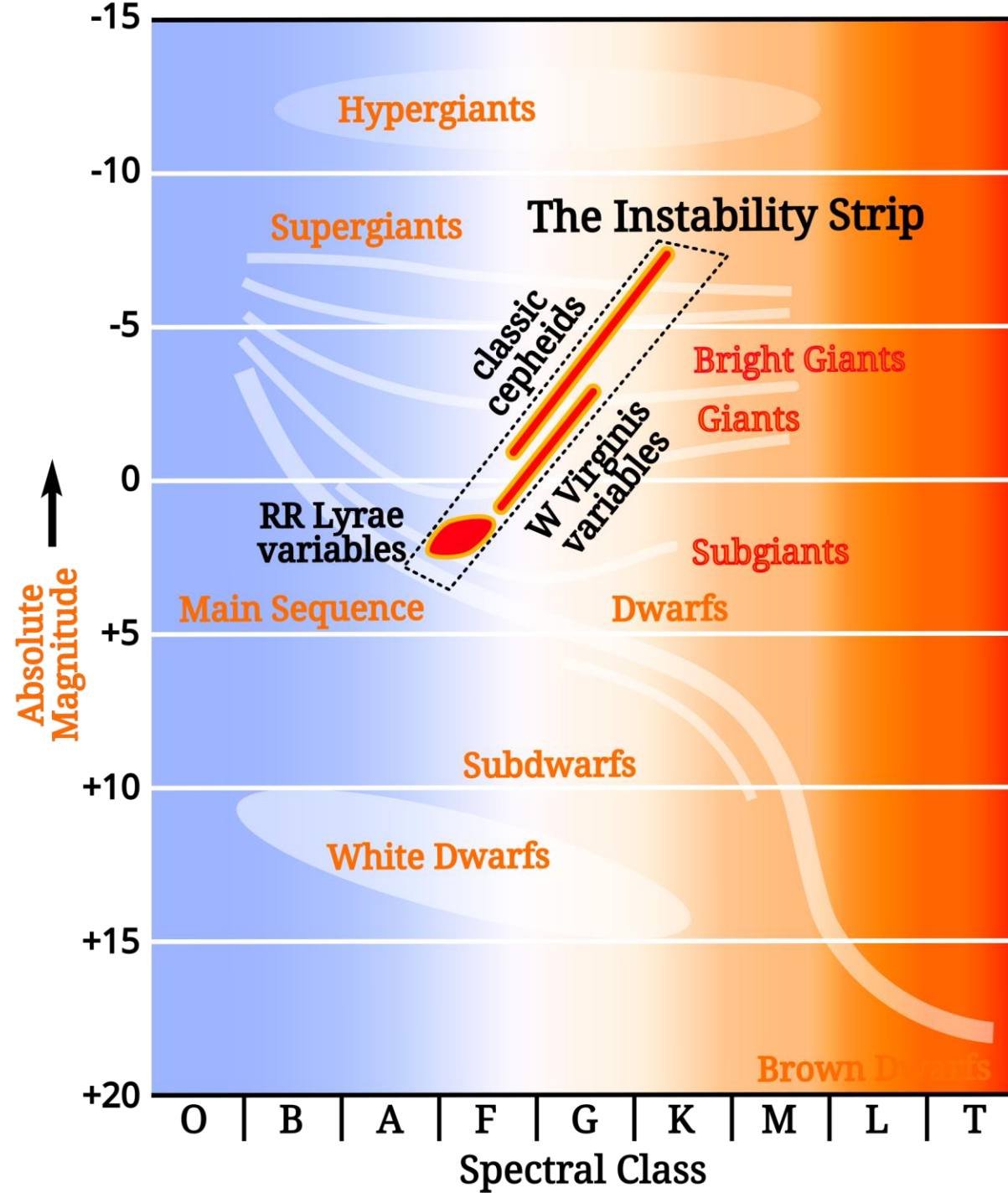




- $\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT$

- $p = mv \approx \sqrt{3mkT}$

- $\Delta x \Delta p \approx \hbar$

$$\rho \approx \frac{m_p}{\Delta x^3} \approx m_p \frac{(3mkT)^{3/2}}{\hbar^3}$$

$$p = (\gamma - 1)\epsilon$$

$$E = \frac{1}{2}m_e v^2 = p^2/2m_e \approx \hbar^2/2m_e a^2 \quad \text{where } \Delta x \approx a$$

$$\epsilon \approx E/a^3 = \hbar^2/2m_e a^5 \quad \text{Since } \rightarrow \quad \rho \sim m_p/a^3$$

$$p = \left(\frac{5}{3} - 1\right)\epsilon \approx \frac{\hbar^2}{3m_e a^5} \approx \frac{\hbar^2}{3m_e} \left(\frac{\rho}{m_p}\right)^{5/3}$$

$$p = \frac{(3\pi^2)^{2/3} \hbar^2}{5 m_e} \left(\frac{\rho}{\mu_e m_u} \right)^{5/3} \quad \text{Non-relativistic}$$



$$p = \rho k T / \mu m_u$$

$$\frac{T}{\rho_{\text{cr}}^{2/3}} = \frac{(3\pi^2)^{2/3} \hbar^2}{5 m_e m_u^{2/3} k} \frac{\mu}{\mu_e^{5/3}} \quad \text{or} \quad \rho_{\text{cr}} = 2.38 \times 10^{-5} \left(\frac{T}{\mu} \right)^{3/2} \mu_e^{5/2} \text{ kg m}^{-3}$$

$$\rho = (n + n_e) \mu m_u = n \mu_0 m_u = n_e \mu_e m_u$$

Relativistic case

$$\Delta p \approx m_e c$$

$$\rho \sim \frac{m_p}{(\Delta x)^3} \sim m_p \left(\frac{m_e c}{\hbar} \right)^3 \sim 3 \times 10^{10} \text{ kg m}^{-3}$$

$$E \approx pc \approx \frac{\hbar c}{a}$$

$$\varepsilon \approx E/a^3 \approx \hbar c/a^4 \quad \longrightarrow \quad \rho \sim m_p/a^3, \quad p \propto \rho^{4/3} \text{ and } \gamma = 4/3.$$

Non-relativistic

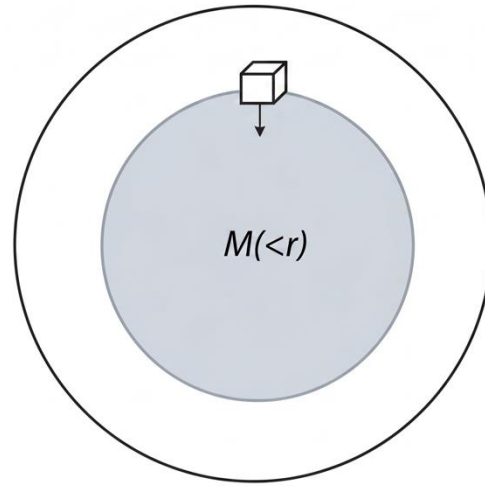
$$p = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_n} \left(\frac{\rho}{m_n} \right)^{5/3}$$

Relativistic

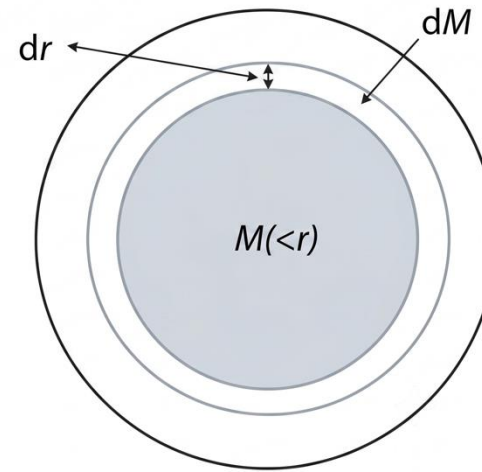
$$p = \frac{(3\pi^2)^{1/3} \hbar c}{4} \left(\frac{\rho}{m_n} \right)^{4/3}$$

See Shapiro and Teukolsky (1983)

(a)



(b)



(a) Illustrating the origin of the equation of hydrostatic support. (b) Illustrating the equation of conservation of mass.

$$F_{\text{gr}} = \frac{GmM(<r)}{r^2} = \frac{GM(<r)\varrho(r)dA dr}{r^2}$$

$$F_p = dA[p(r) - p(r + dr)] = -dA dr \frac{dp}{dr}$$

$$\frac{dp}{dr} = -\frac{GM(<r)\varrho(r)}{r^2}$$

$$M(r + dr) - M(r) = dM = 4\pi r^2 \varrho(r) dr$$

$$\frac{dM}{dr} = 4\pi r^2 \varrho(r)$$

Chandrasekhar Limit

$$\frac{dp}{dr} = -\frac{GM\varrho}{r^2}; \quad \frac{dM}{dr} = 4\pi r^2 \varrho$$



$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dp}{dr} \right) + 4\pi G \rho r^2 = 0 \quad p = \kappa \rho^\gamma \quad ; \quad \rho(r) = \rho_c w^n$$

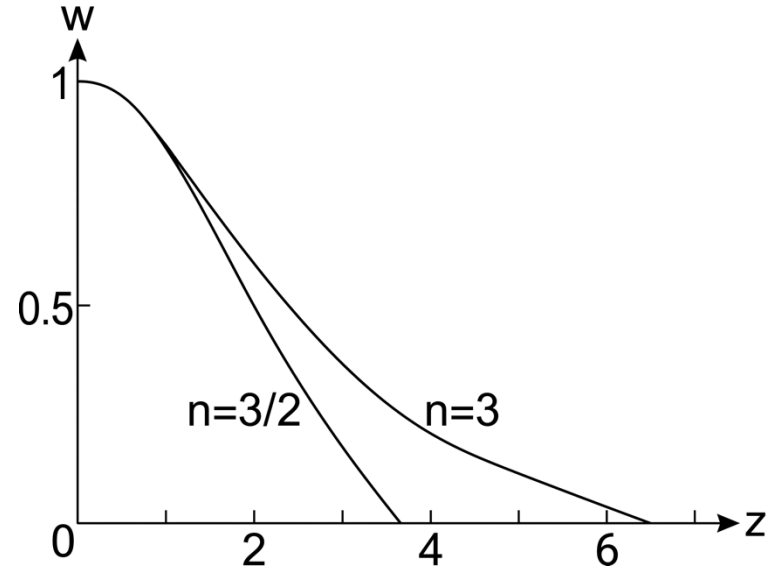
$$\gamma = 1 + (1/n) \quad \leftarrow \quad n \text{ is polytropic index}$$

Let $r = az$ where $a = \left[\frac{(n+1)\kappa\rho_c^{(1/n)-1}}{4\pi G} \right]^{1/2}$

$$\frac{1}{z^2} \left[\frac{d}{dz} \left(z^2 \frac{dw}{dz} \right) \right] + w^n = 0 \quad \text{Lane-Emden equation}$$

