right\}$ $\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{g}{\ell}\sin x_1 - \dfrac{b}{m\ell^2}x_2 + \dfrac{1}{m\ell^2}u \end{array}$ | Let output $y := \theta$

$u := \tau$ $\Rightarrow y = x_1$

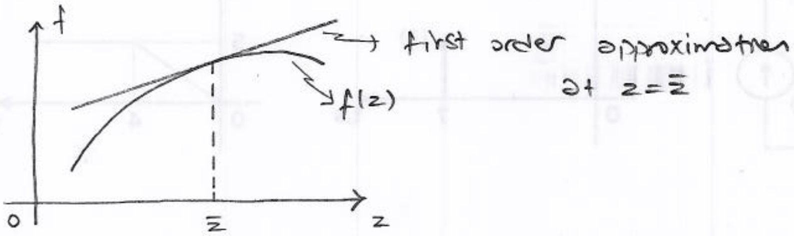
Then $\dot{x} = f(x, u)$ where $f(x,u) = \begin{bmatrix} x_2 \\ \dfrac{g}{\ell}\sin x_1 - \dfrac{b}{m\ell^2}x_2 + \dfrac{1}{m\ell^2}u \end{bmatrix}$ & $g(x) = x_1$

$y = g(x)$

In general, state space representation of a nonlinear system looks like

$$\dot{x} = f(x,u) \qquad \text{where} \qquad f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$$
$$y = g(x,u) \qquad\qquad\qquad g : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^m$$

Consider function $f(\cdot)$ differentiable



Around $\bar{z}$: $\underbrace{f(z) \cong f(\bar{z}) + \left.\frac{df}{dz}\right|_{z=\bar{z}} \cdot (z-\bar{z})}_{1^{st} \text{ order app. of } f \text{ around } \bar{z}}$

Idea : for values of z close to $\bar{z}$ we can replace $f(z)$ with its approximation.

This has a generalization to vector-valued functions.

Ex: Consider function $f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$

$$f(x,u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Approximation of $f(x,u)$ at $(x,u) = (\bar{x}, \bar{u})$?

$$f_1(x_1, x_2, u) \cong f_1(\bar{x}_1, \bar{x}_2, \bar{u}) + \frac{\partial f_1}{\partial x_1} \cdot (x_1 - \bar{x}_1) + \frac{\partial f_1}{\partial x_2} \cdot (x_2 - \bar{x}_2) + \frac{\partial f_1}{\partial u}(u - \bar{u})$$

evaluated at $(\bar{x}, \bar{u})$

Likewise, $f_2(x_1, x_2, u) \cong f_2(\bar{x}_1, \bar{x}_2, \bar{u}) + \frac{\partial f_2}{\partial x_1}(x_1 - \bar{x}_1) + \frac{\partial f_2}{\partial x_2}(x_2 - \bar{x}_2) + \frac{\partial f_2}{\partial u}(u - \bar{u})$

$$\Rightarrow f(x,u) \cong f(\bar{x}, \bar{u}) + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}}_{=: \frac{\partial f}{\partial x}} (x - \bar{x}) + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}}_{=: \frac{\partial f}{\partial u}} (u - \bar{u})$$

approximation of f at $(x,u) = (\bar{x}, \bar{u})$

Now, consider the system $\begin{cases} \dot{x} = f(x,u) \\ y = g(x,u) \end{cases}$

The approximation of the righthand side:

$$f(x,u) \cong f(\bar{x},\bar{u}) + \left[\frac{\partial f}{\partial x}\right]\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \cdot (x-\bar{x}) + \left[\frac{\partial f}{\partial u}\right]\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \cdot (u-\bar{u})$$

$$g(x,u) \cong g(\bar{x},\bar{u}) + \left[\frac{\partial g}{\partial x}\right]\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \cdot (x-\bar{x}) + \left[\frac{\partial g}{\partial u}\right]\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \cdot (u-\bar{u})$$

We let $A = \left[\frac{\partial f}{\partial x}\right]\Big|_{\substack{x=\bar{x}\\u=\bar{u}}}$, $B = \left[\frac{\partial f}{\partial u}\right]\Big|_{\substack{x=\bar{x}\\u=\bar{u}}}$, $C = \left[\frac{\partial g}{\partial x}\right]\Big|_{\substack{x=\bar{x}\\u=\bar{u}}}$, $D = \left(\frac{\partial g}{\partial u}\right)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}}$

$$\Rightarrow \quad \begin{aligned} \dot{x} &\cong f(\bar{x},\bar{u}) + A(x-\bar{x}) + B(u-\bar{u}) \\ y &\cong g(\bar{x},\bar{u}) + C(x-\bar{x}) + D(u-\bar{u}) \end{aligned}$$

Define $\xi := x-\bar{x}$ (variation of state x around $\bar{x}$)
$v := u-\bar{u}$
$w := y - g(\bar{x},\bar{u})$

$$\Rightarrow \quad \dot{\xi} = \frac{d}{dt}(x-\bar{x}) = \dot{x} \quad \Rightarrow \quad \left.\begin{aligned} \dot{\xi} &= f(\bar{x},\bar{u}) + A\xi + Bv \\ w &= C\xi + Dv \end{aligned}\right\} \text{Approximate system.}$$

Solution of actual system : $x(t)$
Solution of approximate system : $\xi(t)$

$\Big\}$ $x(t) \cong \bar{x} + \xi(t)$ as long as $(x(t),u(t)) \cong (\bar{x},\bar{u})$
$y(t) \cong g(\bar{x},\bar{u}) + w(t)$

<u>Question</u>: When is approximate system linear?

<u>Answer</u>: When $f(\bar{x},\bar{u}) = 0$.

When $f(\bar{x},\bar{u}) = 0$ we have $\boxed{\begin{aligned} \dot{\xi} &= A\xi + Bv \\ w &= C\xi + Dv \end{aligned}}$

This is called the linearization of nonlinear system $\begin{cases} \dot{x} = f(x,u) \\ y = g(x,u) \end{cases}$

at the <u>equilibrium point</u> $(\bar{x},\bar{u})$

<u>Definition</u>: For a system $\dot{x} = f(x,u)$, an equilibrium point $(\bar{x},\bar{u})$ is such that $f(\bar{x},\bar{u}) = 0$.

Remark: Linearization of a nonlinear system is meaningful only at an equil. point.

Example: Consider the system $\ddot{y} = -y + y^3 + u$ (nonlinear spring)

a) obtain its SS representation

b) Find equil. point(s) for $u = 0$.

Sol'n: $\left.\begin{array}{l} x_1 = y \\ x_2 = \dot{y} \end{array}\right\}$ $\dot{x} = \begin{bmatrix} x_2 \\ -x_1 + x_1^3 + u \end{bmatrix} =: f(x,u)$

$y = x_1 =: g(x,u)$

At equilibrium $f(x,u) = 0 \Rightarrow \begin{bmatrix} x_2 \\ \underbrace{-x_1 + x_1^3 + u}_{=0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_2 = 0 \\ x_1 = 0 \text{ or } x_1 = \mp 1 \end{array}$

$\Rightarrow$ Three equilibrium points (for $\bar{u} = 0$) $\bar{x} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$

c) Linearize the system at $(\bar{x}, \bar{u}) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right)$

$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + 3x_1^2 & 0 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \frac{\partial g}{\partial x} = [1 \;\; 0], \quad \frac{\partial g}{\partial u} = 0$

$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \;\; 0], \quad D = 0$

$\Rightarrow \boxed{\begin{array}{l} \dot{\xi} = A\xi + Bu \\ w = C\xi \end{array}}$ the linearization at $\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \bar{u} = 0$.

Question: How about discrete-time case $\begin{cases} x^+ = f(x,u) \\ y = g(x,u) \end{cases}$?

How to define equilibrium point?

Transition between different representations of LTI systems

→ From SS to TF

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \xrightarrow[\text{Laplace Transform}]{\mathcal{L}} \quad \begin{aligned} sX(s) - x(0) &= AX(s) + BU(s) \quad (1) \\ Y(s) &= CX(s) + DU(s) \quad (2) \end{aligned}$$

Eq. (1) $\Rightarrow \quad sX(s) - AX(s) = x(0) + BU(s)$

$\Rightarrow \quad (sI - A)X(s) = x(0) + BU(s)$

$\Rightarrow \quad X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s) \quad (3)$

Eq. (2) & (3) $\Rightarrow \quad Y(s) = C(sI-A)^{-1}x(0) + \underbrace{\left[C(sI-A)^{-1}B + D\right]}_{=:G(s)} U(s)$

<u>Note</u>: when init. cond. is zero $(x(0) = 0)$ we have $Y(s) = G(s)U(s)$.

$G(s) = C(sI-A)^{-1}B + D$ is called the <u>transfer function</u> (matrix) of the linear system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

→ $G(s)$ is $\underset{\text{number of outputs}}{m} \times \underset{\text{number of inputs}}{k}$ matrix

→ Entries of $G(s)$: $g_{ij}(s) = \dfrac{q_{ij}(s)}{p_{ij}(s)}$

q_{ij}, p_{ij} : polynomials with $\deg q_{ij} \leq \deg p_{ij}$ (i.e. all g_{ij} are <u>proper</u>)

→ If $\deg q_{ij} < \deg p_{ij}$ then g_{ij} is called <u>strictly proper</u>

If all g_{ij} are strictly proper then $G(s)$ is called strictly proper.

→ From TF to SS

Question: Given $G(s)$ how to obtain A, B, C, D such that $C(sI-A)^{-1}B + D = G(s)$?

Answer: The quadruple (A,B,C,D) is not unique. One way is as follows:

Example: Let the TF of a 2-input 2-output LTI system be

$$G(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

Find a quadruple (A,B,C,D) yielding

$$C(sI-A)^{-1}B + D = G(s)$$

Sol'n (controllable canonical representation)

Step 1

$$G(s) = \underbrace{\begin{bmatrix} -\frac{12}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}}_{\bar{G}(s)} + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{D}$$

Note: $\bar{G}(s)$ is strictly proper

Step 2 Find a monic common denominator poly. $d(s)$

$$d(s) = \left(s+\tfrac{1}{2}\right)(s+2)^2 = s^3 + \tfrac{9}{2}s^2 + 6s + 2$$

Step 3 Write $\bar{G}(s)$ as

$$\bar{G}(s) = \frac{\begin{bmatrix} -6(s+2)^2 & 3(s+\frac{1}{2})(s+2) \\ \frac{1}{2}(s+2) & (s+1)(s+\frac{1}{2}) \end{bmatrix}}{d(s)} = \frac{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}s^2 + \begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}s + \begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix}}{d(s)} \quad (1)$$

Step 4 Define state $x \in \mathbb{R}^6$ → $= 3 \times 2$ → number of inputs
↳ degree of $d(s)$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x_1, x_2, x_3 \in \mathbb{R}^2 \rightarrow \text{number of inputs}$$

$$X_3(s) := \frac{U(s)}{d(s)}, \quad X_2(s) := s\,X_3(s), \quad X_1(s) := s^2 X_3(s)$$

$$\Rightarrow \quad d(s)\, X_3 = U$$

$$\Rightarrow \quad s^3 X_3 + \frac{9}{2} s^2 X_3 + 6s X_3 + 2X_3 = U$$

$$\Rightarrow \quad s^3 X_3 = -\frac{9}{2} s^2 X_3 - 6s X_3 - 2X_3 + U$$

$$\Rightarrow \quad \left.\begin{aligned} sX_1 &= -\frac{9}{2} X_1 - 6X_2 - 2X_3 + U \\ sX_2 &= X_1 \\ sX_3 &= X_2 \end{aligned}\right\} \quad s\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2}I & -6I & -2I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} U \qquad (2)$$

$$I \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

Now, $\quad Y(s) = \left(\bar{G}(s) + D\right) U(s)$

$$(1) \Rightarrow \quad Y = \begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix} \underbrace{s^2 \frac{U}{d(s)}}_{X_1} + \begin{bmatrix} -24 & 15/2 \\ 1/2 & 3/2 \end{bmatrix} \underbrace{s \frac{U}{d(s)}}_{X_2} + \begin{bmatrix} -24 & 3 \\ 1 & 1/2 \end{bmatrix} \underbrace{\frac{U}{d(s)}}_{X_3} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} U \qquad (3)$$

$\mathcal{L}^{-1}\{(2)\} \Rightarrow$
(under zero init. cond.)

$$\dot{x} = \underbrace{\left[\begin{array}{cc|cc|cc} -\frac{9}{2} & 0 & -6 & 0 & -2 & 0 \\ 0 & -\frac{9}{2} & 0 & -6 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]}_{A \rightarrow 6\times 6} x + \underbrace{\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]}_{B \rightarrow 6\times 2} u$$

$$(3) \Rightarrow \quad y = \underbrace{\left[\begin{array}{cc|cc|cc} -6 & 3 & -24 & 15/2 & -24 & 3 \\ 0 & 1 & 1/2 & 3/2 & 1 & 1/2 \end{array}\right]}_{C \rightarrow 2\times 6} x + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{D \rightarrow 2\times 2} u$$

<u>Note</u>: Two different quadruples (A,B,C,D) & $(\bar{A},\bar{B},\bar{C},\bar{D})$ may yield the same TF. If they do (i.e. $C(sI-A)^{-1}B+D = \bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D}$) then the systems

$$\left\{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}\right\} \quad \& \quad \left\{\begin{array}{l}\dot{z} = \bar{A}z + \bar{B}u \\ y = \bar{C}z + \bar{D}u\end{array}\right\}$$

are said to be <u>zero-state equivalent</u>.

→ If two systems are ZSE then, under zero init. conditions, they give the same response at their outputs when excited by the same input.

<u>Algebraic Equivalence</u>

Given the system $\left\{\begin{array}{l}\dot{x} = Ax+Bu \\ y = Cx+Du\end{array}\right\}$ (1) let $T \in \mathbb{R}^{n\times n}$ be a nonsingular matrix

Define the new state $z := Tx$ $\quad (x = T^{-1}z)$

Then $\dot{z} = T\dot{x} = T(Ax+Bu)$
$\quad = TAT^{-1}z + TBu$

Also, $y = Cx+Du$
$\quad = CT^{-1}z + Du$

Now, let
$$\left.\begin{array}{l}\tilde{A} = TAT^{-1} \\ \tilde{B} = TB \\ \tilde{C} = CT^{-1} \\ \tilde{D} = D\end{array}\right\} (*)$$

Then, $\left.\begin{array}{l}\dot{z} = \tilde{A}z + \tilde{B}u \\ y = \tilde{C}z + \tilde{D}u\end{array}\right\}$ (2) is a new system.

This new system (2) has the property that if its init. condition is chosen as $z(0) = Tx(0)$ then for the same input the outputs of sys(1) & sys(2) will be identical.

<u>Definition</u>: Two LTI systems (1) & (2) are said to be <u>algebraically equivalent</u> if there exists a nonsingular matrix such that set of equations $(*)$ holds. (The map $z = Tx$ is called a similarity transformation.)

<u>Exercise</u>: Show that
a) AE ⇒ ZSE
b) ZSE ⇏ AE

SOLUTION TO LTV SYSTEM

Simplest case : (homogeneous, first order, LTI)

$$\left.\begin{aligned} \dot{x} &= \alpha x \\ x(t_0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \Rightarrow \text{solution} \quad x(t) = e^{\alpha(t-t_0)} x_0$$

let $\phi(t,t_0) := e^{\alpha(t-t_0)} \quad \Rightarrow \quad x(t) = \phi(t,t_0)\, x_0$

properties ?

$\rightarrow \phi(t_0,t_0) = 1 \quad \Rightarrow \quad x(t_0) = x_0$ (init. cond. constraint)

$\rightarrow \frac{d}{dt}\phi(t,t_0) = \alpha\, \phi(t,t_0) \quad \Rightarrow \quad \dot{x} = \alpha x$ (diff. eqn. constraint)

Now, consider

$$\left.\begin{aligned} \dot{x} &= A(t)\, x \qquad A(t) \in \mathbb{R}^{n\times n} \\ x(t_0) &= x_0 \in \mathbb{R}^n \end{aligned}\right\} \quad x(t) = ?$$

Guess : suppose there exists $\Phi(t,t_0) \in \mathbb{R}^{n\times n}$ satisfying the properties:

$\rightarrow \Phi(t_0,t_0) = I \qquad (1)$

$\rightarrow \frac{d}{dt}\Phi(t,t_0) = A(t)\, \Phi(t,t_0) \qquad (2)$

Then : $x(t) = \Phi(t,t_0)\, x_0$ would be the solution. (WHY ?)

How to obtain $\Phi(t,t_0)$?

$\int_{t_0}^{t} (2) \Rightarrow \quad \Phi(t,t_0) - \Phi(t_0,t_0) = \int_{t_0}^{t} A(s_1)\, \Phi(s_1,t_0)\, ds_1$

$(1) \Rightarrow \quad \Phi(t,t_0) = I + \int_{t_0}^{t} A(s_1)\, \Phi(s_1,t_0)\, ds_1 \qquad (3)$

$(3) \Rightarrow \quad \Phi(t,t_0) = I + \int_{t_0}^{t} A(s_1) \left\{ I + \int_{t_0}^{s_1} A(s_2)\, \Phi(s_2,t_0)\, ds_2 \right\} ds_1$

$\vdots \qquad = I + \int_{t_0}^{t} A(s_1)\, ds_1 + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)\, \Phi(s_2,t_0)\, ds_2\, ds_1$

$\Rightarrow \quad \Phi(t,t_0) = I + \int_{t_0}^{t} A(s_1)\, ds_1 + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)\, ds_2\, ds_1 + \ldots$ (Peano-Baker series)

<u>Theorem 1</u> The unique sol'n to $\left\{ \begin{array}{l} \dot{x} = A(t)x \\ x(t_0) = x_0 \end{array} \right\}$ is $x(t) = \Phi(t,t_0)x_0$ where $\Phi(t,t_0)$ is constructed through the P.-B. series. The matrix $\Phi(t,t_0)$ is called the <u>state transition matrix</u>.

<u>Properties of Φ</u>

$\boxed{P1}$ $\frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0)$ & $\Phi(t_0,t_0) = I$

$\boxed{P2}$ $\Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau)$ $\forall t,s,\tau \in \mathbb{R}$ (semigroup property)

$\boxed{P3}$ $\Phi(t,s)^{-1} = \Phi(s,t)$ $\forall s,t \in \mathbb{R}$ (follows from P2)

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<u>Theorem 2</u> The solution to $\left\{ \begin{array}{l} \dot{x} = A(t)x + B(t)u \\ x(t_0) = x_0 \end{array} \right\}$ is

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

<u>proof</u>: init. cond. $x(t_0) = \underbrace{\Phi(t_0,t_0)}_{I}x_0 + \underbrace{\int_{t_0}^{t_0}\cdots}_{0} = x_0$

diff. eqn. $\dot{x} = \frac{d}{dt}\Phi(t,t_0)x_0 + \frac{d}{dt}\int_{t_0}^{t}\cdots$

$$= A(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t}\frac{\partial}{\partial t}\Phi(t,\tau)B(\tau)u(\tau)d\tau + \Phi(t,t)B(t)u(t)$$

$$= A(t)\Phi(t,t_0)x_0 + A(t)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau + B(t)u(t)$$

$$= A(t)x(t) + B(t)u(t) \quad \blacksquare$$

Derivative of integral:

$$F(t) = \int_{a(t)}^{b(t)} f(t,\tau)d\tau \Rightarrow \frac{d}{dt}F(t) = f(t,b(t))\frac{d}{dt}b(t) - f(t,a(t))\frac{d}{dt}a(t) + \int_{a(t)}^{b(t)}\frac{\partial}{\partial t}f(t,\tau)d\tau$$

$$F(t) = \int_a^t f(\tau)d\tau \Rightarrow \frac{d}{dt}F(t) = f(t)$$

$$F(t) = \int_a^t f(t,\tau)d\tau \Rightarrow \frac{d}{dt}F(t) = \int_a^t \frac{\partial}{\partial t}f(t,\tau)d\tau + f(t,t)$$

Exercise: $\dot{x} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ t^2 \end{bmatrix} u \quad ; \quad x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$

$y = [\,0 \quad 1\,]\, x \qquad\qquad u \equiv 1$

$y(t) = ?$

Answer: $\Phi(t,\tau) = I + \int_\tau^t A(s_1)\, ds_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(t^2-\tau^2) & 1 \end{bmatrix}$

$$x(t) = \Phi(t,0)\,x(0) + \int_0^t \Phi(t,\tau) B(\tau) u(\tau)\, d\tau = \begin{bmatrix} x_{10} \\ x_{20} + \frac{t^2}{2} x_{10} + \frac{t^3}{3} \end{bmatrix}$$

$$y(t) = x_{20} + \frac{t^2}{2} x_{10} + \frac{t^3}{3}$$

Exercise: Given $A(t) = \begin{bmatrix} 0 & t \\ 0 & 2 \end{bmatrix}$ find $\Phi(t,\tau)$.