Analytical solutions exist for $n = 0, 1, 5$

- Exercise: Derive the Lane-Emden equation



$$\rho(r) = \rho_c w^n \quad , \quad r = az$$

$$a = \left[\frac{(n+1)\kappa\rho_c^{(1/n)-1}}{4\pi G} \right]^{1/2}$$

Solutions of the Lane–Emden equation for values of the polytropic index $n = 3/2$ and 3, corresponding to ratios of specific heat capacities $\gamma = 5/3$ and $4/3$, respectively. In both cases, the density falls to zero at a finite value of z .

$$r = az \quad \text{where} \quad a = \left[\frac{(n+1)\kappa\rho_c^{(1/n)-1}}{4\pi G} \right]^{1/2}$$

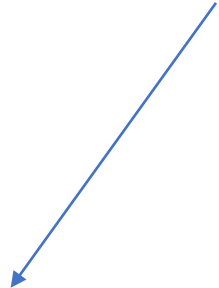
$$\text{Solve for } \rho_c \longrightarrow \rho_c \propto R^{2n/(1-n)}$$

- Exercise: Find analytical solution for mass for $n = 1$

$$\begin{aligned}
 M &= \int_0^R 4\pi \rho r^2 \, dr = 4\pi \rho_c \int_0^R w^n r^2 \, dr , \\
 &= 4\pi \rho_c a^3 \int_0^{z_n} w^n z^2 \, dz = 4\pi \rho_c \left(\frac{r}{z}\right)^3 \int_0^{z_n} w^n z^2 \, dz
 \end{aligned}$$

When $r = R$

Use the integral of Lane-Emden eq.



$$M = 4\pi \rho_c \left(\frac{R}{z_c}\right)^3 \left[-z^2 \left(\frac{dw}{dz}\right) \right] \quad \text{evaluated at } r = R$$

$$M \propto \rho_c R^3 \propto R^{(3-n)/(1-n)}$$

Chandrasekhar Mass

$$\int_0^{z_n} z^2 w^n dz = - \left(z^2 \frac{dw}{dz} \right)_R$$

2.71406 for $n = 2/3$

2.01824 for $n = 3$

$$M = 2.018244 \times \pi \left(\frac{\kappa}{\pi G} \right)^{3/2} = \frac{(3\pi)^{3/2}}{2} \left(\frac{\hbar c}{G} \right)^{3/2} \times \frac{2.01824}{(\mu_e \mu_u)^2} = \frac{5.836}{\mu_e^2} M_{\text{Sun}}$$

where $\mu_e = 2$ for
white dwarfs and
 $\mu_e = 1$ for neutron
stars

- For $n = 3$, the mass is independent of $R \rightarrow M_{\text{Ch}} = 1.46 M_{\text{Sun}}$

Simple approach

$$U = V\varepsilon = 3Vp \approx V\hbar c \left(\frac{\rho}{m_p} \right)^{4/3}$$

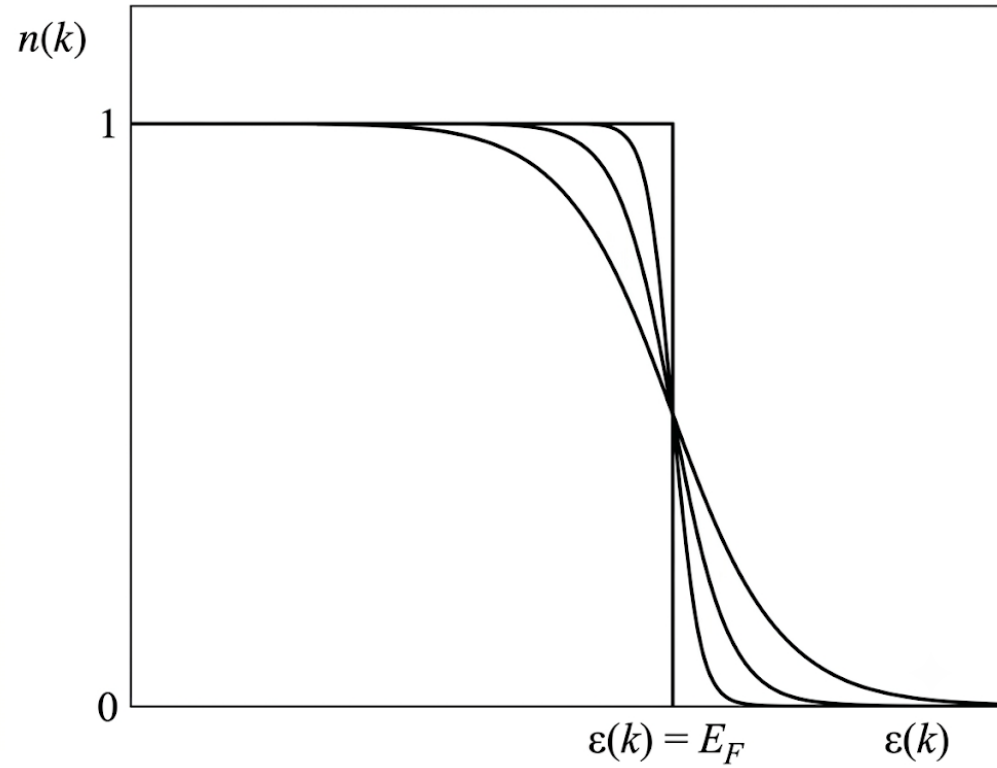
$$2U = |\Omega| \longrightarrow 2V\hbar c \left(\frac{\rho}{m_p} \right)^{4/3} = \frac{1}{2} \frac{GM^2}{R} \quad \text{Virial theorem}$$

$$V \approx R^3 \text{ and } \rho = \frac{M}{R^3}$$

The $1/R$ factors on the left-hand side and right-hand side of the equation cancel out

$$\longrightarrow M \approx \frac{1}{m_p^2} \left(\frac{\hbar c}{G} \right)^{3/2} \approx 2M_\odot$$

Exercise: Derive the Chandrasekhar mass limit using pressure of Fermi gas and gravitational pressure



$$n(k) = \frac{1}{e^{(\epsilon(k) - \mu)/k_b T} + 1}$$

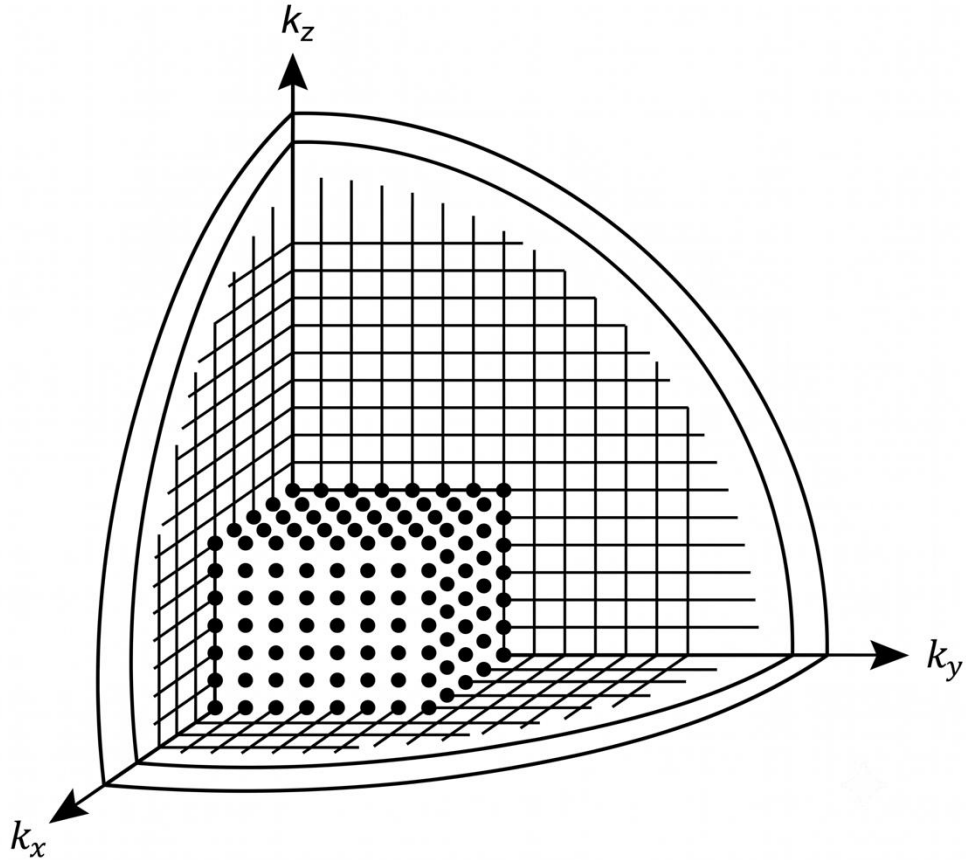
Fermi–Dirac distribution

The Fermi–Dirac distribution function, $n(k)$, for low temperatures.

For $T = 0$ there is a sharp step in $n(k)$ at the Fermi energy, $\epsilon(k) = E_F$.

For finite temperatures the step is smoothed out.

The curves shown here are for $\mu/k_B T = 50, 20$, and 10 with the curve for 50 lying closest to the step function.



The single-particle states in k space are all occupied for wave vector less than k_F . The single-particle states in the range k to $k+dk$ are occupied by the next batch of Fermi particles. Eventually, when the last fermions enter, all states with wave vector up to k_F are occupied, all states with larger wave vector are empty.

$$D(k) = V k^2 / 2\pi^2$$

$$N = 2 \frac{V}{2\pi^2} \int_0^{k_F} k^2 dk = \frac{V}{3\pi^2} k_F^3$$

$$k_F = (3\pi^2 n)^{1/3} \quad \xrightarrow{\quad} \quad n = N/R^3$$

$$E_{total} \approx N \hbar c k_F \sim \frac{1}{R} \quad \xrightarrow{\quad} \quad N = M/m$$

$$\Rightarrow P_{Fermi} \approx \left(\frac{M}{m}\right)^{4/3} \left(\frac{1}{R}\right) \left(\frac{1}{R}\right) \left(\frac{1}{R^2}\right) \approx \left(\frac{M}{m}\right)^{4/3} \frac{1}{R^4}$$

Energy, force, pressure

$$P_{Fermi} \approx \left(\frac{M}{m}\right)^{\frac{4}{3}} \left(\frac{1}{R}\right) \left(\frac{1}{R}\right) \left(\frac{1}{R^2}\right) \approx \hbar c \left(\frac{M}{m}\right)^{4/3} \frac{1}{R^4}$$

$$P_{gravitation} \approx \frac{GM^2}{R^2} \frac{1}{R^2} \approx G \frac{M^2}{R^4}$$

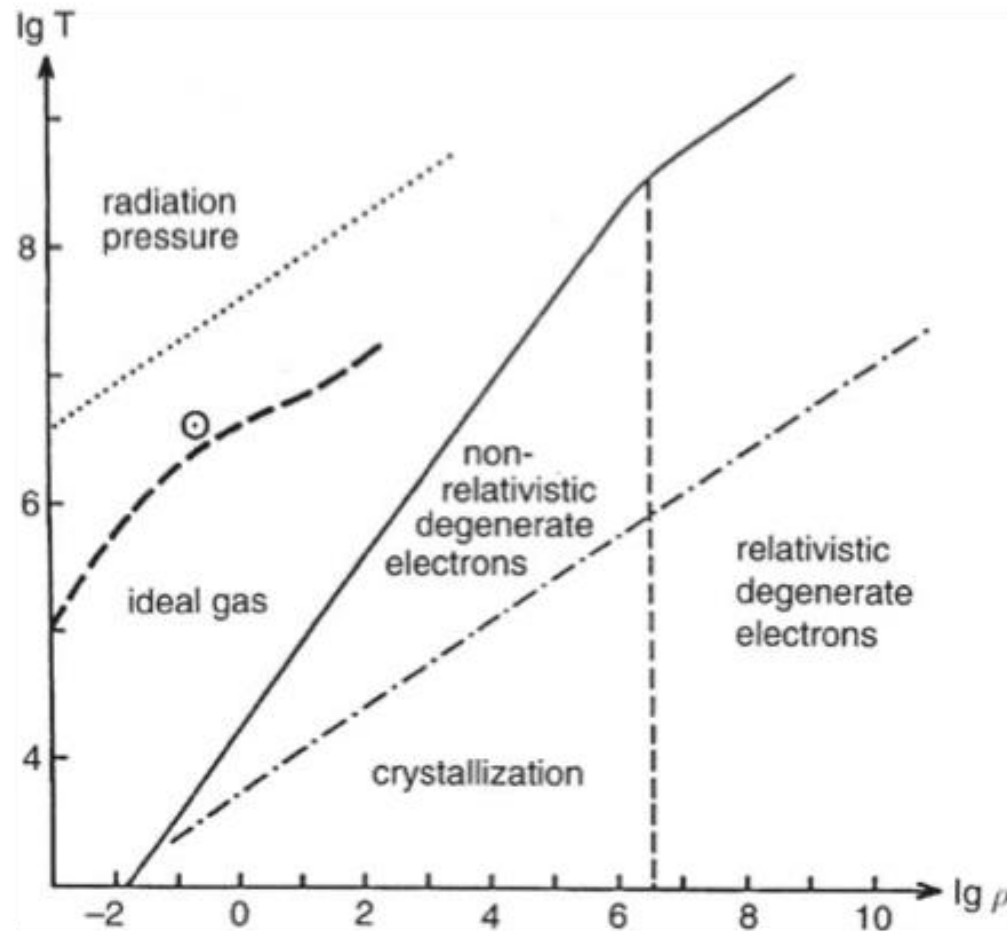
HW: estimate radius
of a white dwarf and
neutron star

$$P_{Fermi} = P_{Gravitation}$$

$$M \approx \left(\frac{\hbar c}{m^{\frac{4}{3}} G}\right)^{\frac{3}{2}} \sim 1.45 M_{sun}$$

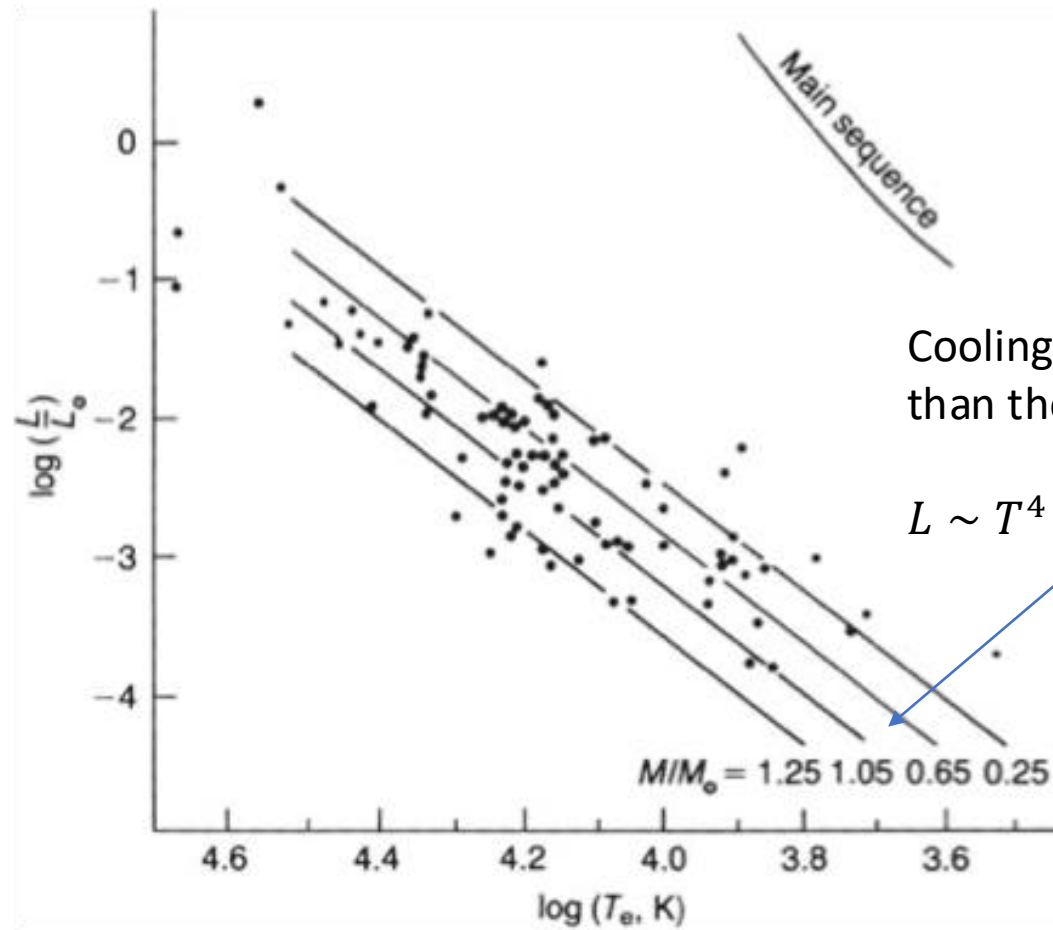
Chandrasekhar mass

M does not depend on R, this is valid
only in relativistic case



Exercise: Find the crystallization and melting temperature of a white dwarf

A sketch of the density–temperature plane showing the regions in which different types of equation of state are applicable. In addition to the regions discussed in the text, the diagram also shows the regions in which radiation pressure exceeds the gas pressure and also the region in which the degenerate gas is expected to become a solid, that is, it represents the melting temperature of the stellar material. The heavy dashed line shows the location of the Sun from its core to envelope (Kippenhahn and Weigert, 1990).

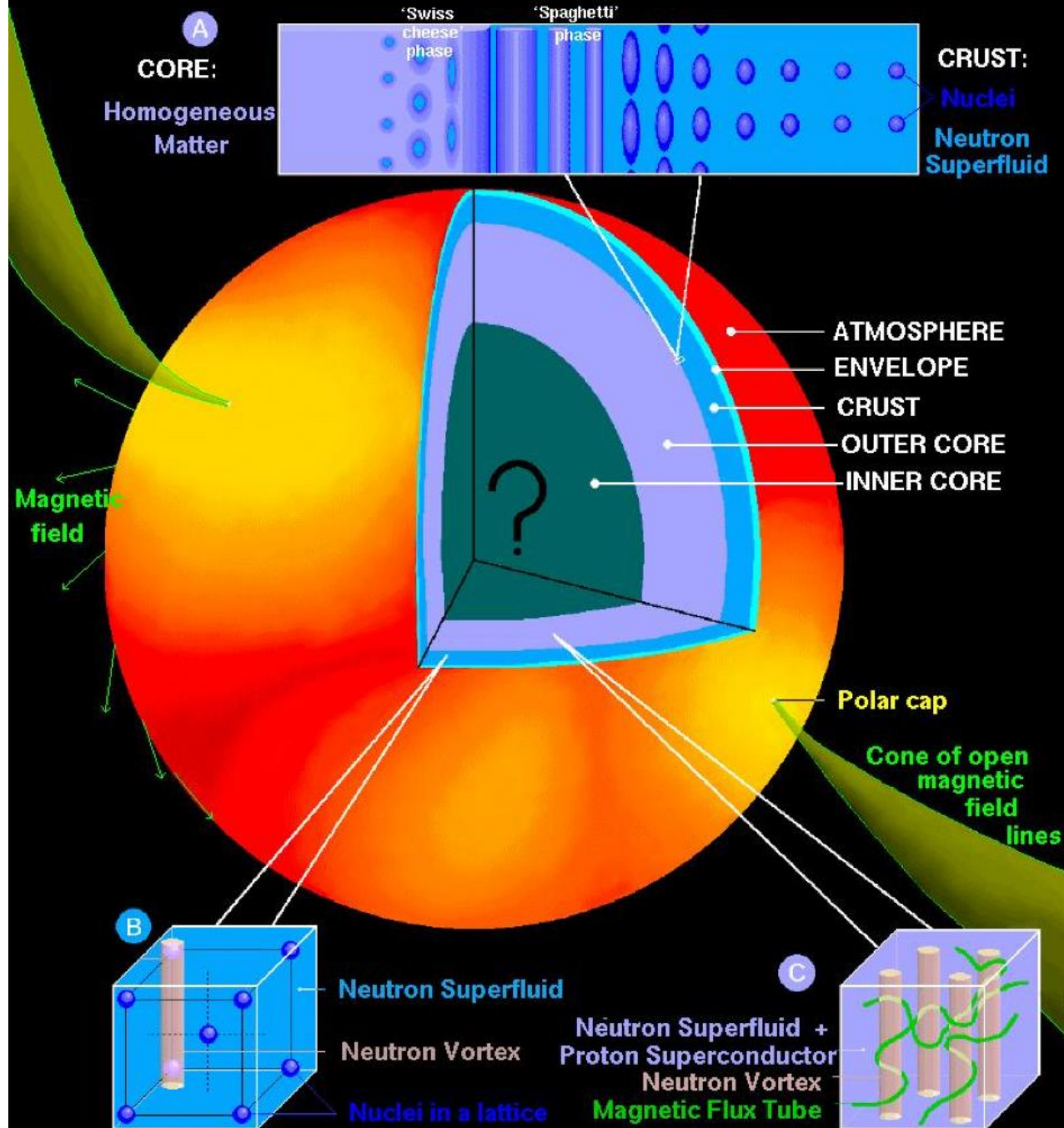


Cooling time $10^9 - 10^{10}$ years: much longer than those of a Sun-like star

$L \sim T^4$ cooling curves

Comparison of the theoretical Hertzsprung–Russell diagram for white dwarfs with their observed properties. The location of the cooling curve on the H-R diagram depends upon the mass of the white dwarf (Shapiro and Teukolsky, 1983).