Answer: $\dot{x} = A(t)\,x \Rightarrow \begin{array}{l} \dot{x}_1 = t x_2 \\ \dot{x}_2 = 2x_2 \Rightarrow x_2(t) = e^{2(t-\tau)} x_2(\tau) \qquad (1) \end{array}$

$$\Rightarrow \dot{x}_1 = t e^{2t} \left\{ e^{-2\tau} x_2(\tau) \right\}$$

$$\Rightarrow x_1(t) - x_1(\tau) = e^{-2\tau} x_2(\tau) \int_\tau^t s e^{2s}\, ds \qquad \left(\int s e^{2s}\, ds = \left(\frac{s}{2} - \frac{1}{4}\right) e^{2s} \right)$$

$$\Rightarrow x_1(t) = x_1(\tau) + \left[\left(\frac{t}{2} - \frac{1}{4}\right) e^{2(t-\tau)} - \left(\frac{\tau}{2} - \frac{1}{4}\right) \right] x_2(\tau) \qquad (2)$$

$$(1)\,\&\,(2) \Rightarrow \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 1 & \left(\frac{t}{2} - \frac{1}{4}\right) e^{2(t-\tau)} - \left(\frac{\tau}{2} - \frac{1}{4}\right) \\ 0 & e^{2(t-\tau)} \end{bmatrix}}_{\text{this matrix has to be } \Phi(t,\tau)} \underbrace{\begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix}}_{x(\tau)}$$

Exercise: Check whether $\dot{\Phi}(t,\tau) \stackrel{?}{=} A(t)\Phi(t,\tau)$ & $\Phi(t,t) \stackrel{?}{=} I$

SOLUTION TO LTI SYSTEM

$$\dot{x} = Ax + Bu \qquad , \quad x(t_0) = x_0 \in \mathbb{R}^n .$$
$$y = Cx + Du$$

State transition matrix $\Phi(t, t_0) = ?$ (use P.B. series)

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A\, ds_1 + \int_{t_0}^{t} A \int_{t_0}^{s_1} A\, ds_2\, ds_1 + \cdots$$

$$= I + A\int_{t_0}^{t} ds_1 + A^2 \int_{t_0}^{t}\int_{t_0}^{s_1} ds_2\, ds_1 + \cdots = I + A(t-t_0) + A^2\frac{(t-t_0)^2}{2} + A^3\frac{(t-t_0)^3}{3!} + \cdots$$

$$\Rightarrow \boxed{\Phi(t,t_0) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} A^k}$$

Note that matrix exponential is defined as $e^M := \sum_{k=0}^{\infty} \frac{1}{k!} M^k$ (M square)

Hence, $\boxed{\Phi(t,t_0) = e^{A(t-t_0)}}$ for the LTI case.

Now, we can write the solution as:

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$y(t) = \underbrace{C e^{A(t-t_0)} x_0}_{\text{hmg. response}} + \underbrace{C \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)}_{\text{forced response}}$$

Properties of matrix exp. e^{At}:

$\boxed{P1}$ e^{At} is the unique solution $\Psi(t) \in \mathbb{R}^{n\times n}$ to

$$\frac{d}{dt}\Psi(t) = A\Psi(t) \quad \text{with} \quad \Psi(0) = I$$

$\boxed{P2}$ $e^{At}e^{A\tau} = e^{A(t+\tau)}$ for all $t,\tau \in \mathbb{R}$ $\Rightarrow$ $\boxed{P3}$ $[e^{At}]^{-1} = e^{-At}$

$\boxed{P4}$ There exists scalar functions $\alpha_0(t), \alpha_1(t), \ldots, \alpha_{n-1}(t)$ such that

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$$ (this follows from Cayley-Hamilton Thm.)

$\boxed{P5}$ $Ae^{At} = e^{At}A$

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Computing e^{At} by Laplace transform

$$\frac{d}{dt}e^{At} = Ae^{At} \quad \overset{\mathcal{L}}{\Rightarrow} \quad s\mathcal{L}\{e^{At}\} - \underbrace{e^{At}\Big|_{t=0}}_{I} = \mathcal{L}\{Ae^{At}\} = A\mathcal{L}\{e^{At}\}$$

$$\Rightarrow [sI - A]\mathcal{L}\{e^{At}\} = I$$

$$\Rightarrow \mathcal{L}\{e^{At}\} = [sI - A]^{-1}$$

$$\Rightarrow \boxed{e^{At} = \mathcal{L}^{-1}\{[sI-A]^{-1}\}}$$

Example : Compute e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

→ physical instances for $\dot{x} = Ax$

(mass spring sys.)

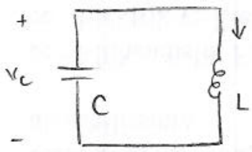
no friction $\mapsto y$

$m=1,\ k=1$

$m\ddot{y} + ky = 0 \Rightarrow \ddot{y} = -y$

let $x_1 = y,\ x_2 = \dot{y} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(LC oscillator)



no resistance

$L=1,\ C=1$

$L\dot{i}_L = v_C$

$C\dot{v}_C = -i_L$

let $x_1 = i_L,\ x_2 = v_C \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

→ what can we say about e^{At} without computing it?

mass spring sys. & no friction ⇒ (mechanical) energy is conserved (?)

$E = \text{pot. energy} + \text{kin. energy} = \frac{1}{2}ky^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 = \frac{1}{2}\|x\|^2 = \frac{1}{2}x^T x$

$\dot{E} = x^T\dot{x} = x^T A x = x^T A^T x = \frac{1}{2}x^T(A + A^T)x$

$A + A^T = 0$ (A skew symmetric) $\Rightarrow$ $\dot{E} = 0$ (energy conserved)

$\dot{E} = 0 \Rightarrow x(t)^T x(t) = x(0)^T x(0)$

$\Rightarrow x(0)^T \underbrace{(e^{At})^T e^{At}}_{M(t)} x(0) = x(0)^T x(0) \qquad (*)$

$\{M(t) \text{ symmetric}\} + \{x(0) \text{ arbitrary}\} + \{(*)\} \overset{(\text{WHY?})}{\Rightarrow} M(t) = I$

That is, $(e^{At})^T e^{At} = I \Rightarrow$ $\boxed{e^{At} \text{ must be orthogonal}}$

[Q orthogonal ⇔ $Q^T Q = I$; orthogonal matrices are norm preserving

i.e. $\|Qx\| = \|x\|$ for all $x \in \mathbb{R}^n$]

In general : $A \in \mathbb{R}^{n\times n}$ skew symmetric ⇒ e^{At} orthogonal

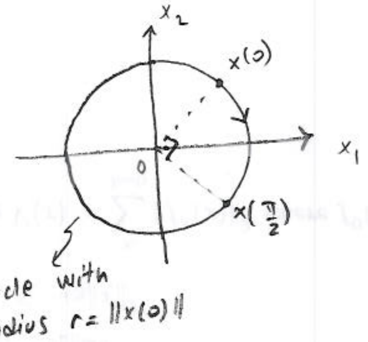
Now, let's actually compute e^{At}:

$$[sI-A] = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \Rightarrow [sI-A]^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ -\frac{1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} & \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ -\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} & \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$[e^{At}]^T e^{At} = \begin{bmatrix} \cos^2 t + \sin^2 t & 0 \\ 0 & \cos^2 t + \sin^2 t \end{bmatrix} = I \quad \text{as expected.}$$

Remark: e^{At} is a rotation matrix.



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Recall solution in DT (LTI case)

$$\text{sys.} \begin{cases} x^+ = Ax + Bu & \& \quad x(0) = x_0 \\ y = Cx + Du \end{cases}$$

$$\text{sol.} \begin{cases} x(t) = A^t x_0 + \sum_{k=0}^{t-1} A^k B u(t-k-1) \\ y(t) = CA^t x_0 + C\sum_{k=0}^{t-1} A^k B u(t-k-1) + Du(t) \qquad t = 0, 1, 2, \ldots \end{cases}$$

Remark: The counterpart of e^{At} (in continuous time) is the matrix A^t in discrete time.

(2015)

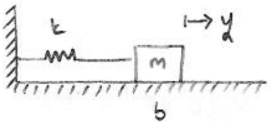
JORDAN FORM & THE (UN)BOUNDEDNESS of A^k & e^{At}

hmg. LTI sys. $\dot{x} = Ax$, $x(0) = x_0 \in \mathbb{R}^n$

solution $x(t) = e^{At}x_0$ $\left(e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots = \mathcal{L}^{-1}\{[sI-A]^{-1}\} \right)$

Possible behaviours (convergence, unboundedness, boundedness)

1) $x(t) \to 0$ for all x_0 $\Rightarrow$ $e^{At} \to 0$ as $t \to \infty$



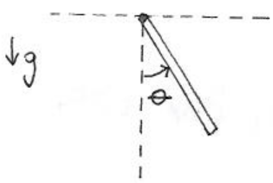
$m, k, b > 0$

$$\left.\begin{array}{l} m\ddot{y} + b\dot{y} + ky = 0 \\ x_1 = y \,,\; x_2 = \dot{y} \end{array}\right\} \quad \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} x$$

$\underbrace{\text{eigenvalues of } A}_{\text{roots of } \det(sI-A)} : \lambda_1, \lambda_2$

$\det(sI-A) = s^2 + \frac{b}{m}s + \frac{k}{m}$ $\Rightarrow$ $Re\{\lambda_i\} < 0$ for all i (nec. & suf. for $e^{At} \to 0$)

2) $\|x(t)\| \to \infty$ for some x_0 $\Rightarrow$ $\|e^{At}\| \to \infty$



$\underbrace{\ddot{\theta} + \sin\theta = 0}_{\text{normalized pendulum eqn.}}$

linearization at $\theta = \pi$

$$\left.\begin{array}{l} x_1 = \theta - \pi \\ x_2 = \dot{x}_1 \end{array}\right\} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x \qquad \lambda_1 = 1, \lambda_2 = -1$$

$Re\{\lambda_i\} > 0$ for some i (suf. for $\|e^{At}\| \to \infty$) [WHY NOT NEC.? (see Remark 2)]

<u>Remark 1</u>: Not all solutions are unbounded. Take for instance $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

then $x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$ $\Rightarrow$ $x(t) \to 0$

3) $x(t)$ remain bounded for all x_0 $\Rightarrow$ $\|e^{At}\| < c$

linearization (of pend.) at $\theta = 0$

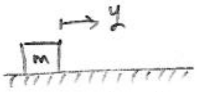
$$\left.\begin{array}{l} x_1 = \theta \\ x_2 = \dot{x}_1 \end{array}\right\} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

$\lambda_1 = j$, $\lambda_2 = -j$ & $\|e^{At}\| = 1$

$Re\{\lambda_i\} \leq 0$ for all i (nec. for $\|e^{At}\| < c$)

[WHY NOT SUF.? (see Remark 2)]

<u>Remark 2</u>: In general $Re\{\lambda_i\} \leq 0$ $\forall i$ is not sufficient for boundedness. Take for instance:



(no friction)

$\ddot{y} = 0$ $\Rightarrow$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 0$

$y(t) = y(0) + \dot{y}(0)t$

$\dot{y}(t) = $ constant

We know from EE501

<u>Theorem</u>: For every matrix $A \in \mathbb{C}^{n\times n}$, there exists a nonsingular change of basis $P \in \mathbb{C}^{n\times n}$ that transforms A into J

$$J = PAP^{-1} = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_\ell \end{bmatrix}$$

where each J_i is a Jordan block of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}_{n_i \times n_i}$$

where each $\lambda_i \in \mathbb{C}$ is an eigenvalue of A, and the number ℓ of Jordan blocks equals the number of lin. ind. eigenvectors of A. Each A has a unique (up to a reordering of the Jordan blocks) J. The matrix J is called the Jordan normal form.

<u>Remark</u>: We do NOT assume $\lambda_i \neq \lambda_j$ for $i \neq j$ here. For instance let $A = I_{n\times n}$. Then A is already in Jordan form with

$$A = I = J = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ & & \ddots \\ 0 & & & J_n \end{bmatrix} \quad \text{with} \quad J_i = [1]_{1\times 1} \quad \text{for } i = 1, 2, \dots, n$$

<u>Question</u>: $A^k \to ?$ as $k \to \infty$ $(k = 0, 1, 2, \dots)$

$$J = PAP^{-1} \Rightarrow A = P^{-1} J P$$

$$\Rightarrow A^2 = (P^{-1}JP)(P^{-1}JP) = P^{-1}J^2P$$

$$\vdots$$

$$\Rightarrow A^k = P^{-1} J^k P = P^{-1} \begin{bmatrix} J_1^k & & 0 \\ & J_2^k & \\ & & \ddots \\ 0 & & & J_\ell^k \end{bmatrix} P$$

$J_i^k = ?$ By induction one can obtain

$$[J_i^k]_{1\times 1} = [\lambda_i^k] \quad , \quad \underbrace{[J_i^k]_{2\times 2}}_{\begin{bmatrix}\lambda_i & 1\\ 0 & \lambda_i\end{bmatrix}^k} = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} \\ 0 & \lambda_i^k \end{bmatrix} \quad , \quad \underbrace{[J_i^k]_{3\times 3}}_{\begin{bmatrix}\lambda_i & 1 & 0\\ 0 & \lambda_i & 1\\ 0 & 0 & \lambda_i\end{bmatrix}^k} = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & \frac{k(k-1)\lambda_i^{k-2}}{2!} \\ 0 & \lambda_i^k & k\lambda_i^{k-1} \\ 0 & 0 & \lambda_i^k \end{bmatrix}$$

$[J_i^k]_{n_i \times n_i}$ = see the text

Hence we deduce:

1) Suppose $|\lambda_i| < 1$ for all i. Then $J_i^k \to 0$ as $k \to \infty$ for all i.
Then $J^k \to 0 \implies A^k \to 0$ as $k \to \infty$. [Because $A^k = P^{-1} J^k P$]

2) Suppose $|\lambda_i| \le 1$ for all i and the size of a Jordan block J_m is 1×1 whenever $|\lambda_m| = 1$. Then all J_i^k remain bounded, i.e., one can find a constant $c > 0$ such that $\|J_i^k\| < c$ for all k.
Then A^k remains bounded.

3) Suppose there exists some i such that

either a) $|\lambda_i| > 1$

or b) $|\lambda_i| = 1$ and the size of J_i is 2×2 or larger.

Then A^k is unbounded, i.e., $\lim_{k\to\infty} \max\{\|A\|, \|A^2\|, \dots, \|A^k\|\} = \infty$

Example : Some plant modeled in DT

$$\left.\begin{aligned} x^+ &= Ax \\ y &= Cx \end{aligned}\right\} \text{plant}$$

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$

WANT : obtain solution $x(k)$

CONSTRAINT: we cannot measure x directly. we have only access to output y.

SOL'N : construct an "observer"

observer state : $\hat{x} \in \mathbb{R}^n$

dynamics : $\hat{x}^+ = A\hat{x} + L(y - C\hat{x})$ $\leftrightarrows$ this system is running in a computer!

WANT: $\hat{x}(k) \to x(k)$ as $k \to \infty$

Choose (if possible) the observer gain $L \in \mathbb{R}^{n \times p}$ such that all the eigenvalues of $[A - LC]$ satisfy $|\lambda_i| < 1$. Then:

Claim : $\hat{x}(k) \to x(k)$ as $k \to \infty$ for all init. cond. $x(0)$ & $\hat{x}(0)$.

Proof : Define error $e := \hat{x} - x$.

Error dynamics? $e^+ = \hat{x}^+ - x^+ = A\hat{x} + L(y - C\hat{x}) - Ax$

$$\begin{aligned} &= A\hat{x} + LC(x - \hat{x}) - Ax \\ &= A(\hat{x} - x) + LC(x - \hat{x}) \\ &= [A - LC](\hat{x} - x) = [A - LC]e \end{aligned}$$

Therefore, $e(k) = [A - LC]^k e(0)$ for $k = 0, 1, 2, \ldots$

Now, since the eigenvalues of $[A - LC]$ satisfy $|\lambda_i| < 1$, we can write

$$\begin{aligned} [A - LC]^k \to 0 \text{ as } k \to \infty \quad &\Rightarrow e(k) \to 0 \text{ as } k \to \infty \\ &\Rightarrow \hat{x}(k) \to x(k) \text{ as } k \to \infty. \quad \blacksquare \end{aligned}$$

Hence, we eventually obtain $x(k)$ by just measuring the output $y(k)$.

Exercise : What happens if $\lambda_i = 0$ for all i? [This case is called "Deadbeat observer". Try, for instance, $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, $L = \begin{bmatrix} 10.5 \\ 12.75 \\ 15 \end{bmatrix}$

$C = [0 \ 0 \ 1]$.

Question $e^{At} \to ?$ as $t \to \infty$ $(t \in \mathbb{R})$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \qquad A = P^{-1} J P$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} (P^{-1} J P)^k$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} P^{-1} J^k P = P^{-1} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} J^k \right) P = P^{-1} e^{Jt} P$$

$$\Rightarrow e^{At} = P^{-1} e^{Jt} P = P^{-1} \begin{bmatrix} e^{J_1 t} & & & 0 \\ & e^{J_2 t} & & \\ & & \ddots & \\ 0 & & & e^{J_\ell t} \end{bmatrix} P$$

It turns out that
$$e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots \\ 0 & 0 & 1 & t & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{n_i \times n_i} \qquad (1)$$

Exercise: Verify eq. (1) by showing:

$\rightarrow e^{J_i t}\big|_{t=0} = I$ &

$\rightarrow \frac{d}{dt} e^{J_i t} = J_i e^{J_i t}$

From (1) we deduce:

1) Suppose $Re\{\lambda_i\} < 0$ for all i. Then

$e^{J_i t} \to 0$ as $t \to \infty \Rightarrow e^{Jt} \to 0$ as $t \to \infty \Rightarrow e^{At} \to 0$ as $t \to \infty$

[Because $e^{At} = P^{-1} e^{Jt} P$]

2) Suppose $Re\{\lambda_i\} \leq 0$ for all i and the size of a Jordan block J_m is 1×1 whenever $Re\{\lambda_m\} = 0$. Then all $e^{J_i t}$ remain bounded $\Rightarrow e^{At}$ remains bounded.

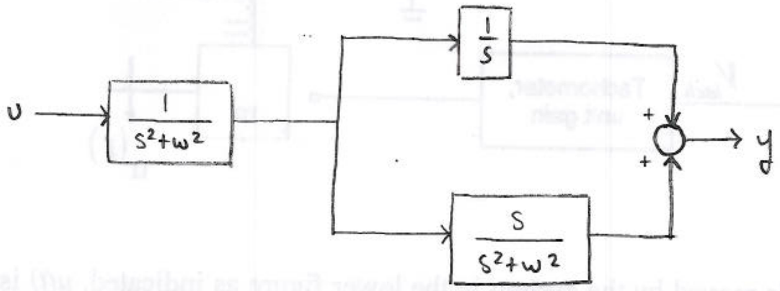
3) Suppose there exists some i such that

either a) $Re\{\lambda_i\} > 0$

or b) $Re\{\lambda_i\} = 0$ & $\text{size}(J_i) \geq 2 \times 2$.