A NEUTRON STAR: SURFACE and INTERIOR

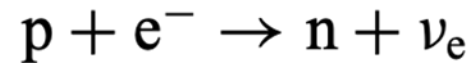


Neutron Stars

When degenerate electron gas becomes relativistic and their energy exceeds the mass difference between neutron and proton:

$$E = \gamma m_e c^2 \geq (m_n - m_p) c^2 = 1.29 \text{ MeV}$$

Inverse β decay stars



$$p_F = \gamma m_e v = \left(\frac{E_{\text{tot}}^2}{c^2} - m_e^2 c^2 \right)^{1/2} \longrightarrow n_e = \left(\frac{8\pi}{3h^3} \right) p_F^3 \longrightarrow \text{neutronization}$$

$$\rho = n_e m_\mu \mu_e \approx 1.2 \times 10^{10} \text{ kg m}^{-3}$$



A representative model showing the internal structure of a $1.4 M_{\odot}$ neutron star.

Sauls 1988

Neutron stars are cold (*i.e.* $T \sim 10^8 K \ll T_{\text{Fermi}} \sim 10^{12} K$) and the same theoretical arguments that lead to the conclusions that terrestrial matter should be superconducting with transition temperatures $T_c \simeq 10^{-3} T_{\text{Fermi}}$ also predict that neutron stars should have superfluid interiors. The necessary ingredient for the formation of a condensate of Cooper pairs, and hence a superfluid (or superconductor) is an attractive interaction between two neutrons (or protons) on the Fermi surface with *zero* total momentum. The Fermi sea guarantees the formation of a bound-state, *i.e.* a Cooper pair, no matter how weak the interaction, so long as it is attractive.

Sauls 1988

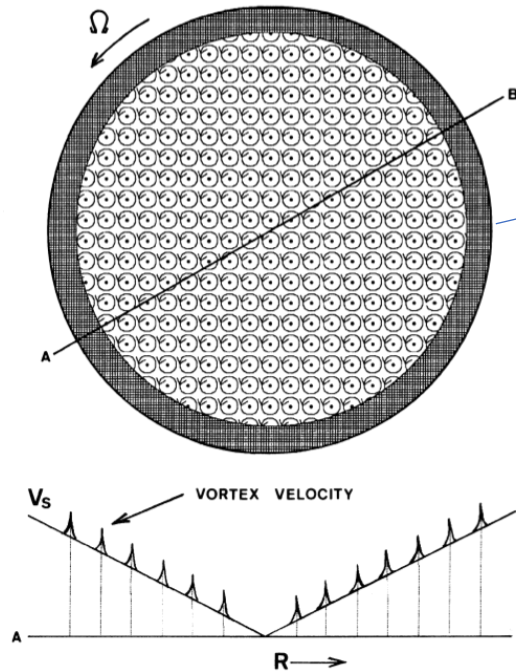
Magnus force radially outward

$$f = \rho \kappa \times (V_s - V_L)$$

Vortex line velocity

Superfluid velocity

Quantum vorticity



$$\vec{\kappa} = \frac{\hbar}{2m_n} \hat{k} \text{ outside the plane}$$

$$n\kappa = 2\Omega$$

density of vortices

Show that total velocity of vortices adds up to 2Ω (Sauls 1988)

Ginzburg & Landau

Order parameter

$$\psi(R) = |\psi| e^{i\varphi(R)}$$

Superfluid velocity

$$\vec{v}_s \equiv \frac{\hbar}{2m} \vec{\nabla} \varphi(R)$$

$$\vec{\nabla} \times \vec{v}_s = 0 \longrightarrow \oint_C \vec{v}_s \cdot d\vec{l} = \frac{h}{2M} N$$

$$\vec{\nabla} \times \vec{v}_s = \frac{h}{2m} \delta(R)$$

$$\oint_{C_R} \vec{v}_n \cdot d\vec{l} = (\Omega R)(2\pi R) = N_v \frac{h}{2M_n},$$

$$\frac{N_v}{\pi R^2} = \frac{4M_n\Omega}{h} \simeq 6.3 \times 10^3 \frac{\text{vortices}}{\text{cm}^2} P^{-1}$$

- For the Vela Pulsar, $P = 0.083$ sec $\rightarrow$ inter vortex distance: 4×10^{-3} cm

$$\vec{v}_p = \frac{\hbar}{2M_p} \vec{\nabla} \vartheta_p - \frac{e}{M_p c} \vec{A}(\vec{R})$$

$$\vec{\nabla} \times \vec{v}_p = 2\vec{\Omega} = -\frac{e}{M_p c} (\vec{\nabla} \times \vec{A})$$

$$\vec{b}_{London} = -\frac{2M_p c}{e} \vec{\Omega}$$

10^{-4} Gauss for Vela

$$\oint A \cdot dl = \oint B \cdot ds = \phi \quad \text{magnetic flux}$$

$$\oint v \cdot dl = \oint \frac{h}{m} \left(\nabla \varphi - \frac{2e}{\hbar c} A \right) \cdot dl = \frac{h}{m} \left(2\pi n - \frac{2e}{\hbar c} \phi \right) = 0$$

$$\phi_0 = \frac{hc}{2e} = 2 \times 10^{19} \text{ G} - \text{fm}^2 \quad 1 \text{ fm} = 1 \times 10^{-15} \text{ m}$$

Quantized flux, Sauls 1988

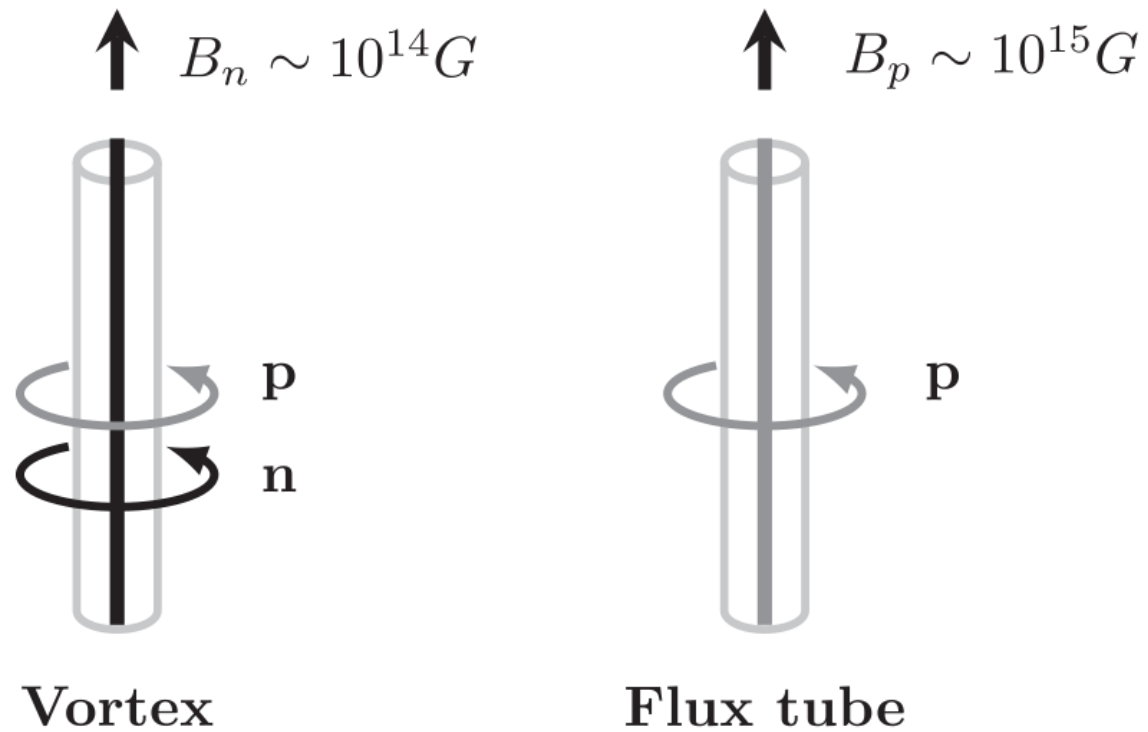


Figure 2. The entrainment of protons by circulating neutrons endows a neutron vortex with a magnetic field B_n . Neutrons are also entrained by protons circulating around a flux tube but, being neutral, they do not contribute to its magnetic field B_p .

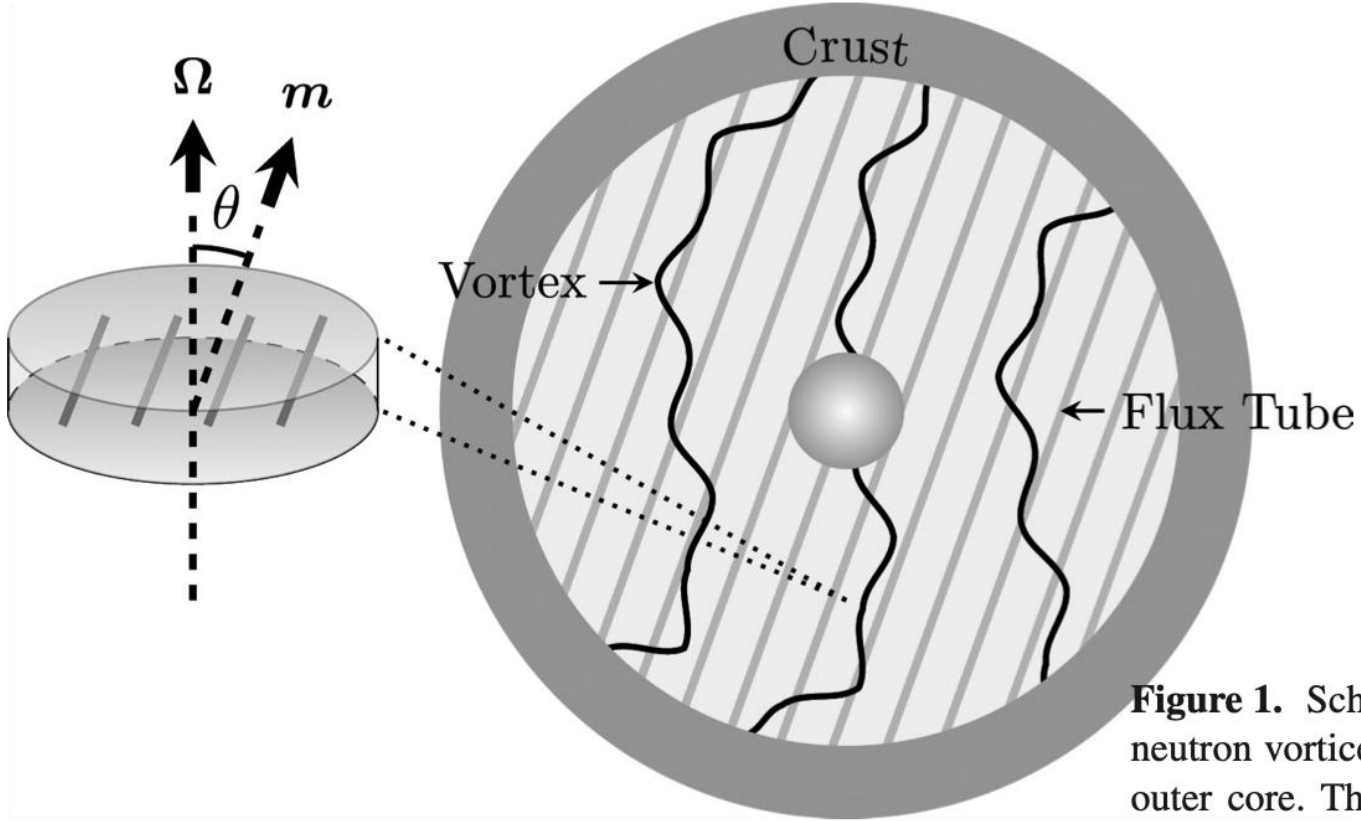


Figure 1. Schematic of a possible, realistic configuration of the superfluid neutron vortices and superconducting proton flux tubes in the neutron star outer core. The vortices lie broadly parallel to the rotation axis Ω . They form a tangled, frustrated equilibrium as they seek a compromise between aligning with Ω globally and pinning to the flux tubes locally. The flux tubes align roughly with the global magnetic dipole moment m but also have a complicated geometry locally (Easson 1979; Melatos 2012; Glampedakis & Lasky 2015), which we do not model in this paper. We solve the GPE in a ‘box’ of fluid centred on the rotation axis (inset). The box is small enough that the flux tubes can be approximated as straight inside it and inclined at an angle θ with respect to Ω . The rigid crust (dark shaded annulus) is not modelled explicitly in this paper; nor is the inner core (shaded circle at centre).

Sauls 1988

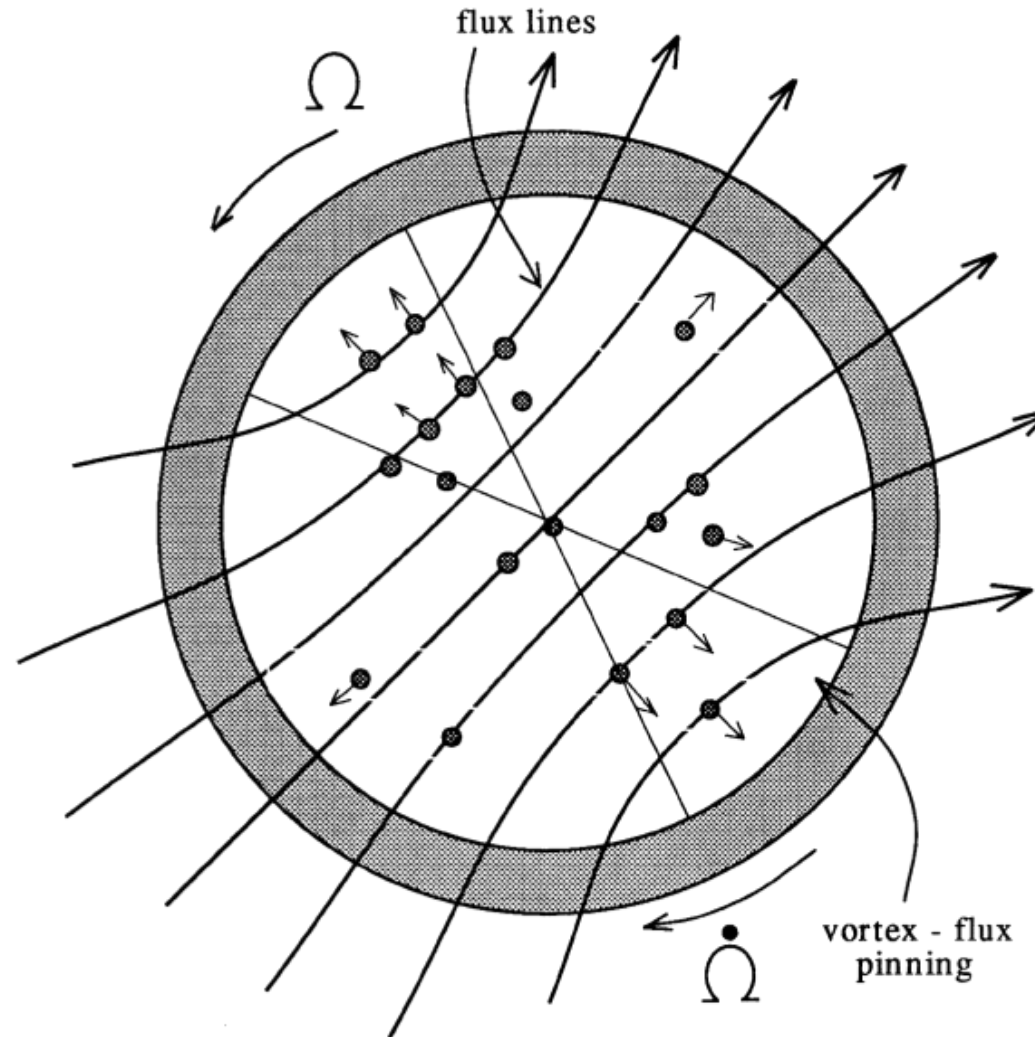
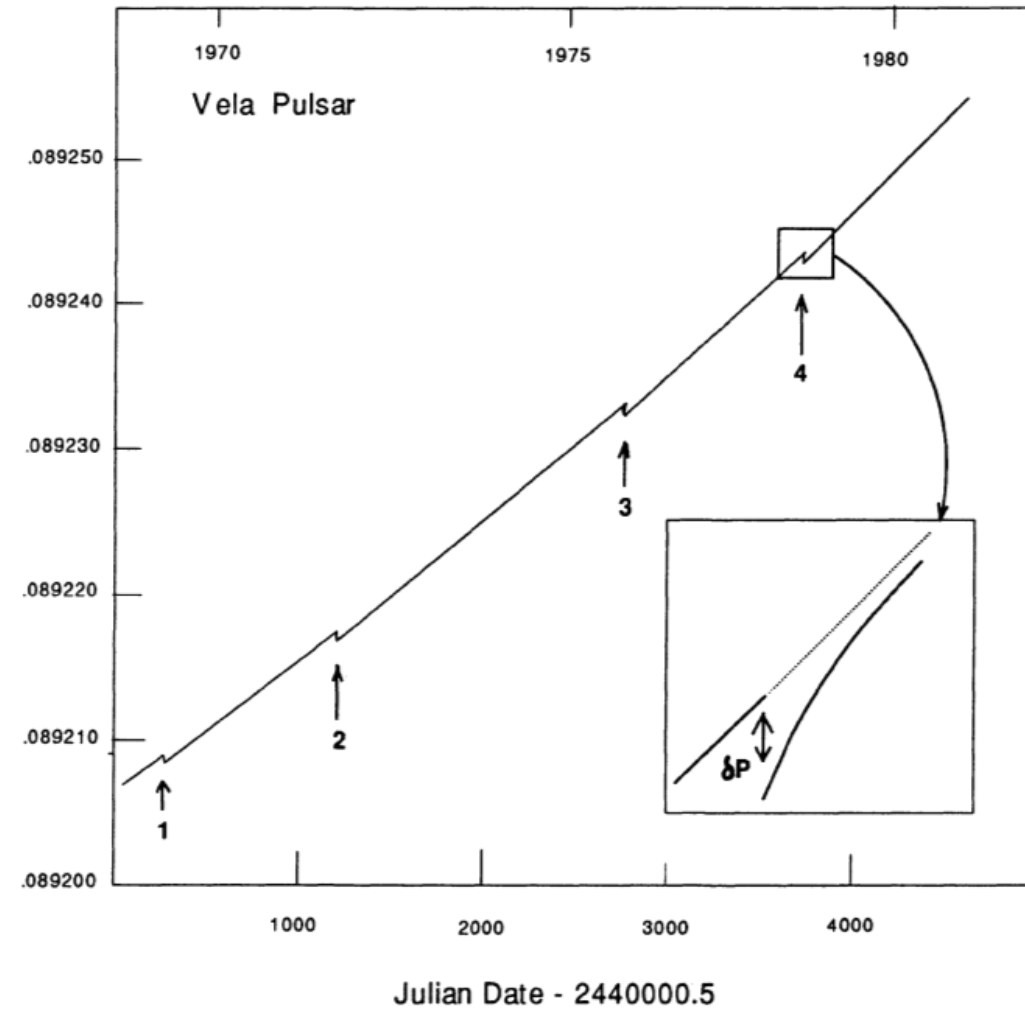
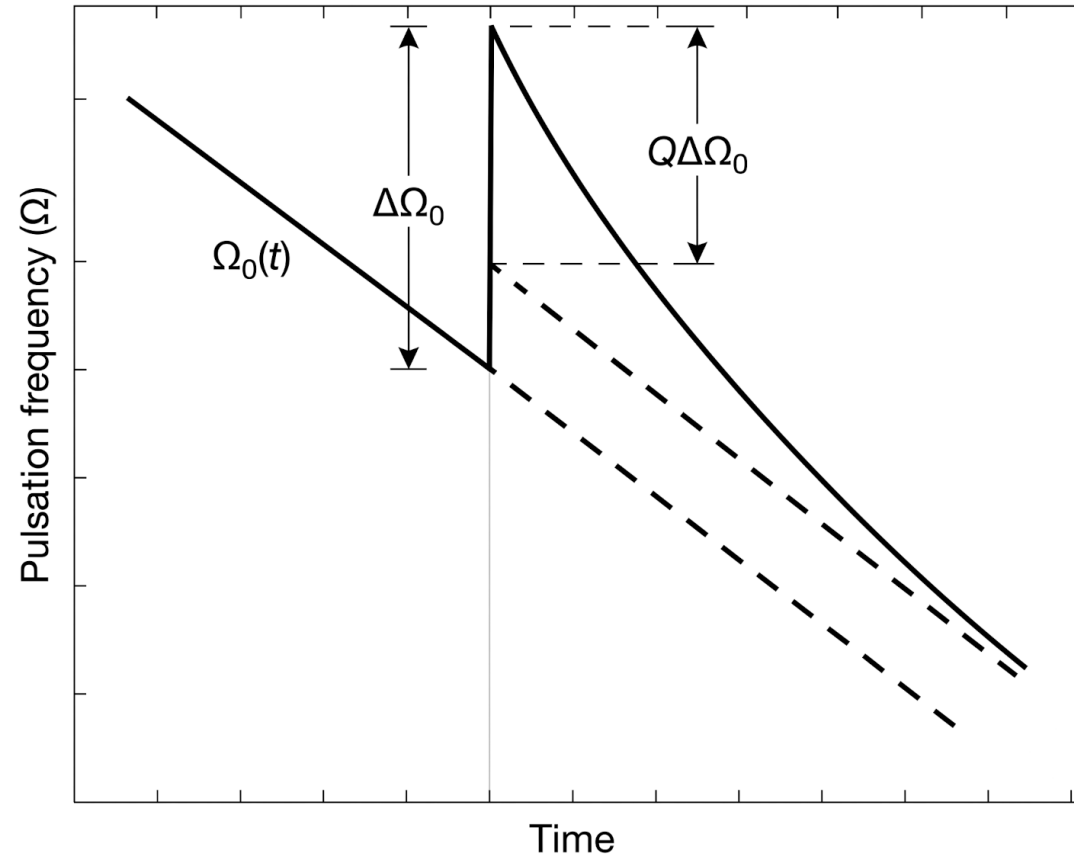


Figure 8. Vortex lines in the core superfluid may pin on the proton flux lines. The region of strongest pinning is the cone where the radial flow of vortex lines is nearly perpendicular to the flux lines.