Then $\|e^{J_i t}\|$ is unbounded $\Rightarrow \|e^{At}\|$ is unbounded.

Example: Consider the below block diagram



a) obtain a state space representation $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

b) For the matrix A in part a, determine the Jordan form J. What can be said about $e^{At} \to ?$ as $t \to \infty$.

Sol'n a) First, obtain the transfer function $G(s) = \frac{Y(s)}{U(s)}$.

$$Y(s) = \frac{1}{s}\left\{ \frac{1}{s^2+w^2} U(s) \right\} + \frac{s}{s^2+w^2}\left\{ \frac{1}{s^2+w^2} U(s) \right\}$$

$$= \left(\frac{1}{s} + \frac{s}{s^2+w^2} \right) \frac{1}{s^2+w^2} U(s) = \frac{2s^2+w^2}{s(s^2+w^2)} \cdot \frac{1}{s^2+w^2} U(s)$$

$$\Rightarrow G(s) = \frac{2s^2+w^2}{s(s^2+w^2)^2} = \frac{2s^2+w^2}{s^5+2w^2s^3+w^4 s}$$

Let us obtain the controllable canonical representation for this G(s).

$$\# \text{ of states} = \underbrace{\# \text{ of inputs}}_{1} \times \underbrace{\text{degree of den.}\{G(s)\}}_{5} = 5$$

Let $X_5 := \dfrac{1}{s^5 + 2w^2 s^3 + w^4 s} U$, $X_4 := s X_5$, $X_3 := s X_4 = s^2 X_5$, $X_2 := s X_3 = s^3 X_5$, $X_1 := s X_2 = s^4 X_5$

$\Rightarrow s^5 X_5 + 2w^2 s^3 X_5 + w^4 s X_5 = U$

$\Rightarrow s^5 X_5 = -2w^2 s^3 X_5 - w^4 s X_5 + U$

$\Rightarrow s X_1 = -2w^2 X_2 - w^4 X_4 + U$

$s X_2 = X_1$

$s X_3 = X_2$

$s X_4 = X_3$

$s X_5 = X_4$

Also,

$Y = 2s^2 X_5 + w^2 X_5$

$= 2 X_3 + w^2 X_5$

Hence,

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & -2w^2 & 0 & -w^4 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 2 & 0 & w^2 \end{bmatrix}}_{C} x$$

b) Char. poly. $d(s) = |sI - A|$

Den. poly. $den(s) = s(s^2 + w^2)^2$

} since $\deg\{d(s)\} = \deg\{den(s)\} = 5$ we have $d(s) = den(s)$ (WHY?)

Therefore, $d(s) = s\,(s + jw)^2 (s - jw)^2$

$\Rightarrow$ eigenvalues of $A = (jw, jw, -jw, -jw, 0)$

Jordan form?

$$J = \begin{bmatrix} j\omega & a & 0 & 0 & 0 \\ 0 & j\omega & 0 & 0 & 0 \\ 0 & 0 & -j\omega & b & 0 \\ 0 & 0 & 0 & -j\omega & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are four possibilities $(a,b) = \{ \underbrace{(0,0)}_{\ell=5}, \underbrace{(0,1), (1,0)}_{\ell=4}, \underbrace{(1,1)}_{\ell=3} \}$ (ℓ: # of Jord. blocks)

<u>Claim</u> : $a = 1$

<u>proof</u> : Suppose not. Then $a = 0$.

$$\Rightarrow \text{rank}\,(J - j\omega I) = \text{rank} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -j2\omega & b & 0 \\ 0 & 0 & 0 & -j2\omega & 0 \\ 0 & 0 & 0 & 0 & -j\omega \end{bmatrix} = 3$$

$$\text{Also, } \text{rank}\,(A - j\omega I) = \text{rank} \begin{bmatrix} -j\omega & -2\omega^2 & 0 & -\omega^4 & 0 \\ 1 & -j\omega & 0 & 0 & 0 \\ 0 & 1 & -j\omega & 0 & 0 \\ 0 & 0 & 1 & -j\omega & 0 \\ 0 & 0 & 0 & 1 & -j\omega \end{bmatrix} \geq 4$$

$$\begin{aligned} \text{Now, } 3 = \text{rank}\,(J - j\omega I) &= \text{rank}\,(PAP^{-1} - j\omega I) \\ &= \text{rank}\, P(A - j\omega I)P^{-1} \\ &= \text{rank}\,(A - j\omega I) \\ &\geq 4 \quad \Rightarrow \text{contradiction} \end{aligned}$$

because P is full rank.

$\square$

Likewise, $b = 1$.

Hence, $J = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_3 \end{bmatrix}$ where $J_1 = \begin{bmatrix} j\omega & 1 \\ 0 & j\omega \end{bmatrix}$, $J_2 = \begin{bmatrix} -j\omega & 1 \\ 0 & -j\omega \end{bmatrix}$, $J_3 = [0]$

$$e^{J_1 t} = \begin{bmatrix} e^{j\omega t} & te^{j\omega t} \\ 0 & e^{j\omega t} \end{bmatrix}$$

unbounded term

<u>$e^{At} \to ?$</u>

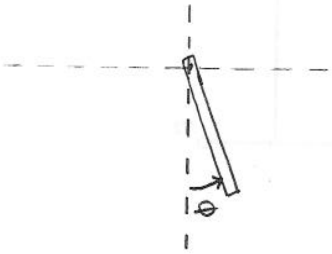
Note that $\text{Re}\{\lambda_1 = j\omega\} = 0$ & $\text{size}\,(J_1) = 2 \times 2$

$\Rightarrow e^{At}$ is unbounded

STABILITY

→ internal stability : do solutions (of autonomous system $\dot{x} = f(x)$) starting close to an equilibrium stay close to that equilibrium ?

ex : pendulum



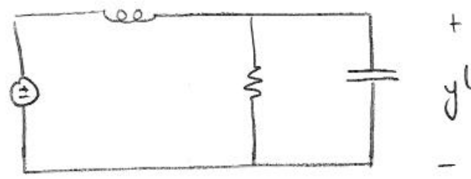
equilibrium	character
$\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	stable
$\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$	unstable

(Note : Stability is a property of equil.)

→ input-output stability : do small inputs result in small outputs ?

ex :

$u(t) = \varepsilon \cos \omega t$



small input magnitude $\overset{?}{\Longrightarrow}$ small output magnitude

Lyapunov (Internal) Stability

$$\left.\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}\right\} (1) \qquad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ u \in \mathbb{R}^k$$

Definition : system (1) is said to be

1) stable in the sense of Lyapunov or internally stable if for each t_0 & initial condition $x(t_0) = x_0$, the homogeneous state response

$$x(t) = \Phi(t, t_0)\, x_0$$

is bounded. That is, given $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ we can find some $c > 0$ such that $\|x(t)\| < c$ for all $t \geq t_0$.

2) asymptotically stable (in the sense of Lyapunov) if (it is stable and) for every initial condition $x(t_0) = x_0$ we have $x(t) \to 0$ as $t \to \infty$.

3) exponentially stable if there exist constants $k, \lambda > 0$ such that

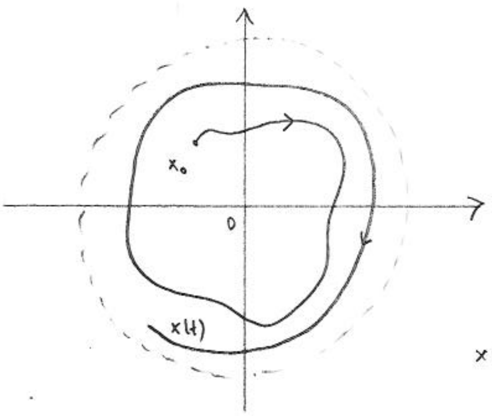
$$\|x(t)\| \leq k e^{-\lambda(t-t_0)} \|x(t_0)\| \quad \text{for all} \quad t \geq t_0.$$

[Remark: $1 \supset 2 \supset 3$]

4) unstable if not stable.

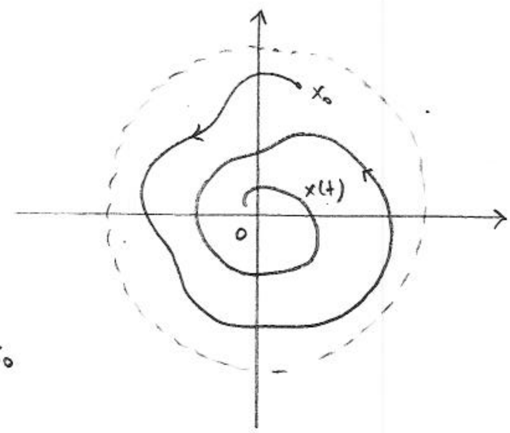
— o —

<u>Remark</u>: This definition has nothing to do with matrices $B(t), C(t), D(t)$. Hence, when studying Lyapunov stability we can simply consider $\dot{x} = A(t)x$ instead of the system (1).

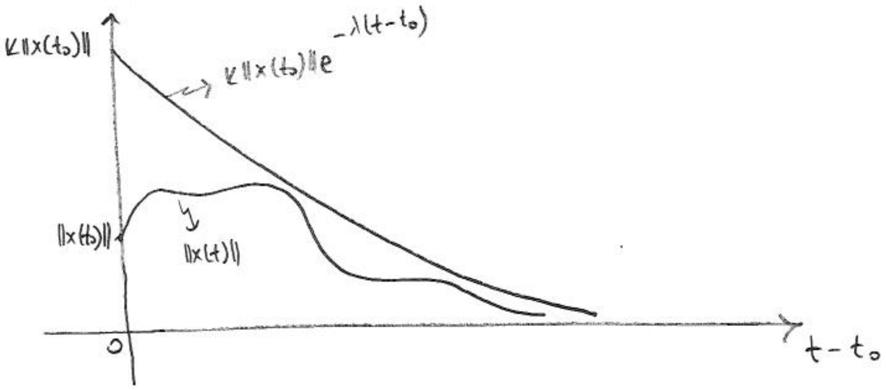


$x(t) = \Phi(t, t_0) x_0$

stable



asy. stable



exp. stable

<u>Theorem</u>: LTI system $\dot{x} = Ax$ is

$\rightarrow$ stable iff a) $\mathrm{Re}\{\lambda_i\} \leq 0$ for all i
and b) $\mathrm{Re}\{\lambda_i\} = 0 \Rightarrow \mathrm{size}(J_i) = 1\times 1$

$\rightarrow$ asy. stable (exp. stable) iff $\mathrm{Re}\{\lambda_i\} < 0$ for all i.

$\rightarrow$ unstable iff either a) $\exists i$ such that $\mathrm{Re}\{\lambda_i\} > 0$
or b) $\exists i$ such that $\mathrm{Re}\{\lambda_i\} = 0$ and $\mathrm{size}(J_i) \geq 2\times 2$

<u>Examples</u>:

STABLE		UNSTABLE
	ASY. STABLE	
$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x$		$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$
$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$	$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$	$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x$
$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x$		

<u>Question</u>: What do the eigenvalues of $A(t)$ say about stability of $\dot{x} = A(t)x$?

<u>Answer</u>: Nothing. (In general)

<u>Example</u>: Consider

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t\cos t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$$

For each t, the eigenvalues of $A(t)$ are $\lambda_{1,2} = -\frac{1}{4} \mp j\frac{\sqrt{7}}{4}$ (VERIFY!)

That is λ_i are constant & satisfy $\mathrm{Re}\{\lambda_i\} < 0$.

However ...

we have $\Phi(t,0) = \begin{bmatrix} e^{t/2}\cos t & e^{-t}\sin t \\ -e^{t/2}\sin t & e^{-t}\cos t \end{bmatrix}$ (VERIFY!)

Now, let $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $x(t) = e^{t/2}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \Rightarrow \|x(t)\| = e^{t/2} \Rightarrow \|x(t)\| \to \infty$.

— o —

Stability check without eigenvalue computation (WHY?)

Consider $\dot{x} = Ax$, $x \in \mathbb{R}^n$

Define $V : \mathbb{R}^n \to \mathbb{R}$ as $V(x) = x^T x = \|x\|^2$

Note that $V(x) > 0$ for all $x \neq 0$ & $V(0) = 0$.

Question : $\frac{d}{dt} V(x(t)) = ?$

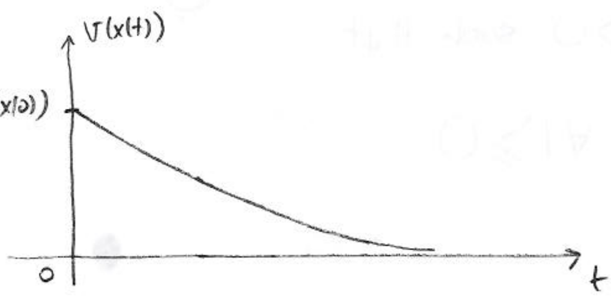
$$\dot{V} = \frac{d}{dt}(x^T x) = \dot{x}^T x + x^T \dot{x}$$
$$= x^T A^T x + x^T A x$$
$$= x^T (A^T + A) x$$

Suppose A is such that $\dot{V} < 0$ for all $x \neq 0$. For instance, $A = \begin{bmatrix} -1 & 2 \\ -2 & -3 \end{bmatrix}$

$$\Rightarrow A^T + A = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}$$

$\Rightarrow \dot{V} = [x_1 \ x_2] \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2x_1^2 - 6x_2^2$ which is obviously < 0 for $x \neq 0$.

Now, $\dot{V} < 0 \Rightarrow V(x(t))$ decreasing $\Rightarrow V(x(t)) \to 0$ (V cannot be negative!)



Therefore, $\|x(t)\| \to 0 \Rightarrow x(t) \to 0$

$\Rightarrow$ The system is asy. stable!

<u>Observe</u> : To figure out that system $\dot{x} = \begin{bmatrix} -1 & 2 \\ -2 & -3 \end{bmatrix} x$ is asy. stable we have neither computed the solution $x(t)$ nor the eigenvalues λ_i. Instead we showed that a positive definite function $V(x)$ always decreases along the solutions of the system $\dot{x} = Ax$. This idea is very powerful and due to Russian scientist Aleksandr Lyapunov (1857-1918).

— o —

<u>Remark</u> : What was essential about $V(x)$ in the above analysis? That it was positive for $x \neq 0$. Hence the generalization: $V(x) = x^T P x$ where $P = P^T$ is positive definite.

— o —

<u>Positive-Definite Matrices</u> (Review)

A symmetric $n \times n$ real matrix Q is

→ positive definite if $x^T Q x > 0$ for all $0 \neq x \in \mathbb{R}^n$

→ positive semidefinite if $x^T Q x \geq 0$ for all x

→ negative (semi) definite if $-Q$ is positive (semi) definite.

— o —

The following statements are equivalent for $Q^T = Q \in \mathbb{R}^{n \times n}$

1) Q is positive definite (we write $Q > 0$)

2) All eigenvalues of Q are strictly positive.

3) The determinants of all upper left submatrices of Q are positive.

<u>ex</u>: $Q = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ $a > 0$, $ad - b^2 > 0$, $\det Q > 0$

4) There exists an $n \times n$ nonsingular real matrix H such that $Q = H^T H$.

— o —

For a symmetric pos. def. Q we have

$$\underbrace{\lambda_{min}(Q)}_{\substack{\text{smallest} \\ \text{eigenvalue} \\ \text{of } Q}} \|x\|^2 \leq x^T Q x \leq \underbrace{\lambda_{max}(Q)}_{\substack{\text{largest} \\ \text{eigenvalue} \\ \text{of } Q}} \|x\|^2$$

LYAPUNOV STABILITY THEOREM.

Consider LTI system

$$\dot{x} = Ax \quad , \quad x \in \mathbb{R}^n \qquad (1)$$

Theorem: The following five conditions are equivalent.

1) System (1) is asy. stable.

2) System (1) is exp. stable.

3) All the eigenvalues of A have strictly negative real parts.

4) For each symmetric pos. def. matrix Q, there exists a unique sol'n $P \in \mathbb{R}^{n\times n}$ to the following "Lyapunov Equation"

$$A^T P + PA = -Q.$$

Moreover, P is symmetric and pos. def.

5) There exists $P^T = P > 0$ for which the following Lyapunov matrix inequality holds

$$A^T P + PA < 0.$$

Proof. We have already established $1 \Leftrightarrow 2 \Leftrightarrow 3$. (By Jordan form arguments.) What remain are: $2 \Rightarrow 4$, $4 \Rightarrow 5$, $5 \Rightarrow 2$.

$\boxed{2 \Rightarrow 4}$ Define matrix $P := \int_0^\infty e^{A^T t} Q e^{At} dt$. (for a given $Q = Q^T > 0$)

$\rightarrow$ P is well-defined: because $e^{At} \to 0$ exponentially fast $\Rightarrow$ $e^{A^T t} Q e^{At} \to 0$ exp. fast $\Rightarrow$ the integral is well-defined.

$\rightarrow$ P solves $A^T P + PA = -Q$

$$A^T P + PA = \int_0^\infty (A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A) dt$$

$$= \int_0^\infty \left\{ \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) \right\} dt$$

$$= e^{A^T t} Q e^{At} \Big|_{t=0}^{t\to\infty} = \underbrace{\lim_{t\to\infty} e^{A^T t} Q e^{At}}_{0} - \underbrace{e^{A^T \cdot 0}}_{I} Q \underbrace{e^{A \cdot 0}}_{I}$$

$$= -Q$$

$\rightarrow$ P is symmetric & $P > 0$:

$$P^T = \int_0^\infty (e^{A^Tt} Q e^{At})^T dt = \int_0^\infty (e^{At})^T Q^T (e^{A^Tt})^T dt = \int_0^\infty e^{A^Tt} Q e^{At} dt = P$$

Hence, symmetry is established.

To show P is pos. def. choose an arbitrary vector $z \in \mathbb{R}^n$ $(z \neq 0)$

$$\Rightarrow z^T P z = \int_0^\infty z^T e^{A^Tt} Q e^{At} z \, dt = \int_0^\infty w(t)^T Q \, w(t) \, dt$$

where $w(t) = e^{At} z$ for $t \geq 0$.

Since $Q > 0$ we have $w(t)^T Q \, w(t) \geq 0$ $\forall t$, which implies $z^T P z \geq 0$.

Also, $z^T P z \neq 0$ because if $z^T P z = 0$ then

$$\int_0^\infty w(t)^T Q w(t) dt = 0 \Rightarrow w(t)^T Q w(t) = 0 \ \forall t \geq 0 \Rightarrow w(t) = 0 \ \forall t \geq 0$$

But $w(t) = e^{At} z \Rightarrow w(0) = e^{A \cdot 0} z = z \neq 0 \Rightarrow$ contradiction.

Hence $\{z^T P z \geq 0\}$ & $\{z^T P z \neq 0\} \Rightarrow z^T P z > 0$

Since z was arbitrary $\Rightarrow P > 0$.

— o —

$\rightarrow$ P is unique : We prove this by contradiction.

Suppose not. Then there exists $\bar{P} \neq P$ such that

$$\left. \begin{array}{l} A^T \bar{P} + \bar{P} A = -Q \\ A^T P + P A = -Q \end{array} \right\} \Rightarrow A^T (P - \bar{P}) + (P - \bar{P}) A = 0$$

Note that $\frac{d}{dt} \left\{ e^{A^Tt} (P - \bar{P}) e^{At} \right\} = e^{A^Tt} \underbrace{\left\{ A^T (P - \bar{P}) + (P - \bar{P}) A \right\}}_{=0} e^{At} = 0$

$$\Rightarrow e^{A^Tt} (P - \bar{P}) e^{At} = \text{constant for all } t \geq 0.$$

$$\Rightarrow \underbrace{e^{A^Tt} (P - \bar{P}) e^{At} \Big|_{t=0}}_{= P - \bar{P}} = \underbrace{\lim_{t \to \infty} e^{A^Tt} (P - \bar{P}) e^{At}}_{= 0} \quad \Rightarrow \bar{P} = P \quad \text{contradiction.}$$

Hence $2 \Rightarrow 4$ is established.