Period [sec]





Illustrating the phenomenon of *glitches*. The pulse period increases smoothly as the rotation rate of the neutron star decreases but there are sudden discontinuities in the pulse period, following which the steady increase in period continues. The variation of the pulse period with time during the glitches provides information about the internal structure of the neutron star (Shapiro and Teukolsky, 1983).

$$I_n \dot{\Omega} = -\alpha - \frac{I_n(\Omega - \Omega_s)}{\tau_c}; \quad I_s \dot{\Omega}_s = \frac{I_n(\Omega - \Omega_s)}{\tau_c}$$

$$\Omega(t) = \Omega_0(t) + \Delta\Omega_0 [Qe^{-t/\tau_c} + 1 - Q]$$

$$\frac{\Delta\Omega}{\Omega} \sim 10^{-7} - 10^{-6}$$

Exercise: Show that glitch function satisfy the above equation

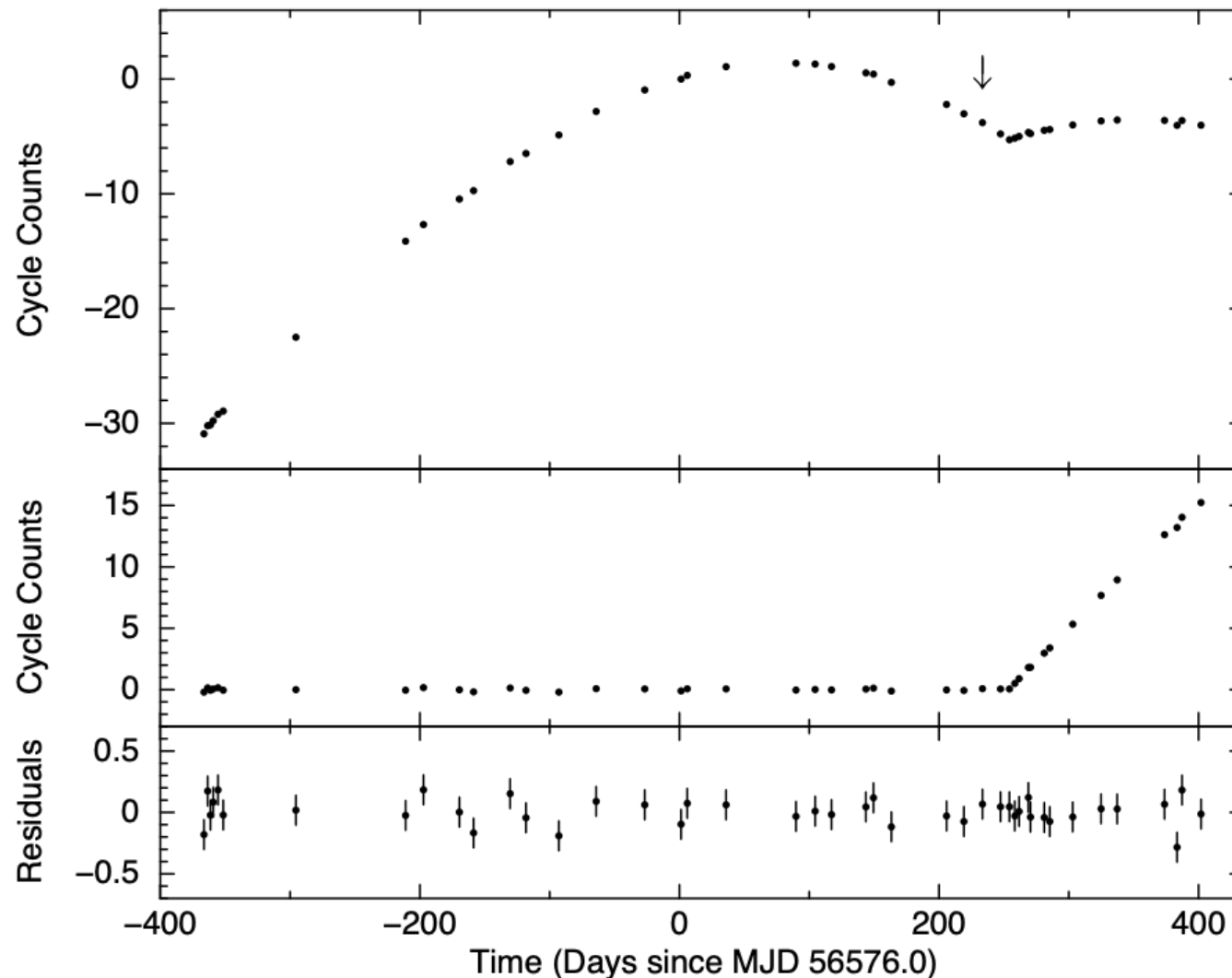
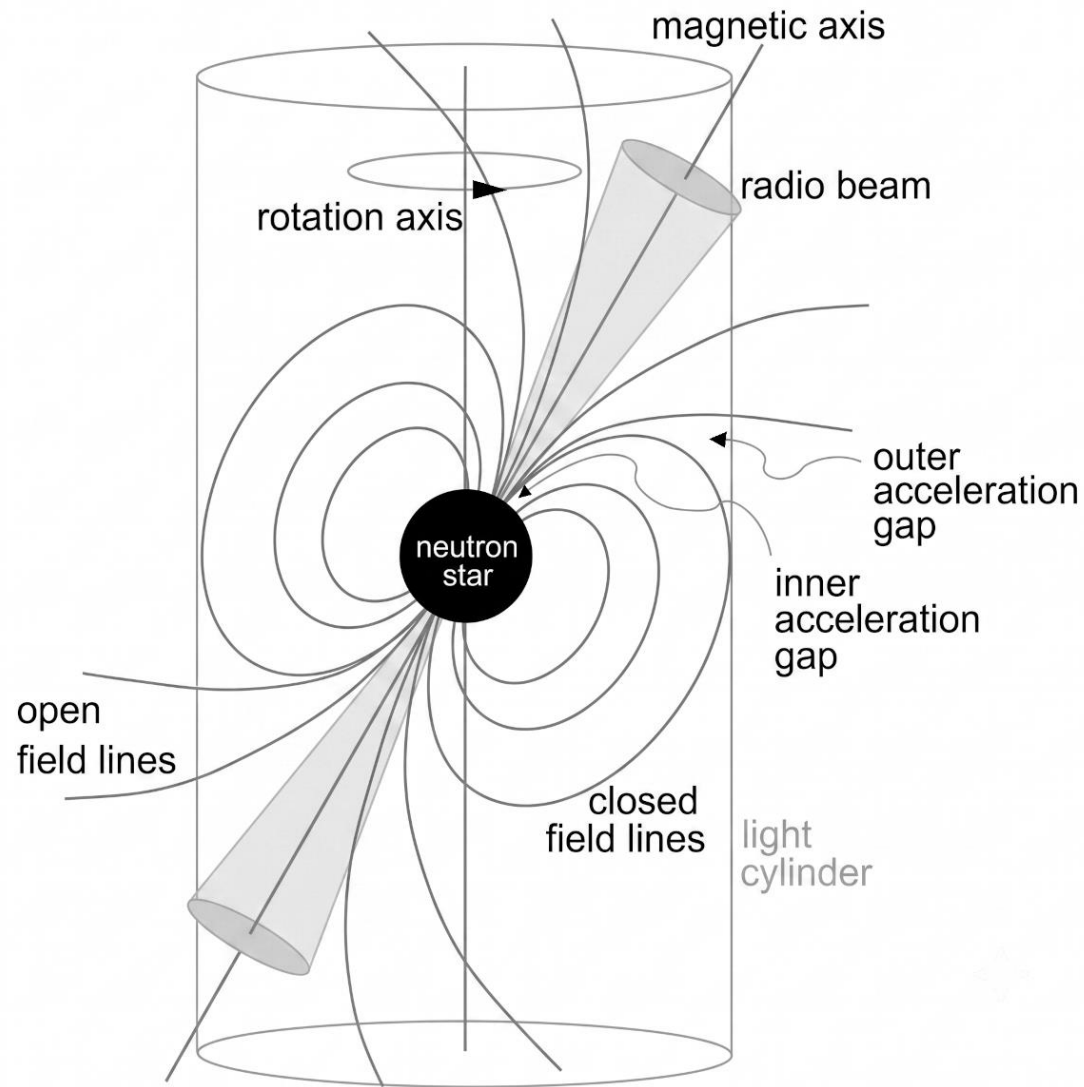


Figure 2. *Upper panel:* Cycle counts of pulse phase obtained by expanding the TOAs. The time denoted with an arrow mark corresponds to the X-ray outburst of SXP 1062, that is MJD 56809.5. *Middle panel:* Phase offset series after the removal of the secular spin-down trend and the orbital model given in Table 1. The glitch event is evident on MJD 56834.5 and it is 25 days after the X-ray outburst. *Lower panel:* Residuals after an additional removal of the glitch event. **The final model has a reduced χ^2 of 1.0.**



A schematic model of a pulsar as a magnetised rotating neutron star in which the magnetic and rotation axes are misaligned. The radio pulses are assumed to be due to beams of radio emission from the poles of the magnetic field distribution and are associated with the passage of the beam across the line of sight to the observer (Lorimer and Kramer, 2005). Typical neutron star parameters are $M \approx 1.4 M_{\odot}$, radius ≈ 10 km, magnetic flux density $10^5 - 10^9$ T.

$$-\langle \frac{d\frac{1}{2}I\Omega^2}{dt} \rangle = -I\Omega \frac{d\Omega}{dt} = \frac{\mu_0 \Omega^4 p^2}{6\pi c^3}$$

$$\dot{\Omega} = -\kappa \Omega^n, \quad \ddot{\Omega} = -n\kappa \dot{\Omega}^{(n-1)} \longrightarrow n = \frac{\Omega \ddot{\Omega}}{\dot{\Omega}^2} \quad \text{Braking index}$$

integrate

$$\int_{\Omega_0}^{\Omega} d\Omega \Omega^{-n} = -\kappa \int_0^{\tau} dt \quad \Omega \gg \Omega_0 \quad \longrightarrow \quad \tau = -\frac{\Omega}{(n-1)(d\Omega/dt)} = \frac{P}{(n-1)dP/dt}$$

Exercise: Show that $n=3 \rightarrow \tau = \frac{P}{2\dot{P}}$

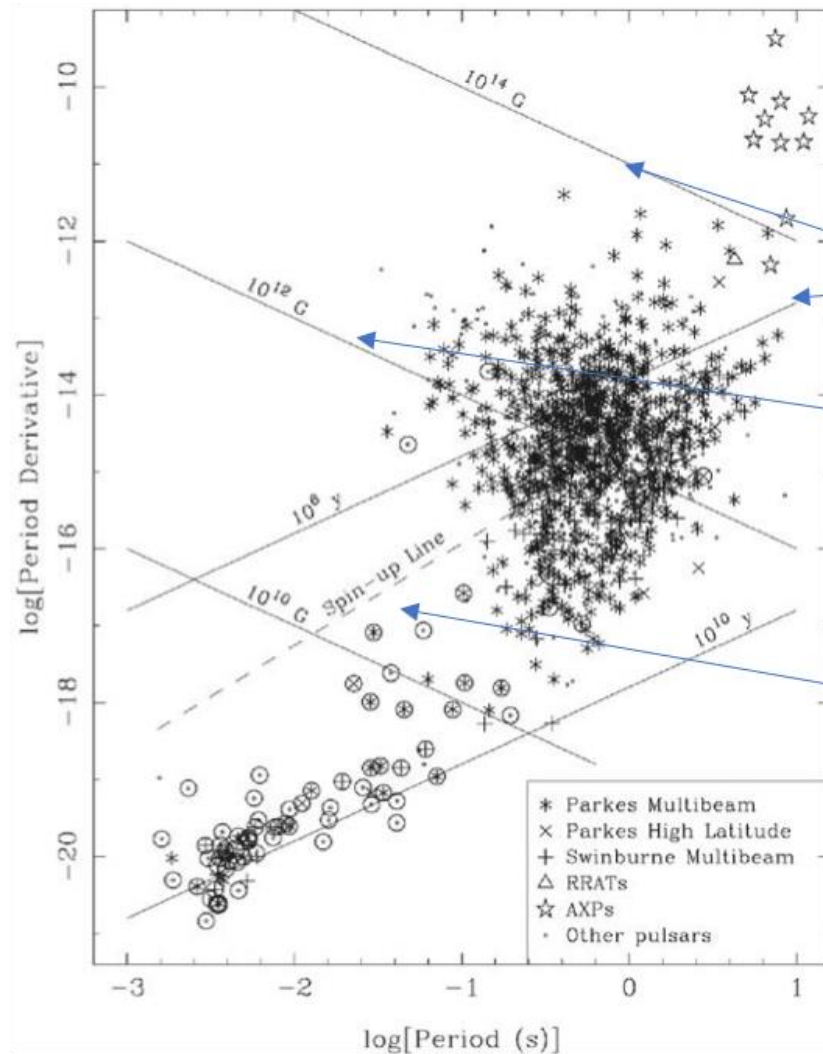
$$\mathbf{B} = \frac{\mu_0 p_{m0}}{4\pi r^3} [2 \cos \theta \mathbf{i}_r + \sin \theta \mathbf{i}_\theta] \quad \text{Magnitude of B at } r=R \rightarrow B_s \approx \mu_0 p_{m0} / 4\pi R^3$$

$$-\frac{d\Omega}{dt} = \frac{\mu_0 \Omega^3 p_{m0}^2}{6\pi c^3 I} = \frac{\mu_0 \Omega^3}{6\pi c^3 I} \left(\frac{4\pi R^3 B_s}{\mu_0} \right)^2 = \frac{8\pi \Omega^3 R^6 B_s^2}{3\mu_0 c^3 I}$$

$$B_s = - \left(\frac{3\mu_0 c^3 M \dot{\Omega}}{20\pi \Omega^3 R^4} \right)^{1/2} = \left(\frac{3\mu_0 c^3 M}{80\pi^3 R^4} \right)^{1/2} (P \dot{P})^{1/2} \approx 3 \times 10^{15} (P \dot{P})^{1/2} \quad \text{T}$$

$$\tau = \frac{P}{2\dot{P}}$$

Exercise: Derive the magnetic dipole field

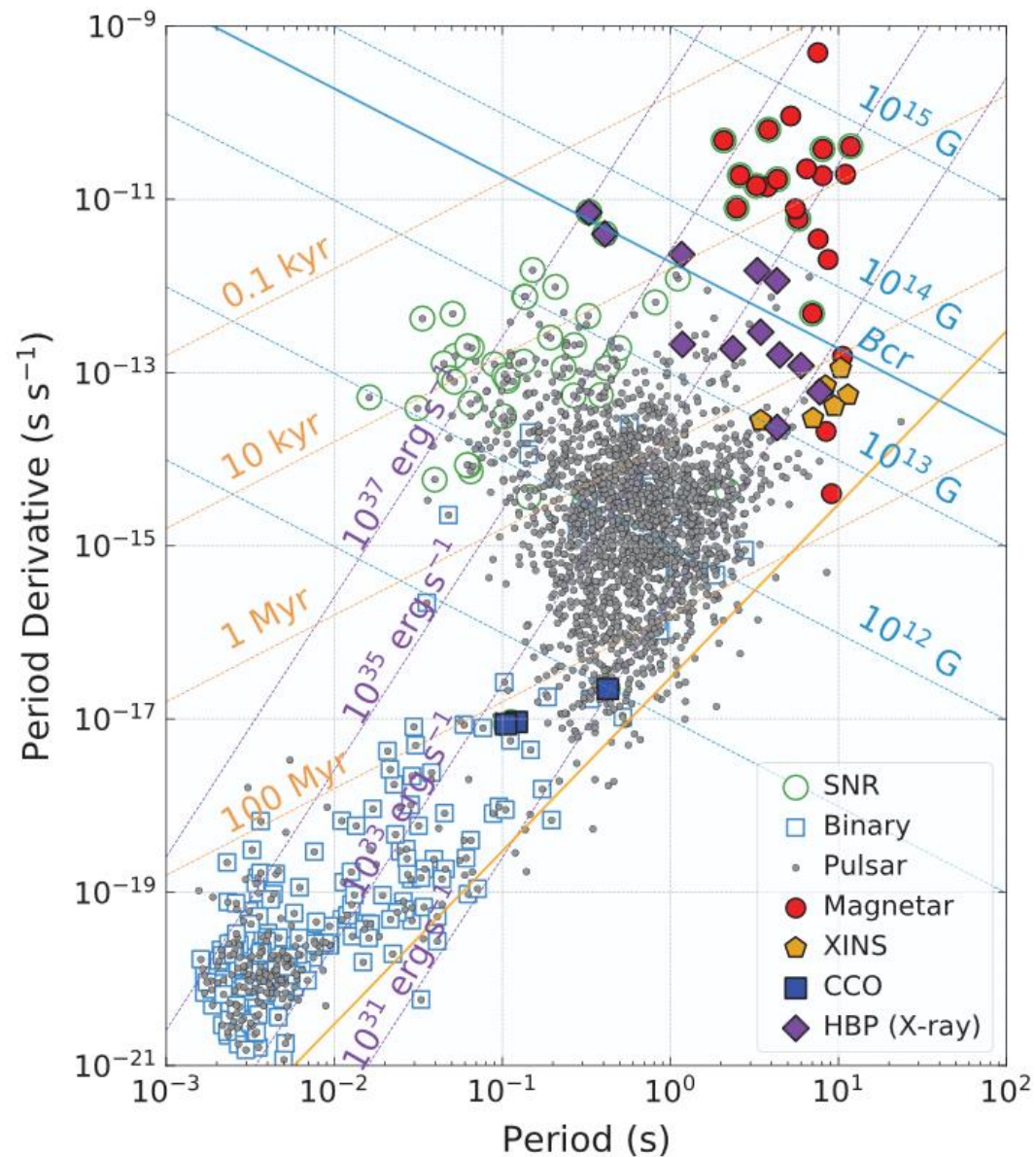


$$\tau = \frac{P}{2\dot{P}}$$

$$B_s \approx 3 \times 10^{15} (P\dot{P})^{1/2}$$

$$P \approx B_s^{6/7} \approx \dot{P}^{3/4} \quad \text{Maximum spin up line, van den Heuvel 1987}$$

A plot of $\dot{P}$ versus P for pulsars, known as the P - $\dot{P}$ diagram (Manchester, 2005, from data described in Manchester *et al.*, 2005). The different symbols refer to different large pulsar surveys. The symbols enclosed in circles represent pulsars which are members of binary systems. Lines of constant age derived from the formula $\tau = P/2\dot{P}$ are shown. The magnetic flux densities are derived from (13.43), assuming the deceleration of the pulsar is due to magnetic braking. The upper limit to the spin-up periods for dead pulsars according to the models of van den Heuvel is also shown (van den Heuvel, 1987).



Enoto et al. 2019

Known pulsars in the $P - \dot{P}$ diagram (period and period-derivative plane)

Dashed lines:

- The characteristic age $\tau_c = P / 2\dot{P}$ (orange)
- Surface magnetic fields B (cyan)
- Spin-down luminosity L_{sd} (purple)

Solid lines:

- The critical field, $B_{cr} = 4.4 \times 10^{13}$ G, (cyan)
- One of the theoretical death-lines (orange)

HBP: high-B pulsars with x-ray emission reported

XINS: X-ray isolated neutrons stars

CCO: compact central objects

Each point is marked with additional symbols if associated with a SNR (green open circle) or if found in a binary system (blue open square).