$\boxed{4 \Rightarrow 5}$ Obvious.

$\boxed{5 \Rightarrow 2}$ Let $Q := -(A^T P + PA) > 0$. Given any solution $x(t)$ of system $\dot{x} = Ax$, define the scalar signal $v(t)$ as

$$v(t) := x(t)^T P x(t).$$

Then, $\dot{v} = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x$

$$= -x^T Q x$$

Now, $\dot{v}(t) = -x(t)^T Q x(t)$ $\qquad x^T Q x \geq \lambda_{min}(Q) \|x\|^2$

$$\leq -\lambda_{min}(Q) \|x(t)\|^2 \qquad \lambda_{max}(P) \|x\|^2 \geq x^T P x$$

$$\leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} x^T(t) P x(t)$$

$$= -\alpha v(t) \qquad \text{where } \alpha := \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} > 0$$

Comparison Lemma: $\dot{v} \leq -\alpha v \Rightarrow v(t) \leq e^{-\alpha t} v(0)$
proof Define $w(t) := e^{\alpha t} v(t)$
$\Rightarrow \dot{w} = \alpha e^{\alpha t} v + e^{\alpha t} \dot{v} \leq \alpha e^{\alpha t} v - e^{\alpha t} \alpha v = 0$
$\Rightarrow w(t) \leq w(0) = v(0)$
$\Rightarrow e^{\alpha t} v(t) \leq v(0)$
$\Rightarrow v(t) \leq e^{-\alpha t} v(0)$

Therefore,

$$\|x(t)\|^2 \leq \frac{1}{\lambda_{min}(P)} \underbrace{x(t)^T P x(t)}_{v(t)} \leq \frac{e^{-\alpha t}}{\lambda_{min}(P)} \underbrace{x(0)^T P x(0)}_{v(0)}$$

$$\leq \frac{\lambda_{max}(P)}{\lambda_{min}(P)} e^{-\alpha t} \|x(0)\|^2$$

$$\Rightarrow \|x(t)\| \leq \left(\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\right)^{1/2} e^{-\frac{\alpha}{2} t} \|x(0)\| \Rightarrow \text{exp. stability} \qquad \blacksquare$$

Quesiton: Geometric meaning of $A^T P + PA < 0$?

Stability in Discrete Time

$$\left.\begin{aligned} x^+ &= A(k)x + B(k)u \\ y &= C(k)x + D(k)u \end{aligned}\right\} \quad (1)$$

Definition System (1) is said to be

1) Stable in the sense of Lyapunov or internally stable if for each k_0 and initial cond. $x(k_0) = x_0$, the homogeneous state response

$$x(k) = A(k-1)A(k-2)\dots A(k_0)x(k_0) \qquad k = k_0, k_0+1, k_0+2, \dots$$

is bounded.

2) Asy. stable if every (homogeneous) sol'n satisfies $x(k) \to 0$ as $k \to \infty$

3) Exp. stable if there exist constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|x(k)\| \leq c\,\lambda^{k-k_0}\|x(k_0)\| \quad \text{for all } k \geq k_0.$$

4) Unstable if not stable.

— o —

Theorem : The LTI system $x^+ = Ax$ is

1) Stable iff a) $|\lambda_i| \leq 1$ for all i

and b) $|\lambda_i| = 1 \Rightarrow$ size $(J_i) = 1\times 1$

2) Asy. stable (exp. stable) iff $|\lambda_i| < 1$ for all i.

3) Unstable iff either a) $\exists i$ such that $|\lambda_i| > 1$

or b) $\exists i$ such that $|\lambda_i| = 1$ & size $(J_i) \geq 2\times 2$

Example : CT system: $\dot{x} = Ax$, solution $x(t)$

Euler discretization: $z(k+1) = z(k) + \varepsilon A z(k)$, $\varepsilon > 0$ step size, $k = 0, 1, 2, \dots$

$z(0) = x(0) \Rightarrow z(k) \cong x(k\varepsilon)$ for ε, k small WHY?

$$z^+ = (I+\varepsilon A)z \quad \Rightarrow \quad z(k) = (I+\varepsilon A)^k z(0)$$

$$x(k\varepsilon) = e^{A\varepsilon k}x(0) = (e^{A\varepsilon})^k x(0) = \underbrace{[I+\varepsilon A + \frac{\varepsilon^2 A^2}{2!} + \dots]}_{\cong I+\varepsilon A \text{ for } \varepsilon \ll 1}{}^k x(0) \cong [I+\varepsilon A]^k x(0)$$

Show that if the CT system is exp. stable then its euler discret. is also exp. stable for $\varepsilon > 0$ small enough.

Sol'n: CT system : $\dot{x} = Ax$, eigenvalues λ_i , given : $Re\{\lambda_i\} < 0$

DT system: $z^+ = (I+\varepsilon A)z$, eigenvalues α_i

$\alpha_i = ?$ Let v_i be the eigenvector for λ_i , i.e., $Av_i = \lambda_i v_i$

Then $(I+\varepsilon A)v_i = v_i + \varepsilon A v_i = v_i + \varepsilon\lambda_i v_i = (1+\varepsilon\lambda_i)v_i \Rightarrow \alpha_i = 1+\varepsilon\lambda_i$

let $\lambda_i = -\delta_i + j\omega_i$ with $\delta_i > 0$ & $\omega_i \in \mathbb{R}$

$$\Rightarrow |\alpha_i|^2 = |1+\varepsilon(-\delta_i + j\omega_i)|^2 = (1-\varepsilon\delta_i)^2 + \varepsilon^2\omega_i^2 = 1 - 2\varepsilon\delta_i + \varepsilon^2(\delta_i^2+\omega_i^2)$$

$$= [1-\varepsilon\delta_i] - \varepsilon\underbrace{[\delta_i - \varepsilon(\delta_i^2+\omega_i^2)]}_{\text{WANT} \geq 0}$$

$$\delta_i - \varepsilon(\delta_i^2+\omega_i^2) \geq 0 \iff \varepsilon \leq \frac{\delta_i}{\delta_i^2+\omega_i^2} = \frac{-Re\{\lambda_i\}}{|\lambda_i|^2}$$

Choose $\varepsilon \leq \min_{i=1,2,\dots,n} -\frac{Re\{\lambda_i\}}{|\lambda_i|^2}$ then $|\alpha_i|^2 \leq 1-\varepsilon\delta_i < 1$ for all i.

$\Rightarrow$ $|\alpha_i| < 1$ for all i

$\Rightarrow$ $z^+ = [I+\varepsilon A]z$ is exp. stable.

— o —

Lyapunov Stability Thm : For the system $x^+ = Ax$ the following cond. are equiv.

1) system is asy. stable / 2) system is exp. stable / 3) $|\lambda_i| < 1$ for all i.

4) For each $Q^T = Q > 0$ there exists a unique sol'n $P \in \mathbb{R}^{n\times n}$ to the following Stein equation (or discrete-time Lyapunov eqn.)

$$A^T P A - P = -Q$$

Moreover, $P = P^T > 0$

$$\left(P := \sum_{k=0}^{\infty} A^{k^T} Q A^k \right.$$

5) There exists $P = P^T > 0$ satisfying $A^T P A - P < 0$.

Example (5 ⇒ 2) For the system $x^+ = Ax$ suppose there exists $P = P^T > 0$ satisfying $A^TPA - P < 0$. Show that the system is exp. stable.

Sol'n Let $Q := P - A^TPA > 0$. Given any solution $x(k)$ to $x^+ = Ax$, define

$v(k) := x(k)^T P x(k) \geq 0$

[Analogue of differentiation is one-step difference in DT]

$$\Rightarrow v(k+1) - v(k) = x(k+1)^T P x(k+1) - x(k)^T P x(k)$$
$$= x(k)^T A^T P A x(k) - x(k)^T P x(k)$$
$$= x(k)^T [A^T P A - P] x(k)$$
$$= -x(k)^T Q x(k)$$

$\lambda_{min}(Q)\|x\|^2 \leq x^T Q x \leq \lambda_{max}(Q)\|x\|^2$

$$\leq -\lambda_{min}(Q)\,\|x(k)\|^2$$
$$\leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(Q)}\, x(k)^T P x(k)$$

$$= -\varepsilon v(k) \quad \text{where} \quad \varepsilon := \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} > 0$$ [Note that $0 < \varepsilon \leq 1$.] ⇒ needn't mention!

$\Rightarrow v(k+1) \leq (1-\varepsilon)v(k)$ Let $\alpha := 1 - \varepsilon$. [Note that $0 \leq \alpha < 1$.]

$\Rightarrow v(k) \leq \alpha^k v(0)$

Then,
$$\|x(k)\|^2 \leq \frac{1}{\lambda_{min}(P)} \underbrace{x(k)^T P x(k)}_{v(k)}$$
$$\leq \frac{\alpha^k}{\lambda_{min}(P)} \underbrace{x(0)^T P x(0)}_{v(0)}$$
$$\leq \frac{\lambda_{max}(P)}{\lambda_{min}(P)} \alpha^k \|x(0)\|^2$$

$$\Rightarrow \|x(k)\| \leq \left[\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\right]^{1/2} \left[\sqrt{\alpha}\right]^k \|x(0)\| \quad \Rightarrow \text{exp. stability.}$$

Determining Local stability of $f(\bar{x}) = 0$

Nonlinear system: $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ (1)

An equilibrium: $f(\bar{x}) = 0$

Linearization of (1) at $x = \bar{x}$: $\dot{\xi} = A\xi$ (2)

where $\xi := x - \bar{x}$ & $A = \left[\frac{\partial f}{\partial x} \right]_{x=\bar{x}}$

<u>Question</u>: What does the stability / instability of system (2) tell us about the stability of the equilibrium $x = \bar{x}$ of system (1) ?

<u>Theorem</u>: (assume: f smooth) If the linearization (2) is exp. stable, then there exist $\varepsilon, c, \lambda > 0$ such that

$$\|x(t_0) - \bar{x}\| < \varepsilon \implies \|x(t) - \bar{x}\| \leq c\, e^{-\lambda(t-t_0)} \|x(t_0) - \bar{x}\| \quad \text{for all } t \geq t_0$$

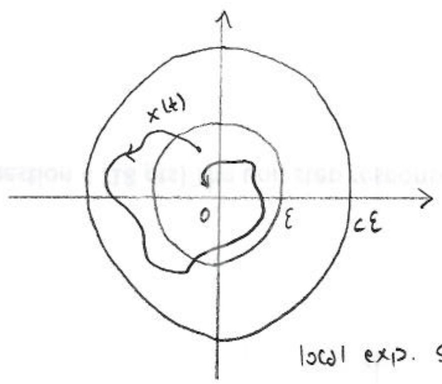
(such equilibrium $\bar{x}$ is called "locally exp. stable equil.")

<u>Proof</u>: See the text.

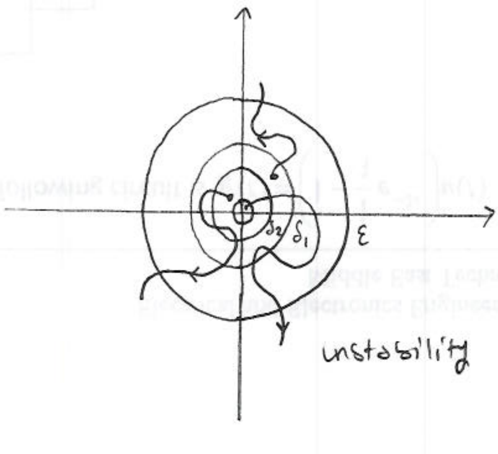
<u>Theorem</u>: [Instability] If the matrix A has an eigenvalue satisfying $\mathrm{Re}\{\lambda_i\} > 0$ then there exists $\varepsilon > 0$ such that for each $\delta > 0$ there exists $T > 0$ and a solution $x(t)$ satisfying

$$\|x(t_0) - \bar{x}\| < \delta \quad \& \quad \|x(t_0 + T) - \bar{x}\| > \varepsilon$$

(such equilibrium $\bar{x}$ is called "unstable")



local exp. stability



unstability

Indeterminate case: when the eigenvalues of the linearization satisfy: $\{\operatorname{Re}\{\lambda_i\} \le 0 \ \forall i \ \& \ \exists i \ \operatorname{Re}\{\lambda_i\} = 0\}$ we cannot determine the behaviour of solutions of the nonlinear system around the equil. $\bar{x}$.

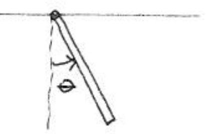
Example [Indeterminate cases]

$\dot{x} = f(x)$	linearization at $\bar{x}=0$	behavior of linearization $\dot{\xi} = A\xi$	behavior of actual system $\dot{x} = f(x)$
$\dot{x} = -x^3$	$\dot{\xi} = 0$	stable $\lvert\xi(t)\rvert = \lvert\xi(0)\rvert$	asy. stable $\lvert x(t)\rvert \to 0$
$\dot{x} = x^3$	$\dot{\xi} = 0$	stable $\lvert\xi(t)\rvert = \lvert\xi(0)\rvert$	unstable $\lvert x(t)\rvert \to \infty$ $\quad x(t) = \frac{x(0)}{\sqrt{1-2x(0)^2 t}}$
$\dot{x}_1 = x_2$ $\dot{x}_2 = -x_1^3$	$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\xi$	unstable	stable $V(x(t)) = V(x(0))$ for $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$
$\dot{x}_1 = x_2$ $\dot{x}_2 = -x_1^3 - x_2^3$	$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\xi$	unstable	asy. stable $\lVert x(t)\rVert \to 0$!!! Thm 8.6

Example [Pendulum] Normalized eqn. (w/ friction)

$$\ddot{\theta} + \varepsilon\dot{\theta} + \sin\theta = 0 \qquad (\varepsilon > 0)$$

$g \downarrow$



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let $\left.\begin{array}{l} x_1 = \theta \\ x_2 = \dot{\theta} \end{array}\right\} \Rightarrow \dot{x} = \begin{bmatrix} x_2 \\ -\sin x_1 - \varepsilon x_2 \end{bmatrix} =: f(x)$

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -\varepsilon \end{bmatrix}$$

equilibria: $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

Linearization at $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (downright equil.)

$$A = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -\varepsilon \end{bmatrix}_{x=\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \Rightarrow \dot{\xi} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} \xi$$

$$|sI - A| = s^2 + \varepsilon s + 1 \Rightarrow \mathrm{Re}\{\lambda_i\} < 0 \quad \forall i$$

$\Rightarrow$ downright equil. is (locally) exp. stable.

Linearization at $\bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ (upright equil.)

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ 1 & -\varepsilon \end{bmatrix} \xi \quad , \quad |sI - A| = s^2 + \varepsilon s - 1$$

$\Rightarrow \exists i$ s.t. $\mathrm{Re}\{\lambda_i\} > 0$

$\Rightarrow$ upright equil. is unstable.

— o —

Discrete time case

System: $x^+ = f(x)$

equilibrium : $f(\bar{x}) = \bar{x}$

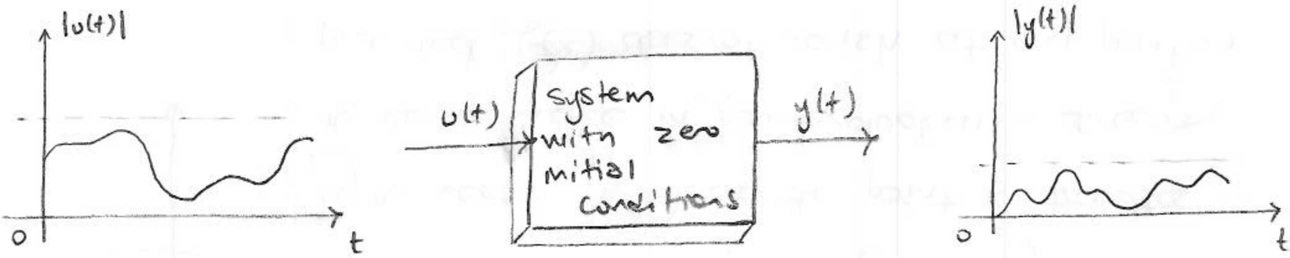
linearization : $\xi^+ = A\xi$

with $A = \left[\dfrac{\partial f}{\partial x}\right]_{x=\bar{x}}$

$\xi^+ = A\xi$	$x^+ = f(x)$
exp. stable	$\bar{x}$ (locally) exp. stable
$\exists \lambda_i$ such that $\lvert\lambda_i\rvert > 1$	$\bar{x}$ unstable
otherwise	?

BOUNDED INPUT BOUNDED OUTPUT (BIBO) STABILITY

Roughly, BIBO stability = small inputs cause small outputs (under zero init. conditions)



Consider LTV system

$$(1)\quad \begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{cases} \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^k, \quad y \in \mathbb{R}^m$$

whose forced response (response for zero init. condition $x(0) = 0$) is

$$y(t) = \int_0^t C(t)\,\Phi(t,\tau) B(\tau) u(\tau)\, d\tau + D(t)u(t) \qquad \text{(forced response)}$$

[$\Phi(t,\tau)$: state transition matrix]

<u>Definition</u>: System (1) is said to be bounded input bounded output (BIBO) stable if there exists a constant $g > 0$ such that for all input signals $u(t)$ the forced response $y(t)$ satisfies

$$\sup_{t \in [0,\infty)} \|y(t)\| \leq g \left\{ \sup_{t \in [0,\infty)} \|u(t)\| \right\}$$

<u>Q</u> How to determine BIBO stability for system (1) ?

<u>A</u> See Thm 9.1.

— o —

<u>Q</u> How to compute minimum g ?

<u>A</u> See Chapter 5.3 of the textbook "Nonlinear Systems" by H. Khalil 3rd ed.

$$g \sim \sup_{\omega} \|G(j\omega)\|_2 \qquad \text{where } G(s) = C(sI - A)^{-1}B + D$$

LTI case

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\right\} \quad (2) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ y \in \mathbb{R}^m$$

the forced response (by definition $x(0) = 0$) is

$$y(t) = \underbrace{\int_0^t C e^{A(t-\tau)} B u(\tau) d\tau}_{\text{How about this term?}} + \underbrace{Du(t)}_{\text{this term is bounded for bounded } u(t)}$$

Theorem: System (2) is BIBO stable if and only if each entry $g_{ij}(t)$ of the $m \times k$ matrix $[Ce^{At}B]$ is absolutely integrable. That is, $\int_0^\infty |g_{ij}(t)| dt < \infty$ for all i,j.

→ How to determine BIBO stability from the TF $G(s)$ of the system?

$$G(s) = C(sI - A)^{-1}B + D$$

Theorem: System (2) is BIBO stable if and only if all the poles of all entries of $m \times k$ matrix $G(s)$ are with negative real parts.

Remark: Matrix D has no effect on pole locations of $G(s)$.
Hence, BIBO stability of system (2) depends only on the triple (A, B, C)

Recall:

$$\{\text{set of poles of } G(s)\} \subset \{\text{set of eigenvalues of } A\}$$

Hence the next thm.

Theorem: If system (2) is exp. stable (in the sense of Lyapunov) then it is BIBO stable. (The converse is NOT true.)

#3 1 Nisan 2014

<u>Example</u>: Consider

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Eigenvalues of A?

$$|sI - A| = \begin{vmatrix} s & -1 \\ -6 & s+1 \end{vmatrix} = s(s+1) - 6 = s^2 + s - 6 = (s+3)(s-2)$$

$\Rightarrow \lambda_{1,2} = 2, -3$

Due to $\lambda_1 = 2 > 0$, system is unstable (in the sense of Lyapunov)

How about BIBO stability?

$$G(s) = C(sI-A)^{-1}B + \cancel{D}^{\,0}$$

$$(sI-A)^{-1} = \frac{1}{(s+3)(s-2)} \begin{bmatrix} s+1 & 1 \\ 6 & s \end{bmatrix}$$

$$G(s) = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ 6 & s \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}}{(s+3)(s-2)}$$

$$\Rightarrow G(s) = \frac{\cancel{s-2}}{(s+3)\cancel{(s-2)}} = \frac{1}{s+3}$$

pole-zero cancellation

poles of G(s)? G(s) has a single pole at $s = -3$ which is negative. Therefore system is BIBO stable.

<u>Observe</u>: B is the eigenvector of the stable ($\lambda_2 = -3$) eigenvalue.

#1 7 Nisan 2014

CONTROLLABILITY & STATE FEEDBACK

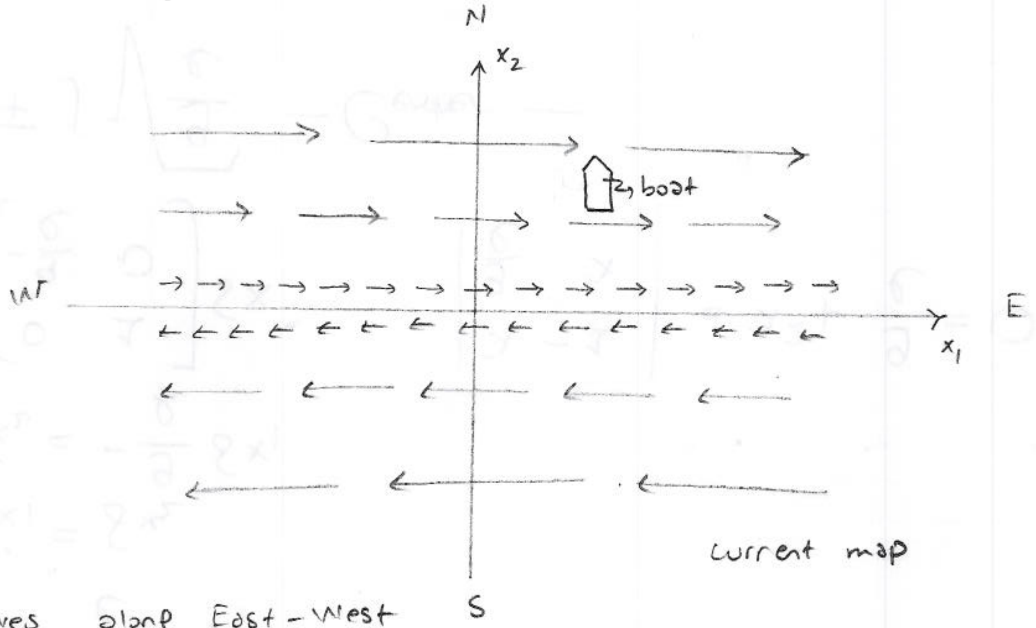
Example : $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu$

A visualization Let x be the position of a boat on sea with currents. Suppose "Ax" term represents the velocity component due to currents and "Bu" represents the component due to the engine (propulsion) of the boat.

Case 1 $(u=0)$ Engines shut off $(\dot{x} = Ax)$

$\dot{x}_1 = x_2$

$\dot{x}_2 = 0$

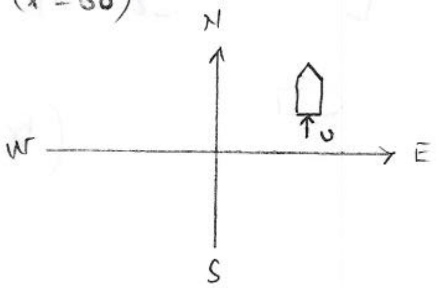


$\Rightarrow$ Boat moves along East-West direction due to currents.

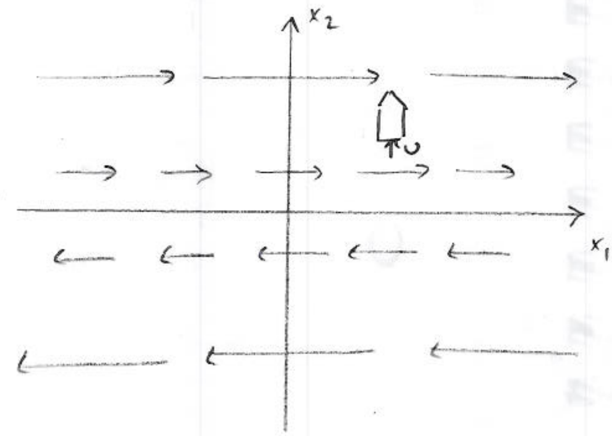
Case 2 Ignore currents $(\dot{x} = Bu)$

$\dot{x}_1 = 0$

$\dot{x}_2 = u$



Case 3 Consider both terms $(\dot{x} = Ax + Bu)$



<u>Question</u>: Given two sets of coordinates $x_{init.}$ & x_{final} and two times $t_0 < t_f$, can we find an input function $u(t)$ such that the boat can be steered to $x(t_f) = x_{final}$ from $x(t_0) = x_{init.}$? (YES)

[This question is directly related to CONTROLLABILITY of the system.]

<u>CONTROLLABLE & REACHABLE SUBSPACES</u>

Consider the LTV system

$$\dot{x} = A(t)x + B(t)u \qquad (x \in \mathbb{R}^n, \ u \in \mathbb{R}^k)$$

whose solution $x(t)$ satisfies

$$x(t) = \Phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

[$\Phi(t,t_0)$: state transition matrix.]

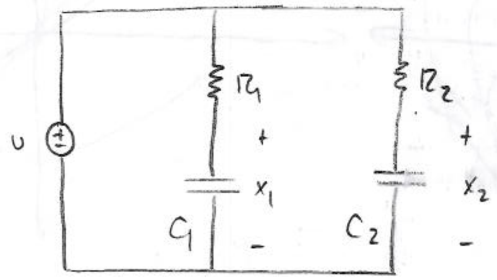
<u>Definition</u>: (Reachable Subspace) Given two times $t_1 > t_0$, the reachable subspace $\mathcal{R}[t_0,t_1]$ consists of all the states $x_1 \in \mathbb{R}^n$ for which there exists an input $u : [t_0,t_1] \to \mathbb{R}^k$ that steers the state from $x(t_0) = 0$ to $x(t_1) = x_1$. In other words,

$$\mathcal{R}[t_0,t_1] = \left\{ x_1 \in \mathbb{R}^n : \exists u(\cdot),\ x_1 = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau \right\}$$

<u>Definition</u> (Controllable Subspace) Given two times $t_1 > t_0$, the controllable subspace $\mathcal{C}[t_0,t_1]$ consists of all the states $x_0 \in \mathbb{R}^n$ for which there exists an input $u : [t_0,t_1] \to \mathbb{R}^k$ that steers the state from $x(t_0) = x_0$ to $x(t_1) = 0$. That is,

$$\mathcal{C}[t_0,t_1] = \left\{ x_0 \in \mathbb{R}^n : \exists u(\cdot),\ 0 = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau \right\}$$

Example



x_i : voltage of capacitor C_i $i=1,2$

system?

$$C_1\dot{x}_1 = \frac{u-x_1}{R_1} \Rightarrow \dot{x}_1 = -\frac{1}{R_1C_1}x_1 + \frac{1}{R_1C_1}u$$

Likewise,
$$\dot{x}_2 = -\frac{1}{R_2C_2}x_2 + \frac{1}{R_2C_2}u$$

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix}}_{B} u$$

$$A \text{ diagonal} \Rightarrow e^{At} = \begin{bmatrix} e^{-t/R_1C_1} & 0 \\ 0 & e^{-t/R_2C_2} \end{bmatrix}$$

let $w_1 := \frac{1}{R_1C_1}$ & $w_2 := \frac{1}{R_2C_2}$

Then,
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

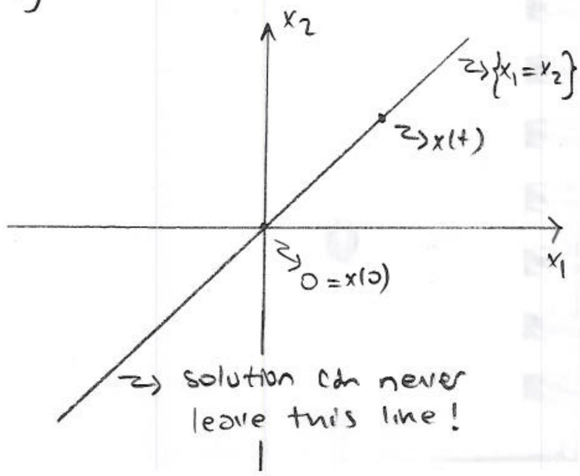
$$= \begin{bmatrix} e^{-w_1t}x_1(0) \\ e^{-w_2t}x_2(0) \end{bmatrix} + \begin{bmatrix} \int_0^t e^{-w_1(t-\tau)} w_1 u(\tau)d\tau \\ \int_0^t e^{-w_2(t-\tau)} w_2 u(\tau)d\tau \end{bmatrix}$$

Now, consider the case with equal time constants: $R_1C_1 = R_2C_2 = 1/w$

$$\Rightarrow x(t) = e^{-wt}x(0) + \underbrace{\left\{ w\int_0^t e^{-w(t-\tau)}u(\tau)d\tau \right\}}_{=:\alpha(t)} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (1)$$

(1) $\Rightarrow$ for $x(0)=0$ we have $x(t) = \alpha(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (2)

(2) implies that for $x(0)=0$ we can never steer the state $x(t)$ to a point that is NOT on $x_1=x_2$ line.



The reachable subspace for this system is therefore

$$\mathcal{R}[t_0, t_1] = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \text{for all} \quad t_1 > t_0.$$

How about the controllable subspace? (Again for equal time constants.)

That is, from which points can we transfer $x(t)$ to the origin?

$$(1) \Rightarrow \quad 0 \overset{?}{=} e^{-\omega t} x(0) + \alpha(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (3)$$

Clearly, Eq. (3) can be solved only for $x(0) \in \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Hence, $C[t_0, t_1] = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for all $t_1 > t_0$.

[We will later see that $\mathcal{R}[t_0, t_1] = C[t_0, t_1] = \mathbb{R}^2$ for this example when $R_1C_1 \neq R_2C_2$.]

— o —

Characterization of Reachable & Controllable Subspaces

Definition Given two times $t_1 > t_0$, the reachability & controllability Gramians of the system $\dot{x} = A(t)x + B(t)u$ respectively are

$$W_R(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B(\tau)^T \Phi(t_1, \tau)^T d\tau$$

$$W_C(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) B(\tau)^T \Phi(t_0, \tau)^T d\tau$$

What's the use of this def.? See Thm. below.

Theorem Given two times $t_1 > t_0$ we have $\mathcal{R}[t_0, t_1] = \text{Range}\{W_R(t_0, t_1)\}$. Moreover, if $x_1 \in \text{Range}\{W_R(t_0, t_1)\}$ with $x_1 = W_R(t_0, t_1)\eta_1$ then

the control input $u(t) = B(t)^T \Phi(t_1, t)^T \eta_1$ for $t \in [t_0, t_1]$ can be used

to transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$.

Proof [A common way of proving the equality of two sets is to demonstrate that each is a subset of the other]

(part I) $\text{Range}\, W_R(t_0,t_1) \subset R[t_0,t_1]$

Let $x_1 \in \text{Range}\, W_R$. Then $\exists \eta_1 \in \mathbb{R}^n$ such that $x_1 = W_R \eta_1$. Now, $x(t_0) = 0$ & $u(t) = B(t)^T \Phi(t_1,t)^T \eta_1$ imply

$$x(t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) u(\tau)\, d\tau$$

$$= \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) B(\tau)^T \Phi(t_1,\tau)^T \eta_1 \, d\tau$$

$$= W_R(t_0,t_1)\eta_1 = x_1$$

Hence $x_1 \in R[t_0,t_1]$.

(part II) $R[t_0,t_1] \subset \text{Range}\, W_R(t_0,t_1)$

Let $x_1 \in R[t_0,t_1]$. Then we can find some input $u(t)$ such that $x_1 = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau$. Now, choose arbitrary $\eta_1 \in \text{null}\, W_R$ (i.e. $W_R\eta_1 = 0$)

Then $$0 = \eta_1^T W_R \eta_1 = \int_{t_0}^{t_1} \eta_1^T \Phi(t_1,\tau) B(\tau) B(\tau)^T \Phi(t_1,\tau)^T \eta_1 \, d\tau$$

$$= \int_{t_0}^{t_1} \| B(\tau)^T \Phi(t_1,\tau)^T \eta_1 \|^2 d\tau$$

$$\Rightarrow B(t)^T \Phi(t_1,t)^T \eta_1 = 0 \quad \forall t \in [t_0,t_1]$$

Therefore $$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T \underbrace{B(\tau)^T \Phi(t_1,\tau)^T \eta_1}_{0} \, d\tau = 0$$

which implies x_1 is orthogonal to η_1. Since $\eta_1 \in \text{null}\, W_R$ was arbitrary we have

$$x_1 \in (\text{null}\, W_R)^\perp = \text{range}\, W_R^T = \text{range}\, W_R .$$

▨

Likewise, we can also establish:

S: subspace of $\mathbb{R}^n$

$S^\perp = \{x \in \mathbb{R}^n : \langle x, y\rangle = 0 \ \ \forall y \in S\}$

Theorem Given two times $t_1 > t_0$ we have $\mathcal{C}[t_0, t_1] = \text{Range}\{W_c(t_0,t_1)\}$. Moreover, if $x_0 \in \text{Range}\{W_c(t_0,t_1)\}$ with $x_0 = W_c(t_0,t_1)\eta_0$ then the control input $u(t) = -B(t)^T \Phi(t_0,t)^T \eta_0$ for $t \in [t_0, t_1]$ can be used to transfer the state from $x(t_0) = x_0$ to $x(t_1) = 0$.

proof: Exercise.

LTI case

system $\dot{x} = Ax + Bu \qquad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^k)$

The gramians boil down to

$$W_R(t_0,t_1) = \int_{t_0}^{t_1} e^{A(t_1-\tau)} BB^T e^{A^T(t_1-\tau)} d\tau = \int_0^{t_1-t_0} e^{At} BB^T e^{A^Tt} dt$$

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} e^{A(t_0-\tau)} BB^T e^{A^T(t_0-\tau)} d\tau = \int_0^{t_1-t_0} e^{-At} BB^T e^{-A^Tt} dt$$

We define the controllability matrix $\mathcal{C} \in \mathbb{R}^{n\times(nk)}$ as

$$\mathcal{C} := [B \ AB \ A^2B \ \dots \ A^{(n-1)}B]$$

Theorem For any two times $t_1 > t_0$ we have

$$R[t_0,t_1] = \text{Range } W_R(t_0,t_1) = \text{Range } \mathcal{C} = \text{Range } W_c(t_0,t_1) = \mathcal{C}[t_0,t_1]$$

Proof: $R[t_0,t_1] \subset \text{Range } \mathcal{C}$

Let $x_1 \in R[t_0,t_1]$. Then we can find control input $u(t)$ such that $x_1 = \int_{t_0}^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$. By Cayley-Hamilton Thm there exist scalar functions $\alpha_i(t)$ such that $e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$

Hence,

$$x_1 = \underbrace{[B \;\; AB \;\; \cdots \;\; A^{n-1}B]}_{C} \begin{bmatrix} \int_{t_0}^{t_1} \alpha_0(t_1-\tau)u(\tau)d\tau \\ \vdots \\ \int_{t_0}^{t_1} \alpha_{n-1}(t_1-\tau)u(\tau)d\tau \end{bmatrix}_{nk\times 1}$$

$\Rightarrow x_1 \in \text{Range } C$

<u>Range $C \subset \mathbb{R}[t_0,t_1]$</u> Let $x_1 \in$ Range C. Then we can find $v \in \mathbb{R}^{nk}$ such that $x_1 = Cv$. Choose an arbitrary $\eta_1 \in$ null $W_R(t_0,t_1)$. Then

$$0 = \eta_1^T W_R \eta_1 = \int_0^{t_1-t_0} \eta_1^T e^{At} BB^T e^{A^Tt} \eta_1 \, dt = \int_0^{t_1-t_0} \| B^T e^{A^Tt} \eta_1 \|^2 dt$$

$\Rightarrow \eta_1^T e^{At} B = 0$

$\Big\downarrow \frac{d}{dt}\{\cdots\}$

$\Rightarrow \eta_1^T A e^{At} B = 0$

$\Big\downarrow \frac{d}{dt}\{\cdots\}$

$\Rightarrow \eta_1^T A^2 e^{At} B = 0$

$\vdots$

$\Rightarrow \eta_1^T A^{n-1} e^{At} = 0$

at $t=0$ $\left(e^{At}\big|_{t=0} = I\right)$ we can write

$$\left.\begin{aligned} \eta_1^T B &= 0 \\ \eta_1^T AB &= 0 \\ &\vdots \\ \eta_1^T A^{n-1}B &= 0 \end{aligned}\right\} \eta_1^T \underbrace{[B \;\; AB \cdots A^{n-1}B]}_{C} = 0$$

Hence we've obtained $\eta_1^T C = 0$

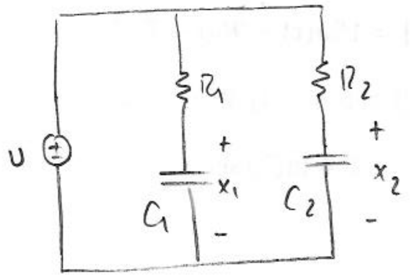
Now, $\eta_1^T x_1 = \eta_1^T C v = 0$

$\Rightarrow x_1 \in \left(\text{null } W_R(t_0,t_1)\right)^\perp$

$\Rightarrow x_1 \in \text{Range } W_R(t_0,t_1)^T = \text{Range } W_R(t_0,t_1)$ ▨

[To show Range $C = C[t_0,t_1]$ we follow similar steps.]

Example [Parallel RC network revisited]



$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

controllability matrix $\mathcal{C} = [B \;\; AB] = \begin{bmatrix} \frac{1}{R_1C_1} & -\frac{1}{(R_1C_1)^2} \\ \frac{1}{R_2C_2} & -\frac{1}{(R_2C_2)^2} \end{bmatrix}$

$$\det \mathcal{C} = -\frac{1}{R_1C_1(R_2C_2)^2} + \frac{1}{R_2C_2(R_1C_1)^2} = \frac{1}{R_1C_1R_2C_2}\left(\frac{1}{R_1C_1} - \frac{1}{R_2C_2}\right)$$

Different time constants $R_1C_1 \neq R_2C_2 \iff \det \mathcal{C} \neq 0 \iff \text{Range}\, \mathcal{C} = \mathbb{R}^2$

Conclusion: If $R_1C_1 \neq R_2C_2$, for all $x_0, x_f \in \mathbb{R}^2$ and $t_f > t_0$ we can find an input voltage $u(t)$ that steers the capacitor voltage vector from $x(t_0) = x_0$ to $x(t_f) = x_f$.

CONTROLLABLE SYSTEMS

Definition (Reachable system) Let $t_1 > t_0$. The system $\dot{x} = A(t)x + B(t)u$ (with $x \in \mathbb{R}^n$) [or the pair $(A(\cdot), B(\cdot))$] is said to be reachable on the interval $[t_0, t_1]$ if $\mathcal{R}[t_0, t_1] = \mathbb{R}^n$, i.e., if the origin can be transfered to any point in $\mathbb{R}^n$.

Definition (Controllable system) The system $\dot{x} = A(t)x + B(t)u$ is said to be controllable on the interval $[t_0, t_1]$ if $\mathcal{C}[t_0, t_1] = \mathbb{R}^n$, i.e., if every state can be transfeed to the origin.

LTI Case

System: $\dot{x} = Ax + Bu$ (1) $(x \in \mathbb{R}^n)$

Previously established: $\text{Range}\, \mathcal{C} = \mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1]$ for any $t_1 > t_0$

where $\mathcal{C} = [B \ AB \ \cdots \ A^{n-1}B]$ is the controllability matrix

Note that system (1) controllable $\Leftrightarrow \mathcal{C}[t_0, t_1] = \mathbb{R}^n$

$\Leftrightarrow \text{Range}\, \mathcal{C} = \mathbb{R}^n$

$\Leftrightarrow \text{rank}\, \mathcal{C} = n$

Hence,

Theorem: System (1), or the pair (A, B), is controllable if and only if $\text{rank}\, \mathcal{C} = n$.

Another controllability test:

Theorem: (Popov-Belevitch-Hautus (PBH) Test). Pair $(\overset{n\times n}{A}, \overset{n\times k}{B})$ is controllable if and only if

$$\text{rank}\, [A - \lambda I \quad B] = n \qquad \forall \lambda \in \mathbb{C} \qquad (1)$$

proof: From the previous thm. we know that controllability is equivalent to

$\operatorname{rank}[B \; AB \; \cdots \; A^{n-1}B] = n. \quad (2)$

Therefore we have to show $(1) \Leftrightarrow (2)$.

Part I: $(2) \Rightarrow (1)$. Suppose not. That is, (2) holds but (1) doesn't.

$[A-\lambda I \;\; B]$ has n rows. Hence if $\operatorname{rank}[A-\lambda I \;\; B] \neq n$ for some λ then the rows must be lin. dep. $\Rightarrow$ columns of $\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix}$ must be lin. dep.

$\Rightarrow$ we can find a nonzero $v \in \mathbb{C}^n$ such that

$$\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} v = 0 \Rightarrow \underbrace{A^T v = \lambda v}_{\substack{v \text{ is an eigenvector} \\ \text{of } A^T}} \quad \& \quad \underbrace{B^T v = 0}_{v \text{ belongs to } \operatorname{null}(B^T)}$$

Now, observe that

$$C^T v = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T A^{(n-1)T} \end{bmatrix} v = \begin{bmatrix} B^T v \\ B^T A^T v \\ \vdots \\ B^T A^{(n-1)T} v \end{bmatrix} = \begin{bmatrix} B^T v \\ \lambda B^T v \\ \vdots \\ \lambda^{n-1} B^T v \end{bmatrix} = 0$$

$\Rightarrow \operatorname{rank} C \neq n \Rightarrow$ contradiction.

Part II: $(1) \Rightarrow (2)$. Suppose not. $\operatorname{rank} C \neq n \Rightarrow \exists$ nonzero $v \in \mathbb{R}^n$ such that

$$C^T v = 0 \Rightarrow \left.\begin{aligned} B^T v &= 0 \\ B^T A^T v &= 0 \\ &\vdots \\ B^T A^{(n-1)T} v &= 0 \end{aligned}\right\} \Rightarrow \text{Define } S := \operatorname{span}\{v, A^T v, \ldots, A^{(n-1)T} v\}$$

Observe:
- $\rightarrow S \subset \operatorname{null}(B^T)$
- $\rightarrow S$ is invariant under A^T (by Cayley-Hamilton Thm.)

Let now $V \in \mathbb{R}^{n \times \ell}$ be a matrix whose columns make a basis for S. Since S is invariant under A^T we can find $\Lambda \in \mathbb{R}^{\ell \times \ell}$ such that

$$A^T V = V \Lambda \quad (3)$$

Let $w \in \mathbb{C}^\ell$ be an eigenvector of Λ. That is $\Lambda w = \alpha w$ for some $\alpha \in \mathbb{C}$

Then $(3) \Rightarrow \left.\begin{aligned} A^T V w &= V \Lambda w \\ &= V \alpha w \\ &= \alpha (V w) \end{aligned}\right\} \Rightarrow$ $z := Vw \in \mathbb{C}^n$ is an eigenvector of A^T.
Moreover, $z \in S \Rightarrow z \in \text{null}(B^T)$

Finally, $\begin{bmatrix} A^T - \alpha I \\ B^T \end{bmatrix} z = \begin{bmatrix} (A^T - \alpha I) z \\ B^T z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{rank}\,[A - \alpha I \;\; B] \neq n$

$\Rightarrow$ contradiction. ∎

Remark: We can restate the PBH test as: "System $\dot{x} = Ax + Bu$ is controllable if and only if there is no eigenvector of A^T in $\text{null}(B^T)$."
Hence, PBH test is also called the eigenvector test for controllability.

Yet another controllability test (for exp. stable systems):

Theorem (Lyapunov test) Let the system

$$\dot{x} = Ax + Bu \quad (1)$$

be exponentially stable. Then the system (1) is controllable if and only if there exists $P = P^T \succ 0$ that solves the following Lyapunov eqn.

$$AP + PA^T = -BB^T. \qquad (APA^T - P = -BB^T \text{ for DT})$$

Moreover, this P is unique & given by $P = \int_0^\infty e^{At} BB^T e^{A^T t}\, dt$. ($P = ?$ for DT)

Proof. Part I Let (A, B) be controllable.

$\rightarrow$ does $P = \int_0^\infty e^{At} BB^T e^{A^T t}\, dt$ satisfy Lyap eqn? YES (See text.)

$\rightarrow$ is P unique? YES (See text.)

} compare with the proof of Lyapunov Thm for exp. stability.

$\rightarrow$ is P positive definite? YES

Suppose not. Then we can find a nonzero $x \in \mathbb{R}^n$ such that $x^T P x \leq 0$

$$\Rightarrow \int_0^\infty x^T e^{At} BB^T e^{A^T t} x\, dt \leq 0 \Rightarrow \int_0^\infty \| x^T e^{At} B \|^2 dt \leq 0$$

$$\Rightarrow x^T e^{At} B = 0 \quad \forall t \geq 0$$

$$\Rightarrow x^T e^{At} B \Big|_{t=0} = 0 \Rightarrow x^T B = 0$$

$$\Rightarrow \frac{d}{dt} \left\{ x^T e^{At} B \right\}_{t=0} \Rightarrow x^T A B = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\Rightarrow \frac{d^{n-1}}{dt^{n-1}} \left\{ x^T e^{At} B \right\}_{t=0} \Rightarrow x^T A^{n-1} B = 0$$

$$x^T [B \;\; AB \; \cdots \; A^{n-1}B] = 0$$

$$\Rightarrow \text{rank}\, [B \;\; AB \; \cdots \; A^{n-1}B] \neq n$$

$\Rightarrow$ (A,B) not controllable

$\Rightarrow$ contradiction.

Part II : Let $AP + PA^T = -BB^T$ has a $P = P^T > 0$ solution. Let x be an eigenvector of A^T. That is, $A^T x = \lambda x$ for some $\lambda \in \mathbb{C}$. Then,

$$
\begin{aligned}
-\|B^T x\|^2 &= -x^* B B^T x \qquad (x^* : \text{conjugate transpose of } x) \\
&= x^* (AP + PA^T) x \\
&= (A^T x)^* P x + x^* P A^T x \\
&= \lambda^* x^* P x + \lambda x^* P x \\
&= 2\,\text{Re}\{\lambda\} \underbrace{x^* P x}_{>0}
\end{aligned}
$$

Finally: A exp. stable $\Rightarrow \text{Re}\{\lambda\} < 0 \Rightarrow B^T x \neq 0 \Rightarrow (A,B)$ controllable. ▨

Remark: Note that all the above implications "$\Rightarrow$" can be replaced by equivalences "$\Leftrightarrow$". Therefore, if we had that (A,B) is controllable then we could establish that A is exp. stable. In other words:

Theorem: Let (A,B) controllable. Then A exp. stable if and only if the Lyap eqn. $AP + PA^T + BB^T = 0$ has a $P = P^T > 0$ solution.

An application of the Lyapunov test

Given a controllable pair (A, B) observe the following:

1) For $\mu \in \mathbb{R}$, pair $(-\mu I - A, B)$ is also controllable. Because of

$$\left\{ A^T v = \lambda v \iff (-\mu I - A)^T v = (-\mu - \lambda) v \right\} \ \& \ \left\{ \text{eigenvector test for controllability} \right\}$$

2) By choosing large enough μ we can make $Re\{\lambda_i(-\mu I - A)\} < 0$ for all i.

— o —

Now, choose $\mu > 0$ large enough so that $-\mu I - A$ is exp. stable. Then

$$[-\mu I - A]V + V[-\mu I - A]^T = -BB^T \quad \text{has a sol'n} \quad V = V^T > 0 \quad \text{(Lyap. Test)}$$

$$\Rightarrow \quad AV + VA^T - BB^T = -2\mu V$$

$P := V^{-1}$

$$\Rightarrow \quad P(AV + VA^T - BB^T)P = -2\mu PVP$$

$$\Rightarrow \quad PA + A^T P - PBB^T P = -2\mu P$$

$K := \frac{1}{2} B^T P$

$$\Rightarrow \quad P(A - BK) + (A - BK)^T P = -2\mu P$$

Since $P > 0$ and $\mu P > 0$ by Lyapunov stability thm. we deduce that the system $\dot{x} = [A - BK]x$ is exp. stable. This implies that the system $\dot{x} = Ax + Bu$ can be stabilized by feedback $u = -Kx$.

Exercise: Show that with this feedback, the closed-loop solutions satisfy $\|x(t)\| \leq c\, e^{-\mu t} \|x(0)\|$ for some $c > 0$.

CONTROLLABLE DECOMPOSITION

Example : Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

→ Is this system stable ?

$$|sI - A| = \begin{vmatrix} s & -1 \\ -6 & s+1 \end{vmatrix} = s(s+1) - 6 = (s+3)(s-2) \qquad \lambda_2 = 2 > 0 \Rightarrow \text{unstable}$$

→ Is this system controllable ?

$$[B \ \ AB] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \text{rank } [B \ \ AB] \neq 2 \Rightarrow \text{uncontrollable}$$

→ Can we still stabilize this system with feedback $u = -Kx$?

$$\dot{x} = Ax + Bu \Big|_{u=-Kx} \Rightarrow \dot{x} = \underbrace{[A - BK]}_{A_{cl}} x \quad , \quad K = [k_1 \ \ k_2]$$

Eigenvalues of the closed-loop system ?

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} [k_1 \ \ k_2] = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \\ 6-2k_1 & -1-2k_2 \end{bmatrix}$$

$$|sI - A_{cl}| = \begin{vmatrix} s+k_1 & k_2-1 \\ 2k_1-6 & s+2k_2+1 \end{vmatrix} = s^2 + [k_1 + 2k_2 + 1]s + [3k_1 + 6k_2 - 6]$$

$$= \underbrace{(s+3)}_{\substack{\text{nothing can be} \\ \text{done about this} \\ \text{eigenvalue}}} \underbrace{(s + k_1 + 2k_2 - 2)}_{\substack{\text{we can place } \lambda_2^{cl} \\ \text{as we wish}}}$$

⇒ choose , for instance , $K = [1 \ \ 1]$ ⇒ $|sI - A_{cl}| = (s+3)(s+1)$

$$\Rightarrow \lambda_1^{cl} = \lambda_1^{\text{open-loop}} = -3 \ , \ \lambda_2^{cl} = -1 \Rightarrow \text{exp. stability}$$

Even, though (A,B) is uncontrollable we can still make $x(t) \to 0$ as $t \to \infty$

Such pair (A,B) is called stabilizable . To study stabilizability we first have to go through "controllable decomposition".

Consider the systems

(1) $\dot{x} = Ax + Bu$ $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$

(2) $\dot{z} = \tilde{A}z + \tilde{B}u$ $z \in \mathbb{R}^n$

Suppose that sys (1) & sys (2) are algebraically equivalent. That is, there exists a similarity transformation $T \in \mathbb{R}^{n\times n}$ ($z = T^{-1}x$) such that

$\tilde{A} = T^{-1}AT$ & $\tilde{B} = T^{-1}B$

<u>Claim</u> : Sys (1) is controllable if and only if Sys (2) is controllable.

<u>proof</u> : Let us construct the controllability matrices.

$$C = [B \ AB \ \cdots \ A^{n-1}B]$$

$$\tilde{C} = [\tilde{B} \ \tilde{A}\tilde{B} \ \ldots \ \tilde{A}^{n-1}\tilde{B}] = [T^{-1}B \ \ (T^{-1}AT)T^{-1}B \ \ldots \ (T^{-1}AT)^{n-1}T^{-1}B]$$

$(T^{-1}AT)^k = T^{-1}A^kT$

$$= [T^{-1}B \ \ T^{-1}AB \ \ldots \ T^{-1}A^{n-1}B]$$

$$= T^{-1}[B \ AB \ \ldots \ A^{n-1}B] = T^{-1}C$$

Then $\operatorname{rank} \tilde{C} = \operatorname{rank} T^{-1}C = \operatorname{rank} C$

[This says that the dimension of the controllable subspaces $S_c := \operatorname{range} C$ & $\tilde{S}_c := \operatorname{range} \tilde{C}$ are equal.]

Finally, Sys(1) cont. $\Leftrightarrow$ $\operatorname{rank} C = n$ $\Leftrightarrow$ $\operatorname{rank} \tilde{C} = n$ $\Leftrightarrow$ Sys(2) cont. ▨

Now, for the sys. $\dot{x} = Ax + Bu$ we will construct the so-called controllable decomposition.

Let $S_c = \operatorname{range} C$ be the cont. subspace

& $\bar{n} := \dim S_c$ ($= \operatorname{rank} C$)

Observe : $\rightarrow S_c$ is invariant under A

$\rightarrow S_c \supset \operatorname{range}(B)$

Let $V \in \mathbb{R}^{n\times\bar{n}}$ be such that its columns form a basis for S_c. By invariance we can find $A_c \in \mathbb{R}^{\bar{n}\times\bar{n}}$ such that

$$AV = VA_c$$

Also, since range $(B) \subset S_c$, we can find $B_c \in \mathbb{R}^{\bar{n}\times k}$ satisfying

$$B = VB_c$$

Now, if $\bar{n} = n$ then let $T = V$. Otherwise let $U \in \mathbb{R}^{n\times(n-\bar{n})}$ be such that its columns together with the columns of V span $\mathbb{R}^n$ and define

$$T := [V \ \ U]_{n\times n} \qquad (\text{note that } T^{-1} \text{ exists})$$

We can write

$$AT = A[V \ \ U] = [AV \ \ AU] = [VA_c \ \ TT^{-1}AU]$$

$$= \left[[V \ \ U]\begin{bmatrix} A_c \\ 0 \end{bmatrix} \ \ TT^{-1}AU \right]$$

$$= T\left[\begin{bmatrix} A_c \\ 0 \end{bmatrix} \ \ T^{-1}AU \right]$$

Let us partition

$$\underbrace{T^{-1}AU}_{n\times(n-\bar{n})} = \begin{bmatrix} A_{12} \\ A_u \end{bmatrix}\begin{array}{l} {\scriptstyle \bar{n}\times(n-\bar{n})} \\ {\scriptstyle (n-\bar{n})\times(n-\bar{n})} \end{array}$$

Then $AT = T\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}$ and $B = VB_c = [V \ \ U]\begin{bmatrix} B_c \\ 0 \end{bmatrix}\begin{array}{l} {\scriptstyle \bar{n}\times k} \\ {\scriptstyle (n-\bar{n})\times k} \end{array} = T\begin{bmatrix} B_c \\ 0 \end{bmatrix}$

From which

$$\underbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = T^{-1}AT \quad \& \quad \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1}B}_{\text{controllable decomposition}}$$

<u>Exercise</u>: Obtain the controllable decomp. for $\dot{x} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}x + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$. $\left(T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right.$ arbitrary!

Theorem (controllable decomposition) for each system

(1) $\dot{x} = Ax + Bu \quad , \quad x \in \mathbb{R}^n \quad , \quad u \in \mathbb{R}^k$

whose controllable subspace has dimension $\bar{n}$, there is a similarity transformation ($z = T^{-1}x$) $T \in \mathbb{R}^{n\times n}$ that transforms the sys (1) into

(2) $$\dot{z} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} z + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u \quad , \quad A_c \in \mathbb{R}^{\bar{n}\times\bar{n}} \quad , \quad B_c \in \mathbb{R}^{\bar{n}\times k}$$

for which

(a) the cont. subspace of the trans. sys. (2) is

$$\tilde{S}_c = \text{range} \begin{bmatrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{bmatrix} \quad \text{and}$$

(b) the pair (A_c, B_c) is controllable.

proof: Let us form the controllability matrix for (2)

$$\tilde{C} = \left[\begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}^{n-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right] = \begin{bmatrix} B_c & A_cB_c & \cdots & A_c^{n-1}B_c \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\Rightarrow \underbrace{\text{range}\,\tilde{C}}_{\dim = ?} \subset \underbrace{\text{range} \begin{bmatrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{bmatrix}}_{\dim = \bar{n}} \quad (3)$$

$$\begin{aligned} \dim \text{range}\,\tilde{C} &= \dim \text{range}\, T^{-1}C \\ &= \text{rank}\, T^{-1}C \\ &= \text{rank}\, C \\ &= \bar{n} \quad (4) \end{aligned}$$

$$(3) \,\&\, (4) \Rightarrow \tilde{S}_c = \text{range} \begin{bmatrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{bmatrix}$$

Also, $$\begin{aligned} \bar{n} = \text{rank}\,\tilde{C} &= \text{rank}\,[B_c \;\; A_cB_c \;\cdots\; A_c^{n-1}B_c] \\ &= \text{rank}\,[B_c \;\; A_cB_c \;\cdots\; A_c^{\bar{n}-1}B_c] \end{aligned}$$ by Cayley-Hamilton Thm.

$\Rightarrow$ the pair (A_c, B_c) is controllable.

∎

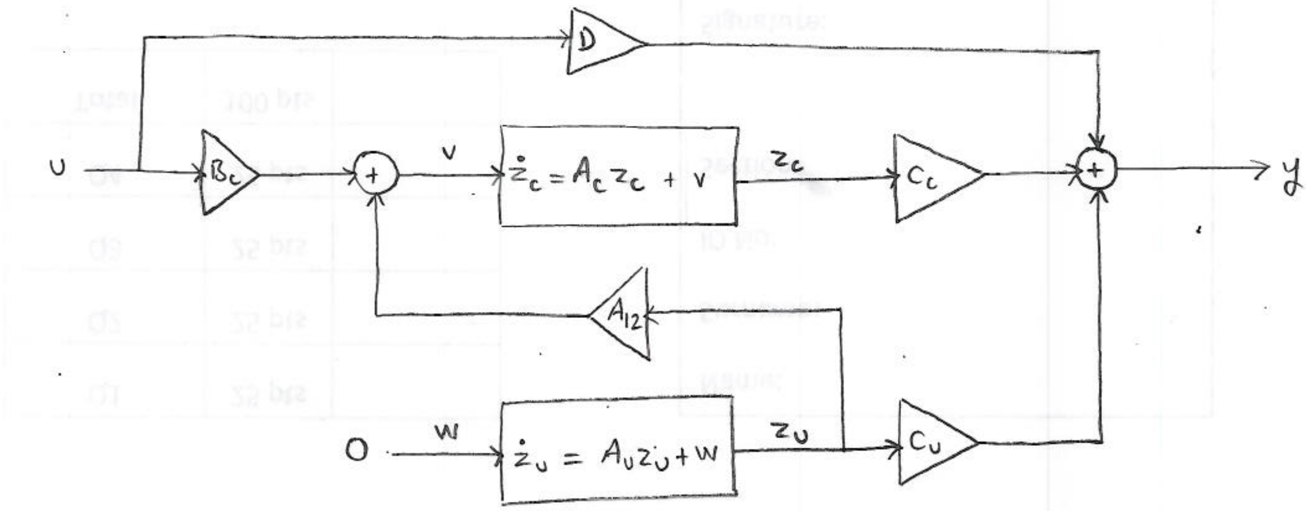
Block diagram interpretation

System : $\left.\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}\right\} \xrightarrow[\text{cont. decomp.}]{z = T^{-1}x}$ $\begin{bmatrix} \dot{z}_c \\ \dot{z}_u \end{bmatrix} = \overbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}}^{\tilde{A}} \begin{bmatrix} z_c \\ z_u \end{bmatrix} + \overbrace{\begin{bmatrix} B_c \\ 0 \end{bmatrix}}^{\tilde{B}} u$

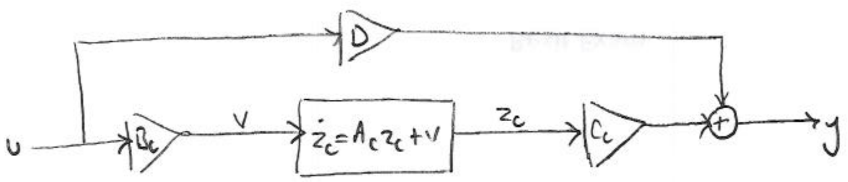
$$y = \underbrace{\begin{bmatrix} C_c & C_u \end{bmatrix}}_{\tilde{C}} \begin{bmatrix} z_c \\ z_u \end{bmatrix} + Du$$

where $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$, $\tilde{C} = CT$

Block diagram for the cont. decomp :



Note that to obtain the transfer function $G(s)$ $\left(Y(s) = G(s)\,U(s) \right)$ we assume zero initial conditions, which implies that all we need to consider is the below block diagram.



which yields $\left.\begin{array}{l}\dot{z}_c = A_c z_c + B_c u \\ y = C_c z_c + Du\end{array}\right\} \Rightarrow G(s) = C_c (sI - A_c)^{-1} B_c + D$

That is, the transfer function of $\left\{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}\right\}$

equals the TF of its controllable part (A_c, B_c, C_c, D).

STABILIZABILITY

There are systems that are uncontrollable but still one can make their state converge to the origin for all initial conditions. Such systems are called stabilizable.

Consider the system

$$(1)\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ y \in \mathbb{R}^m$$

with controllable decomposition

$$(2)\begin{cases} \begin{bmatrix} \dot{z}_c \\ \dot{z}_u \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}\begin{bmatrix} z_c \\ z_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} C_c & C_u \end{bmatrix}\begin{bmatrix} z_c \\ z_u \end{bmatrix} + Du \end{cases} \qquad z_c \in \mathbb{R}^{\bar{n}},\ z_u \in \mathbb{R}^{n-\bar{n}}$$

Definition: system (1) is said to be stabilizable if either $n=\bar{n}$ (that is, A_u nonexistent) or A_u is a stability matrix; that is, $Re\{\lambda_i(A_u)\} < 0 \ \forall i$.

(How about the discrete-time case?)

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Note that $\dot{z}_u = A_u z_u \Rightarrow z_u(t) \to 0$ as $t \to \infty$ if A_u is exp. stable

$$\Rightarrow \dot{z}_c = A_c z_c + \underbrace{A_{12} z_u}_{\text{this term decays exponentially}} + B_c u \quad \Rightarrow \quad \dot{z}_c \cong A_c z_c + B_c u \text{ after some time}$$

Now, pair (A_c, B_c) controllable $\Rightarrow$ $z_c(t)$ can be made $\to 0$ by some $u(t)$

Then $\begin{bmatrix} z_c(t) \\ z_u(t) \end{bmatrix} \to 0$. Sys(1) & Sys(2) algebraically equiv. $\Rightarrow$ $x(t) = \underbrace{T}_{n\times n}\begin{bmatrix} z_c(t) \\ z_u(t) \end{bmatrix}$

$\Rightarrow$ $x(t) \to 0$. In other words, if sys. (1) is stabilizable then (and only then) we can always make $x(t) \to 0$ as $t \to \infty$

Q: Can one check stabilizability without obtaining the cont. decomp.?

A: Below theorem.

Theorem [Eigenvector test for stabilizability] Consider the system

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k \qquad (1)$$

System (1) is stabilizable if and only if

"No eigenvector of A^T with eigenvalue $Re\{\lambda\} \geq 0$ belongs to null (B^T)." (E)

($|\lambda| \geq 1$ for DT)

Proof: Let $T \in \mathbb{R}^{n \times n}$ be a transformation that puts (1) into cont. decomp.

$$\bar{A} := \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = T^{-1} A T \quad \& \quad \bar{B} := \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1} B$$

Stabilizability => (E): Suppose not. That is, the system (1) is stabilizable, but $\exists\, v \in \mathbb{C}^n$ $(v \neq 0)$ & $\lambda \in \mathbb{C}$ with $Re\{\lambda\} \geq 0$ such that $A^T v = \lambda v$ & $B^T v = 0$.

$$\bar{A} = T^{-1} A T \Rightarrow T\bar{A} = AT \Rightarrow \bar{A}^T T^T = T^T A^T \Rightarrow \bar{A}^T T^T v = T^T A^T v$$
$$= \lambda T^T v \quad (2)$$

Define $\begin{bmatrix} w_c \\ w_u \end{bmatrix} = w := T^T v$

$$(2) \Rightarrow \bar{A}^T w = \lambda w \Rightarrow \overbrace{\begin{bmatrix} A_c^T & 0 \\ A_{12}^T & A_u^T \end{bmatrix} \begin{bmatrix} w_c \\ w_u \end{bmatrix} = \begin{bmatrix} \lambda w_c \\ \lambda w_u \end{bmatrix}}^{(*)} \Rightarrow A_c^T w_c = \lambda w_c \quad (3)$$

Also, $B^T v = 0 \Rightarrow B^T T^{-T} w = 0 \Rightarrow (T^{-1} B)^T w = 0 \Rightarrow \bar{B}^T w = 0 \Rightarrow B_c^T w_c = 0 \quad (4)$

Since (A_c, B_c) is controllable, (3) & (4) $\Rightarrow w_c = 0 \Rightarrow w_u \neq 0$

Then $(*) \Rightarrow A_u^T w_u = \lambda w_u \Rightarrow \lambda$ is an eigenvalue of A_u.

Stabilizability $\Rightarrow Re\{\lambda\} < 0 \Rightarrow$ contradiction.

(E) => stabilizability: Suppose not. That is, (E) holds but A_u has an eigenvalue λ satisfying $Re\{\lambda\} \geq 0$. Then we can find $0 \neq x_u \in \mathbb{C}^{n-\bar{n}}$ such that

$$A_u^T x_u = \lambda x_u$$

Then $\bar{A}^T \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} A_c^T & 0 \\ A_{12}^T & A_u^T \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} 0 \\ A_u^T x_u \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ x_u \end{bmatrix}$ (1)

and $\bar{B}^T \begin{bmatrix} 0 \\ x_u \end{bmatrix} = [B_c^T \;\; 0] \begin{bmatrix} 0 \\ x_u \end{bmatrix} = 0$ (2)

Define $z := T^{-T} \begin{bmatrix} 0 \\ x_u \end{bmatrix}$

$\Rightarrow A^T z = (T\bar{A}T^{-1})^T z = T^{-T}\bar{A}^T T^T z = T^{-T}\bar{A}^T \begin{bmatrix} 0 \\ x_u \end{bmatrix} \overset{(1)}{=} T^{-T}\left(\lambda \begin{bmatrix} 0 \\ x_u \end{bmatrix}\right) = \lambda z$ (3)

& $B^T z = (T\bar{B})^T z = \bar{B}^T T^T z = \bar{B}^T \begin{bmatrix} 0 \\ x_u \end{bmatrix} \overset{(2)}{=} 0$ (4)

Since $Re\{\lambda\} \geq 0$, (3)&(4) contradicts (E). ▨

Note that we can restate the eigenvector test as follows

Theorem (PBH Test for stabilizability) System $\dot{x} = Ax + Bu$ $(x \in \mathbb{R}^n)$ is stabilizable if and only if $\operatorname{rank}[A - \lambda I \;\; B] = n$ for all $\lambda \in \mathbb{C}$ satisfying $Re\{\lambda\} \geq 0$.

Lyapunov Test for Stabilizability

Theorem: System $\dot{x} = Ax + Bu$ $(x \in \mathbb{R}^n)$ is stabilizable if and only if there is $P = P^T > 0$ $(P \in \mathbb{R}^{n \times n})$ that solves the following Lyap. matrix ineq.

$$AP + PA^T - BB^T < 0. \quad (1) \qquad (APA^T - P - BB^T < 0 \text{ for DT})$$

Proof: Part I (1) $\Rightarrow$ stabilizability Suppose $\exists P = P^T > 0$ satisfying (1).

Let $x \in \mathbb{C}^n$ be an eigenvector of A^T satisfying

$$A^T x = \lambda x \qquad \text{with } Re\{\lambda\} \geq 0$$

Then $\|B^T x\|^2 = x^* B B^T x$

$$> x^*(AP + PA^T)x$$

$$= (A^T x)^* P x + x^* P (A^T x)$$

$$= \lambda^* (x^* P x) + \lambda (x^* P x)$$

$$= \underbrace{2\mathrm{Re}\{\lambda\}}_{\geq 0} \cdot \underbrace{(x^* P x)}_{> 0} \qquad (2)$$

$(2) \Rightarrow \|B^T x\| > 0 \Rightarrow B^T x \neq 0 \Rightarrow$ system stabilizable (by eigenvector test)

Part II Stabilizability $\Rightarrow$ (1) see the text.

Feedback Stabilization based on Lyapunov Test

Let $\dot{x} = Ax + Bu$ $\quad (x \in \mathbb{R}^n, u \in \mathbb{R}^k)$ be stabilizable.

Then $\exists\, P = P^T > 0$ such that $AP + PA^T - BB^T < 0$

Define $K := \frac{1}{2} B^T P^{-1}$ $\quad (K \in \mathbb{R}^{k \times n})$

Claim: $u = -Kx$ is a stabilizing feedback law. That is, the closed-loop system $\dot{x} = [A - BK]x$ is exp. stable.

Because: $\quad 0 > AP + PA^T - BB^T$

$$= AP + PA^T - \tfrac{1}{2} P P^{-1} B B^T - \tfrac{1}{2} B B^T P^{-1} P$$

$$= [A - \tfrac{1}{2} B B^T P^{-1}] P + P [A - \tfrac{1}{2} B B^T P^{-1}]^T$$

$$= [A - BK] P + P [A - BK]^T \qquad (1)$$

Ineq. (1) implies by Lyap. Stability Thm. that $[A - BK]^T$ is exp. stable. Hence $[A - BK]$ is exp. stable. ☒

Incidentally, we've proven the half of the next theorem:

Theorem: (A, B) stabilizable $\Leftrightarrow$ $\exists\, K$ such that $\dot{x} = [A - BK]x$ exp. stable.

When the system is not just stabilizable, but controllable, then a lot more is possible:

Theorem: Suppose the system $\dot{x} = Ax + Bu$ $(x \in \mathbb{R}^n, u \in \mathbb{R}^k)$ is controllable. Then for all sets of complex numbers $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, where nonreal λ_i's appear as conjugate pairs, we can find a feedback gain $K \in \mathbb{R}^{k \times n}$ such that $|sI - [A - BK]| = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$. This is called pole placement or eigenvalue assignment.

Example Consider

$$\dot{x} = \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

a) Is this system stable?
b) Is this system controllable/stabilizable?
c) Obtain the controllable decomp.

d) Find, if possible, a feedback gain $K = [k_1 \ k_2]$ so that the feedback law $u = -Kx$ renders the closed-loop system $\dot{x} = [A - BK]x$ exp. stable.

Sol'n: Stability : $|sI - A| = \begin{vmatrix} s+5 & -3 \\ 6 & s-4 \end{vmatrix} = (s+5)(s-4) + 18 = (s-1)(s+2)$

$\Rightarrow \lambda_1 = 1, \ \lambda_2 = -2$. $\quad Re\{\lambda_1\} > 0 \Rightarrow$ unstable.

Controllability: Obtain controllability matrix

$C = [B \ \ AB]$, $\quad AB = \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow \text{rank } C = 1$

$\text{rank } C \neq n = 2 \quad \Rightarrow$ uncontrollable

Stabilizability: Eigenvector test. Obtain $v_1 \in \mathbb{R}^2$ satisfying $A^T v_1 = \lambda_1 v_1$ where $\lambda_1 = 1$ is the unstable eigenvalue.

$[\lambda_1 I - A^T] v_1 = 0 \quad \Rightarrow \begin{bmatrix} 6 & 6 \\ -3 & -3 \end{bmatrix} v_1 = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Is $v_1 \in \text{null}(B^T)$? $\qquad B^T v_1 = [1 \ \ 2]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \neq 0 \quad \Rightarrow v_1 \notin \text{null}(B^T)$

$\Rightarrow$ stabilizable.

Controllable decomp.

$$\text{range}\, C = \text{range}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \Rightarrow v := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T = [v \;\; u] \;,\; \text{let } u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \;\&\; T^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$z := T^{-1}x \Rightarrow \dot{z} = T^{-1}(Ax + Bu) = \underbrace{[T^{-1}AT]}_{\bar{A}} z + \underbrace{[T^{-1}B]}_{\bar{B}} u$$

$$\Rightarrow \bar{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}}_{\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix}}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \;\&\; \bar{B} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\dot{z} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u} \quad \text{cont. decomp.}$$

Stabilizing feedback : Use cont. decomp.

$$\text{Let } u = -\bar{k}z = -[\bar{k}_1 \;\; \bar{k}_2]z \Rightarrow \bar{A} - \bar{B}\bar{k} = \begin{bmatrix} 1-\bar{k}_1 & 3-\bar{k}_2 \\ 0 & -2 \end{bmatrix}$$

$$\text{Let } \bar{k}_1 = 2 \;\&\; \bar{k}_2 = 0 \Rightarrow \bar{A} - \bar{B}\bar{k} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \Rightarrow \text{eigenvalues } \bar{\lambda}_{1,2} = -1, -2$$

$\Rightarrow$ exp. stable

$$u = -\bar{k}z = -\bar{k}T^{-1}x = -kx \Rightarrow k = \bar{k}T^{-1} \Rightarrow k = \underbrace{[2 \;\; 0]}_{\bar{k}}\underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_{T^{-1}} = \boxed{[2 \;\; 0]}$$

Now, go check $\lambda_{1,2}$ of $[A - Bk]$.

Remark : Since $[A-Bk] = T[\bar{A} - \bar{B}\bar{k}]T^{-1}$ we expect $\lambda_i = \bar{\lambda}_i$ for all i.

$$[A - Bk] = \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}[2 \;\; 0] = \begin{bmatrix} -7 & 3 \\ -10 & 4 \end{bmatrix} \Rightarrow |sI - [A-Bk]| = (s+1)(s+2)$$

as expected.

OBSERVABILITY

Consider the LTI system

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k, \; y \in \mathbb{R}^m$$
$$y = Cx + Du$$

In practice it is usually the case that

known: $y(t)$, $u(t)$

because: measured, it is us who determine the input

UNKNOWN: (partly known) $x(t)$

because: we cannot directly measure all the state variables $x_1, x_2, \dots, x_n$
(it is either expensive or simply impossible)

WANT: complete state information $x(t)$

WHY: For example, we want to stabilize the system via state feedback $u = -Kx$
which requires the knowledge of $x(t)$

Observability is related to the question: "By looking at the output $y(t)$ during some time interval, can we figure out the state $x(t)$?"

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UNOBSERVABLE SUBSPACE

$$\dot{x} = A(t)x + B(t)u \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k, \; y \in \mathbb{R}^m$$
$$y = C(t)x + D(t)u$$

Suppose that we have the output $y(\cdot)$ & input $u(\cdot)$ info on the interval $[t_0, t_1]$. Recall that $y(t)$ satisfies

$$\underbrace{y(t)}_{\text{known}} = \underbrace{C(t)\Phi(t,t_0)x_0}_{\substack{\text{unknown} \\ \text{because } x_0 \\ \text{is unknown}}} + \underbrace{\int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau}_{\text{known}} + \underbrace{D(t)u(t)}_{\text{known}} \qquad t \in [t_0, t_1]$$

Let's gather the knowns at one side and the unknown at the other

$$\tilde{y}(t) = C(t)\,\Phi(t,t_0)\,x_0 \qquad t \in [t_0, t_1] \qquad (1)$$

where

$$\tilde{y}(t) := y(t) - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau - D(t)u(t)$$

Eqn. (1) suggests the following definition

Definition (Unobservable subspace) Given $t_1 > t_0$, the unobservable subspace on $[t_0, t_1]$, denoted $UO[t_0,t_1]$, consists of all states $x_0 \in \mathbb{R}^n$ for which

$$C(t)\,\Phi(t,t_0)\,x_0 = 0 \qquad \forall t \in [t_0,t_1]$$

Properties: Suppose we are given $t_1 > t_0$ & input / output info $u(\cdot)$, $y(\cdot)$ on the interval $[t_0,t_1]$. Then

P1) When a particular initial state $x_0 = x(t_0)$ satisfies (1), then any init. state $x_0 + x_u$ with $x_u \in UO[t_0,t_1]$ also satisfies (1)

P2) When $UO[t_0,t_1] = \{0\}$, that is, the unobservable subspace contains only the zero vector, the initial state x_0 satisfying (1) is unique.

proof - Suppose not. Then $\exists\, \bar{x}_0 \neq x_0$ such that

$$\left.\begin{aligned} \tilde{y}(t) &= C(t)\,\Phi(t,t_0)\,\bar{x}_0 \\ \tilde{y}(t) &= C(t)\,\Phi(t,t_0)\,x_0 \end{aligned}\right\} \forall t \in [t_0,t_1] \implies C(t)\,\Phi(t,t_0)\,(x_0 - \bar{x}_0) = 0 \quad \forall t \in [t_0,t_1]$$

$$\implies x_0 - \bar{x}_0 = 0 \implies \text{contradiction.}$$

P2 motivates the below def.

Definition (observable system) Given $t_1 > t_0$, the system $\begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{cases}$ [or, simply, the pair $(C(t), A(t))$] is said to be observable if $UO[t_0,t_1] = \{0\}$.

Remark: When studying observability we can disregard $B(t)$ & $D(t)$ matrices and consider only the system $\begin{cases} \dot{x} = A(t)x \\ y = C(t)x \end{cases}$

OBSERVABILITY GRAMIAN

Given $t_1 > t_0$, for the system $\begin{cases} \dot{x} = A(t)x \\ y = C(t)x \end{cases}$, we define:

Observability Gramian : $\underbrace{W_o(t_0, t_1)}_{n\times n,\ \text{symmetric, psd}} := \int_{t_0}^{t_1} \Phi(\tau, t_0)^T C(\tau)^T C(\tau) \Phi(\tau, t_0)\, d\tau$

Theorem : Given $t_1 > t_0$, $UO[t_0, t_1] = \text{null}\ W_o(t_0, t_1)$

Proof : See the text.

This thm implies

Corollary : Given $t_1 > t_0$, for the system $\begin{cases} \dot{x} = A(t)x \\ y = C(t)x \end{cases}$ $(x \in \mathbb{R}^n)$ is observable if and only if $\text{rank}\ W_o(t_0, t_1) = n$.

LTI case

Consider the following LTI systems

(1) $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ $(x \in \mathbb{R}^n)$ & (2) $\begin{cases} \dot{z} = -A^T z + C^T v \\ w = B^T z + D^T v \end{cases}$

For sys. (1) we have

cont. gram. $W_c(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_0-\tau)} BB^T e^{A^T(t_0-\tau)} d\tau$

obs. gram. $W_o(t_0, t_1) = \int_{t_0}^{t_1} e^{A^T(\tau-t_0)} C^T C e^{A(\tau-t_0)} d\tau$

For sys. (2) we have

$$\overline{W}_c(t_0, t_1) = \int_{t_0}^{t_1} e^{A^T(\tau-t_0)} C^T C e^{A(\tau-t_0)} d\tau$$

$$\overline{W}_o(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_0-\tau)} BB^T e^{A^T(t_0-\tau)} d\tau$$

Note that $\overline{W}_c = W_o$ & $\overline{W}_o = W_c$.

Now, we can write:

$$\left.\begin{aligned} \text{sys(1) controllable} &\Leftrightarrow \operatorname{rank} W_c = n \\ &\Leftrightarrow \operatorname{rank} \overline{W}_o = n \\ &\Leftrightarrow \text{sys(2) observable} \end{aligned}\right\} \Rightarrow (A,B) \text{ cont} \Leftrightarrow (B^T, -A^T) \text{ obs} \quad (3)$$

Then, (A,B) cont $\Leftrightarrow$ $(-A,B)$ cont $\overset{(3)}{\Leftrightarrow}$ (B^T, A^T) obs.

Hence we've established $\boxed{(A,B) \text{ controllable} \Leftrightarrow (B^T, A^T) \text{ observable}}$ or, equivalently $\boxed{(C,A) \text{ observable} \Leftrightarrow (A^T, C^T) \text{ controllable}}$. Due to this peculiar relation controllability and observability are said to be dual concepts for LTI systems. Thanks to this duality many observability results are trivially obtained from their dual controllability results.

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Given the pair (C,A) define the <u>observability matrix</u> $\mathcal{O}$ as follows

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

<u>Theorem</u>: Pair $\underbrace{(C,A)}_{n \times n}$ observable $\Leftrightarrow$ $\operatorname{rank} \mathcal{O} = n$.

<u>Proof</u>: (C,A) obs. $\Leftrightarrow$ (A^T, C^T) cont $\Leftrightarrow$ $\operatorname{rank}[C^T \; A^TC^T \dots (A^T)^{n-1}C^T] = n$ $\Leftrightarrow$ $\operatorname{rank} \mathcal{O} = n$

<u>Theorem</u>: $UO[t_0, t_1] = \operatorname{null} \mathcal{O}$

<u>Proof</u>:
$$\begin{aligned}
\operatorname{null} \mathcal{O} = \operatorname{range}(\mathcal{O}^T)^\perp &= \left(\operatorname{range}[C^T \; A^TC^T \dots (A^T)^{n-1}C^T]\right)^\perp \\
&= \left(\operatorname{range}[C^T \; (-A^T)C^T \dots (-A^T)^{n-1}C^T]\right)^\perp \\
&= \operatorname{range}(\overline{W}_c)^\perp \quad (\overline{W}_c : \text{cont. gram for pair } (-A^T, C^T)) \\
&= \operatorname{null}(\overline{W}_c^T) \\
&= \operatorname{null}(\overline{W}_c) \\
&= \operatorname{null}(W_o) \quad (W_o : \text{obs. gram for pair } (C,A)) \\
&= UO[t_0, t_1]
\end{aligned}$$

<u>Remark</u>: Unobservable subspace is time independent for LTI systems.

<u>Theorem</u> [Eigenvector test] The pair (C, A) is observable iff no eigenvector of A belongs to $\text{null}(C)$.

<u>proof</u> (C, A) obs. $\iff$ (A^T, C^T) cont.

$\iff$ no eigenvector of $(A^T)^T$ belongs to $\text{null}((C^T)^T)$

Likewise,

<u>Theorem</u> [PBH Test] The pair (C, A) is observable iff

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \text{for all } \lambda \in \mathbb{C}$$

<u>Theorem</u> [Lyapunov Test] Suppose A is exp. stable. Then the pair (C, A) is observable iff there is a unique $P = P^T > 0$ solution to the Lyap eqn.

$$A^T P + PA = -C^T C \qquad (A^T P A - P = -C^T C \text{ for discrete time})$$

<u>Theorem</u> [Lyapunov Test Backwards] Suppose the pair (C, A) is observable. Then A is exp. stable iff there is a unique $P = P^T > 0$ satisfying $A^T P + PA = -C^T C$ ($A^T P A - P = -C^T C$ for discrete time.)

<u>Example</u> Let (C, A) be observable & $\exists\, P = P^T > 0$ satisfying $A^T P + PA = -C^T C$. Prove that $\dot{x} = Ax$ is exp. stable.

<u>Sol'n</u>: Let $v \in \mathbb{C}^n$ be an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$. $(Av = \lambda v)$

$$
\begin{aligned}
0 &= v^* [A^T P + PA + C^T C] v \\
&= (Av)^* P v + v^* P (Av) + (Cv)^* C v \\
&= \lambda^* (v^* P v) + \lambda (v^* P v) + \|Cv\|^2 \\
&= 2\,\text{Re}\{\lambda\} \underbrace{(v^* P v)}_{\substack{>0 \\ \text{because} \\ P>0}} + \underbrace{\|Cv\|^2}_{\substack{>0 \\ \text{by eigenvector} \\ \text{test}}} \implies \text{Re}\{\lambda\} < 0 \implies \text{exp. stability.} \quad \blacksquare
\end{aligned}
$$

Observable Decomposition

Recall that controllability is invariant under similarity transformation. Same goes for observability. That is, (C,A) observable $\iff$ $(CT, T^{-1}AT)$ obs.

Theorem [obs. decomp.] Given the system

$$(1)\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad x\in\mathbb{R}^n,\ y\in\mathbb{R}^m,\ u\in\mathbb{R}^k$$

let $\bar{n}\le n$ be the dimension of unobservable subspace. We can always find a coordinate change $T\in\mathbb{R}^{n\times n}$ (choose $T=[U\ \underbrace{V}_{n\times\bar{n}}]$ where columns of V span the unobs. subspace) such that in new coordinates $z := T^{-1}x$ the system becomes

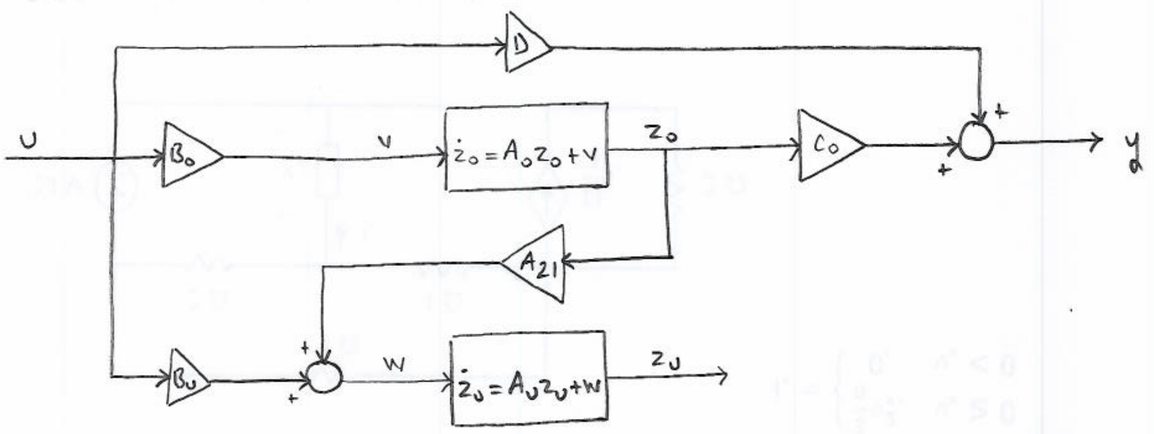
$$\begin{bmatrix}\dot{z}_o\\ \dot{z}_u\end{bmatrix} = \begin{bmatrix} \overset{(n-\bar{n})\times(n-\bar{n})}{A_o} & 0 \\ A_{21} & A_u\end{bmatrix}\begin{bmatrix} z_o\\ z_u\end{bmatrix} + \begin{bmatrix} B_o\\ B_u\end{bmatrix}u \qquad \text{where} \quad \begin{bmatrix} A_o & 0\\ A_{21} & A_u\end{bmatrix} = T^{-1}AT$$

$$y = [\underbrace{C_o}_{m\times(n-\bar{n})}\ \ 0]\begin{bmatrix} z_o\\ z_u\end{bmatrix} + Du \qquad \begin{bmatrix} B_o\\ B_u\end{bmatrix} = T^{-1}B \ \ \& \ \ [C_o\ \ 0] = CT$$

and the pair (C_o, A_o) is observable.

— o —

Block diagram repres.

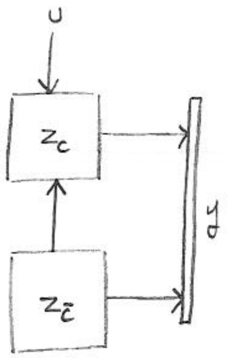


<u>Remark</u>: The TF of system (1) equals the TF of its observable part.

That is, $C(sI-A)^{-1}B + D = C_o(sI-A_o)^{-1}B_o + D$

Unification of Cont. & Obs. Decompositions : Kalman Decomp.

· Cont. Decomp.



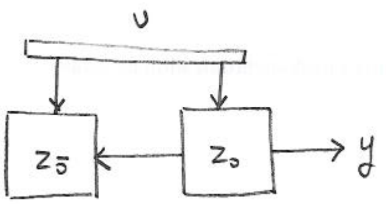
$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_c \\ z_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

(A_c, B_c) controllable

$$y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_c \\ z_{\bar{c}} \end{bmatrix} + Du$$

$TF = C_c(sI - A_c)^{-1}B_c + D$

Obs. Decomp.



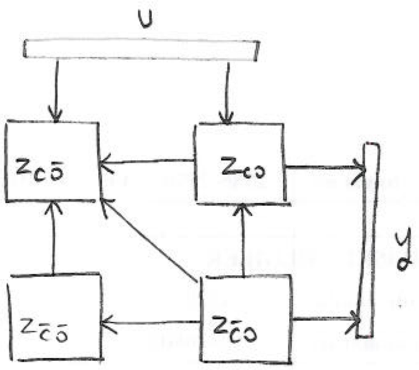
$$\begin{bmatrix} \dot{z}_o \\ \dot{z}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} z_o \\ z_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u$$

(C_o, A_o) observable

$TF = C_o(sI - A_o)^{-1}B_o + D$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} z_o \\ z_{\bar{o}} \end{bmatrix} + Du$$

Kalman Decomp.



$$\underbrace{\begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{c\bar{o}} \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + Du}_{(KD)}$$

<u>Theorem</u> [Kalman Decomp.] For each system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ there is a coordinate change $z = T^{-1}x$ that yields the form (KD), for which

→ the pair $\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$ is controllable

→ the pair $\left(\begin{bmatrix} C_1 & C_3 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix} \right)$ is observable

→ the triple $(\underbrace{C_1, \overbrace{A_{11}}_{\text{obs.}}, B_1}^{\text{con.}})$ is both observable & controllable

→ the TF of the original system equals the TF of the both cont & obs. part. That is, $C(sI - A)^{-1}B + D = C_1(sI - A_{11})^{-1}B_1 + D$.

<u>Detectability</u>

Consider an unobservable system in obs. decomp form

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_u \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + \begin{bmatrix} B_o \\ B_u \end{bmatrix} u$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + Du$$

We can determine [based on our knowledge on $y(t)$ & $u(t)$] the observable part of the state $x_o(t)$. How about $x_u(t)$?

$$\dot{x}_u = A_u x_u + A_{21} x_o + B_u u$$

$$\Rightarrow x_u(t) = \underbrace{e^{A_u t} x_u(0)}_{\text{unknown}} + \underbrace{\int_0^t e^{A_u(t-\tau)} [A_{21} x_o(\tau) + B_u u(\tau)] d\tau}_{\text{known}}$$

Suppose now A_o is exp. stable. Then the term $e^{A_o t} x_o(0) \to 0$ exponentially fast and we can write

$$x_o(t) \cong \int_0^t e^{A_o(t-\tau)} [A_{21} x_o(\tau) + B_o u(\tau)] d\tau \quad \text{for } t \text{ large enough.}$$

That is, $x_o(t)$ becomes eventually known. Hence the whole state $\begin{bmatrix} x_o(t) \\ x_o(t) \end{bmatrix}$ becomes eventually known. This motivates the following def.

Definition The pair (C, A) is said to be detectable if either it is observable or its obs. decomp. is such that A_o is exp. stable.

Remark : Detectability & stabilizability are duals of one another. That is,

$$(C, A) \text{ detectable} \iff (A^T, C^T) \text{ stabilizable}$$

This allows us to readily generate the following detectability tests from the dual stabilizability tests.

Eigenvector test : The pair (C, A) is detectable iff no eigenvector of A with eigenvalue $Re\{\lambda\} \geq 0$ ($|\lambda| \geq 1$ for DT) belongs to null(C).

PBH test : The pair $(C, \overset{\nearrow n\times n}{A})$ is detectable iff

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C} \text{ satisfying } Re\{\lambda\} \geq 0 \quad (|\lambda| \geq 1 \text{ for DT})$$

Lyapunov Test : The pair (C, A) is detectable iff $\exists P = P^T \succ 0$ satisfying

$$A^T P + PA - C^T C \prec 0 \qquad \text{(continuous time)}$$

$$A^T P A - P - C^T C \prec 0 \qquad \text{(discrete time)}$$

Example : Prove $A^T PA - P - C^T C < 0 \Rightarrow (C, A)$ detectable

Let $Av = \lambda v$ & $|\lambda| \geq 1$ $\Rightarrow$

$$0 > v^* (A^T PA - P - C^T C) v$$

$$= (Av)^* PAv - v^* Pv - \|Cv\|^2$$

$$= \underbrace{(|\lambda|^2 - 1)}_{\geq 0} \underbrace{v^* Pv}_{> 0} - \|Cv\|^2 \Rightarrow \|Cv\|^2 > 0 \Rightarrow Cv \neq 0$$

$\Rightarrow$ detectability follows by eigenvector test.

Let (C,A) be detectable. Then $\exists P = P^T > 0$ such that $A^TP + PA - C^TC < 0$.

Define $L := \frac{1}{2}P^{-1}C^T$

$$\Rightarrow \quad 0 > A^TP + PA - C^TC = A^TP + PA - \frac{1}{2}PP^{-1}C^TC - \frac{1}{2}C^TCP^{-1}P$$

$$= (A - \tfrac{1}{2}P^{-1}C^TC)^TP + P(A - \tfrac{1}{2}P^{-1}C^TC)$$

$$= (A - LC)^TP + P(A - LC)$$

$\Rightarrow$ the system $\dot{x} = [A - LC]x$ is exp. stable. ☒

Therefore: when the pair $(\underset{m\times n}{C}, \underset{n\times n}{A})$ is detectable, it is always possible to find $L \in \mathbb{R}^{n\times n}$ such that the matrix $[A - LC]$ is exp. stable.

<u>Question</u>: what's the use of above fact?

<u>Answer</u>: observer design.

<u>State Estimation</u>

system: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$	measured / known: $y(t)$ & $u(t)$	suppose: $(\underset{m\times n}{C}, \underset{n\times n}{A})$ detectable

WANT: compute $x(t)$

<u>Sol'n</u>: Choose gain $L \in \mathbb{R}^{n\times m}$ such that $[A - LC]$ is exp. stable. (We can do this when and only when (C,A) is detectable.) Then construct the following sys.

$$(1) \quad \boxed{\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})} \qquad \text{where} \quad \hat{y} = C\hat{x} + Du$$

<u>Claim</u>: $\hat{x}(t) \to x(t)$ exp. fast as $t \to \infty$ (for all initial conditions $x(0), \hat{x}(0)$)

That is, $\hat{x}(t) \cong x(t)$ for t large enough.

proof: Define error $e := \hat{x} - x$

$$\Rightarrow \dot{e} = \dot{\hat{x}} - \dot{x} = A\hat{x} + \cancel{Bu} + L\{Cx + \cancel{Du} - C\hat{x} - \cancel{Du}\} - Ax - \cancel{Bu}$$

$$= [A - LC](\hat{x} - x) = [A - LC]e$$

$$\Rightarrow \dot{e} = [A - LC]e \Rightarrow e(t) \to 0 \text{ exp. fast}$$

$$\Rightarrow \|\hat{x}(t) - x(t)\| \to 0 \text{ exp. fast}$$

$$\Rightarrow \hat{x}(t) \cong x(t) \text{ for large } t. \quad \blacksquare$$

System (1) is called an observer. It runs within our computer. Note that it is also a linear system with inputs u & y.

$$(1) \Rightarrow \dot{\hat{x}} = \underbrace{[A - LC]}_{\hat{A}}\hat{x} + \underbrace{[B - LD \quad L]}_{\hat{B}}\underbrace{\begin{bmatrix} u \\ y \end{bmatrix}}_{\hat{u}} \Rightarrow \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u}$$

Stabilization via output feedback

system: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$	measured / known: $y(t)$ & $u(t)$	Suppose: (A, B) stabilizable, (C, A) detectable

WANT: $x(t) \to 0$ (stabilization)

Sol'n: Apply state feedback $u(t) = -Kx(t)$ where $[A - BK]$ is exp. stable

problem: we do not have access to the state $x(t)$

Revised sol'n: Construct an observer first.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \quad \text{with } [A - LC] \text{ exp. stable}$$

Then let $u(t) = -K\hat{x}(t)$

Why does this sol'n work?

Let $e := \hat{x} - x$. Then $\dot{e} = [A - LC]\, e$ (1)

How about $\dot{x} = ?$

$$\dot{x} = Ax + Bu \Big|_{u = -K\hat{x}} \quad \Rightarrow \quad \dot{x} = Ax - BK\hat{x}$$
$$= Ax - BK(e + x)$$
$$= [A - BK]x - BKe \qquad (2)$$

$$(1)\ \&\ (2) \Rightarrow \underbrace{\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix}}_{A_{big}\ \text{(}2n \times 2n\text{ matrix)}} \underbrace{\begin{bmatrix} x \\ e \end{bmatrix}}_{x_{big}}$$

$$\Rightarrow \dot{x}_{big} = A_{big}\, x_{big}$$

A_{big} is upper block triangular. Therefore

$$\det(sI - A_{big}) = \det(sI - [A - BK]) \cdot \det(sI - [A - LC])$$

$$\Rightarrow \text{eigenvalues of } A_{big} = \underbrace{\{\text{eigenvalues of } [A - BK]\}}_{Re\{\lambda\} < 0} \cup \underbrace{\{\text{eigenvalues of } [A - LC]\}}_{Re\{\lambda\} < 0}$$

$\Rightarrow A_{big}$ is exp. stable

$\Rightarrow x_{big}(t) \to 0 \Rightarrow x(t) \to 0$ stabilization is achieved. ∎

Minimal Realization

Given a transfer function (matrix) $G(s)$, the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \qquad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ u \in \mathbb{R}^k$$

is said to be a realization of $G(s)$ if $C(sI-A)^{-1}B + D = G(s)$. The number n is the order of the realization. A realization of $G(s)$ is said to be minimal if there is no realization of $G(s)$ of smaller order.

Thm A realization is minimal if and only if it is both observable & controllable.

Thm All minimal realizations are algebraically equivalent.

SISO (single input single output case)

let $G(s) = \dfrac{num(s)}{den(s)}$ with

→ $G(s)$ proper, i.e., $\deg num(s) \le \deg den(s)$

→ $num(s)$ & $den(s)$ coprime, i.e., they do not share a common root.

→ $den(s)$ monic, i.e., $den(s) = s^n + a_1 s^{n-1} + \cdots$
(the highest order term's coefficient is one.)

Let $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ $x \in \mathbb{R}^n,\ u \in \mathbb{R},\ y \in \mathbb{R}$ be a minimal realization of $G(s)$.

Then we have $\boxed{\det(sI-A) = den(s)}$

An implication : Consider a SISO system that is both controllable & observable. This system is BIBO stable if and only if it is exponentially stable (in the sense of Lyapunov).