The data are taken from the Australia Telescope National Facility (ATNF) pulsar catalogue (version 1.60) and McGill magnetar catalogue for pulsars and magnetars, respectively. The source lists of HBPs, XINSs, and CCOs used here are mainly based on and references therein.

$$\Omega = \Omega_K(r_m)$$

$$P = 2\pi \left(\frac{r_M^3}{GM_*} \right)^{1/2} = \frac{2\pi}{(GM_*)^{1/2}} \left(\frac{2\pi^2}{G\mu_0^2} \right)^{3/14} \left(\frac{R_*^{12}}{M_* \dot{m}^2} \right)^{3/14} B_s^{6/7}$$

Exercise: Show that

$$L = G\dot{m}M_*/R_* = 1.3 \times 10^{31} \text{ W}$$

Eddington Limit

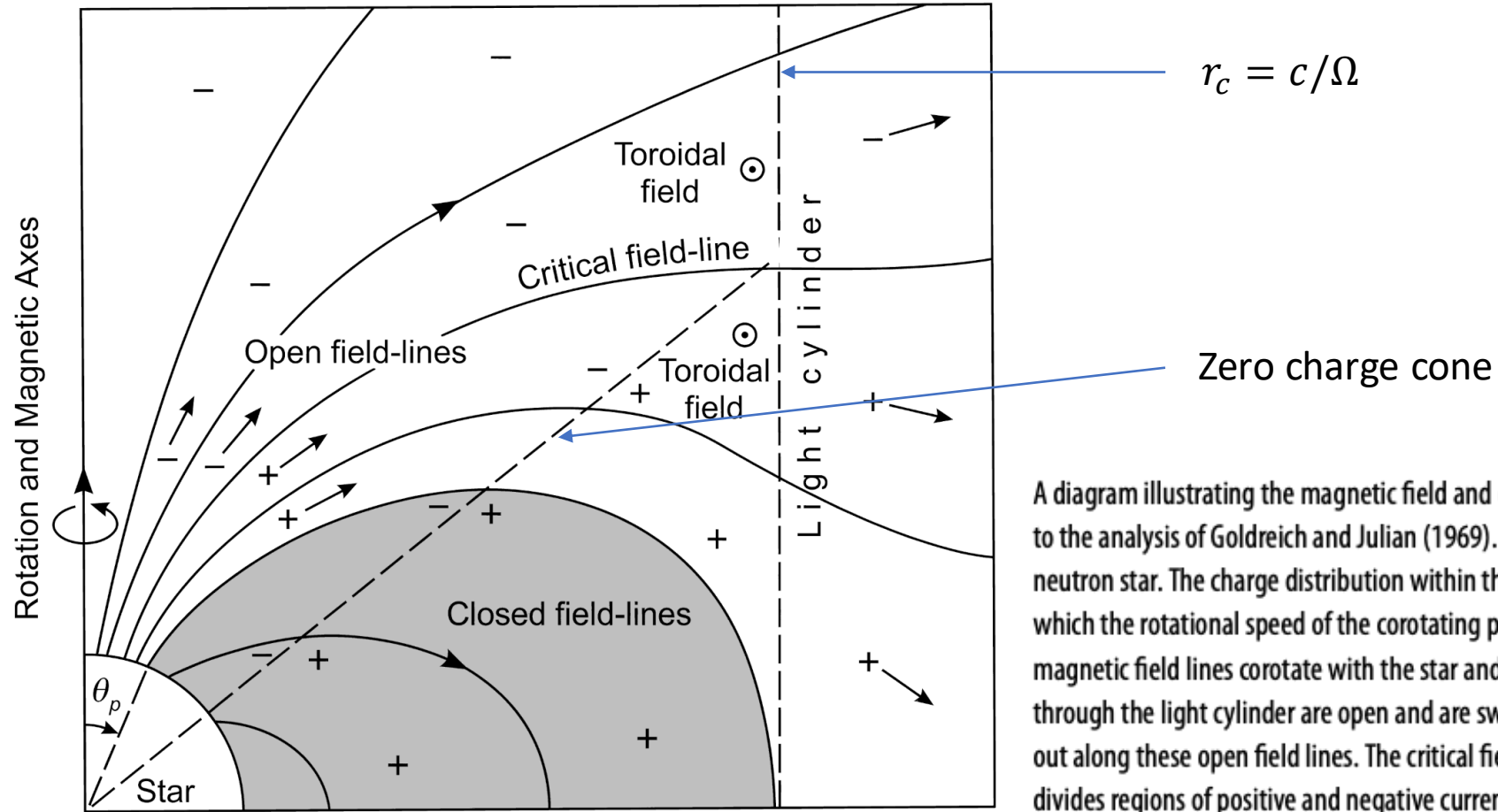
$$B_s = 3 \times 10^{15} (P \dot{P})^{1/2} \text{ T}$$

$$P_{\min} \approx 2B_5^{6/7} \text{ ms}$$

Maximum spin-up line, van den Heuvel 1987

$$\dot{P} \leq 2 \times 10^{-15} P^{4/3}$$

Pulsar Magnetosphere



A diagram illustrating the magnetic field and charge distribution about a rotating magnetised neutron star according to the analysis of Goldreich and Julian (1969). The magnetic axis is taken to be parallel to the rotation axis of the neutron star. The charge distribution within the magnetosphere is shown. The *light cylinder* is defined as that radius at which the rotational speed of the corotating particles is equal to the speed of light. Particles attached to closed magnetic field lines corotate with the star and form a corotating magnetosphere. The magnetic field lines which pass through the light cylinder are open and are swept back to form a toroidal field component. Charged particles stream out along these open field lines. The critical field line is at the same potential as the exterior interstellar medium and divides regions of positive and negative current flows from the star. The plus and minus signs indicate the sign of the electric charges in different regions about the neutron star as given by (13.50). The dashed line shows the *zero charge cone* (Manchester and Taylor, 1977).

$$\mathbf{E} + (\mathbf{v} \times \mathbf{B}) = \mathbf{E} + [(\boldsymbol{\Omega} \times \mathbf{r}) \times \mathbf{B}] = 0$$



$$\mathbf{E}_i = (\mathbf{v} \times \mathbf{B}) \quad \text{Induced electric field}$$

$$\phi = -\frac{B_0 \Omega R^5}{6r^3} (3 \cos^2 \theta - 1)$$

$$\mathbf{E} = -\nabla \phi$$

$$\vec{E} = -\vec{\nabla}\phi$$

$$e(\mathbf{v} \times \mathbf{B})/(GMm/R^2) \sim 10^{12}$$

Ratio of Lorentz force to gravitational force

$$\rho_e = e(n_+ - n_-)$$



$$\rho_e = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 B_0 \left(\frac{R^3}{r^3} \right) (1 - 3 \cos^2 \theta)$$

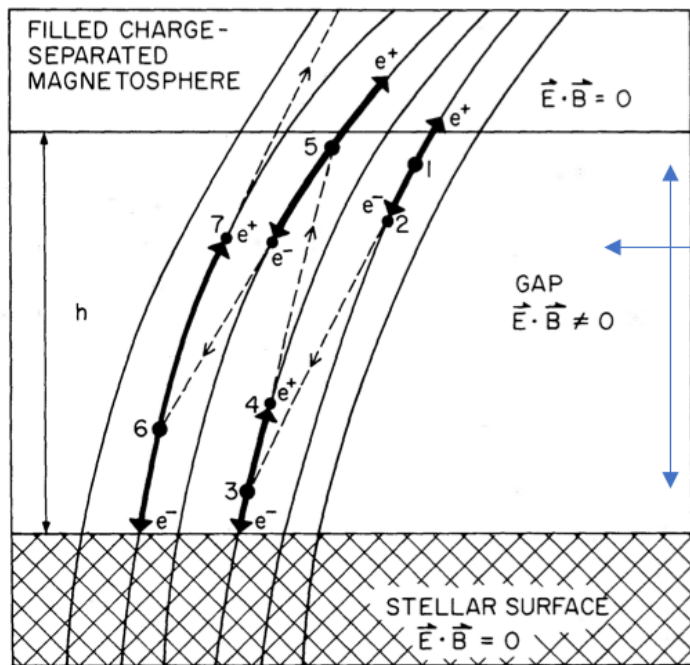
$$\frac{(\sin \theta)^2}{r} \equiv \textit{constant}$$

Exercise: Show this equation

$$\theta_{\text{pc}} = (R/r_c)^{1/2} = (\Omega R/c)^{1/2}$$

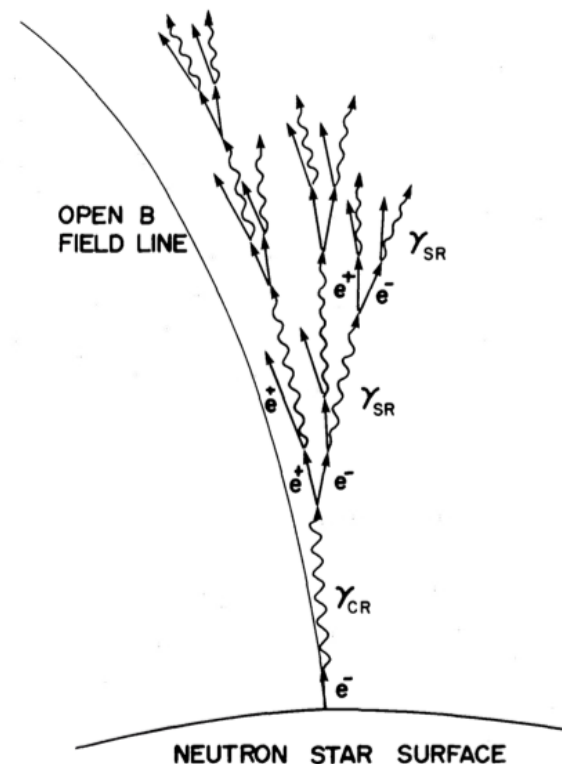
$$\Delta\phi \approx \frac{\Omega B_0 R^2 \theta_{\text{pc}}^2}{2} \approx \frac{\Omega^2 R^3 B_0}{2c} \approx 6 \times 10^{12} / P^2 \text{ Volt}$$

Ruderman 1975, Daugherty and Harding 1982



$$h \approx 10^4 \text{ cm}$$

FIG. 3.—Breakdown of the polar gap. The solid lines are polar field lines of average radius of curvature ρ ; for a pure dipole field $\rho \sim (Rc/\Omega)^{1/2} \sim 10^8 P^{1/2} \text{ cm}$, but for a realistic pulsar one expects $\rho \sim 10^6 \text{ cm}$ if many multipoles contribute near the surface. A photon (of energy $> 2mc^2$) produces an electron-positron at 1. The electric field of the gap accelerates the positron out of the gap and accelerates the electron toward the stellar surface. The electron moves along a curved field line and radiates an energetic photon at 2 which goes on to produce a pair at 3 once it has a sufficient component of its momentum perpendicular to the magnetic field. This cascade of pair production—acceleration of electrons and positrons along curved field lines—curvature radiation—pair production results in a “spark” breakdown of the gap.

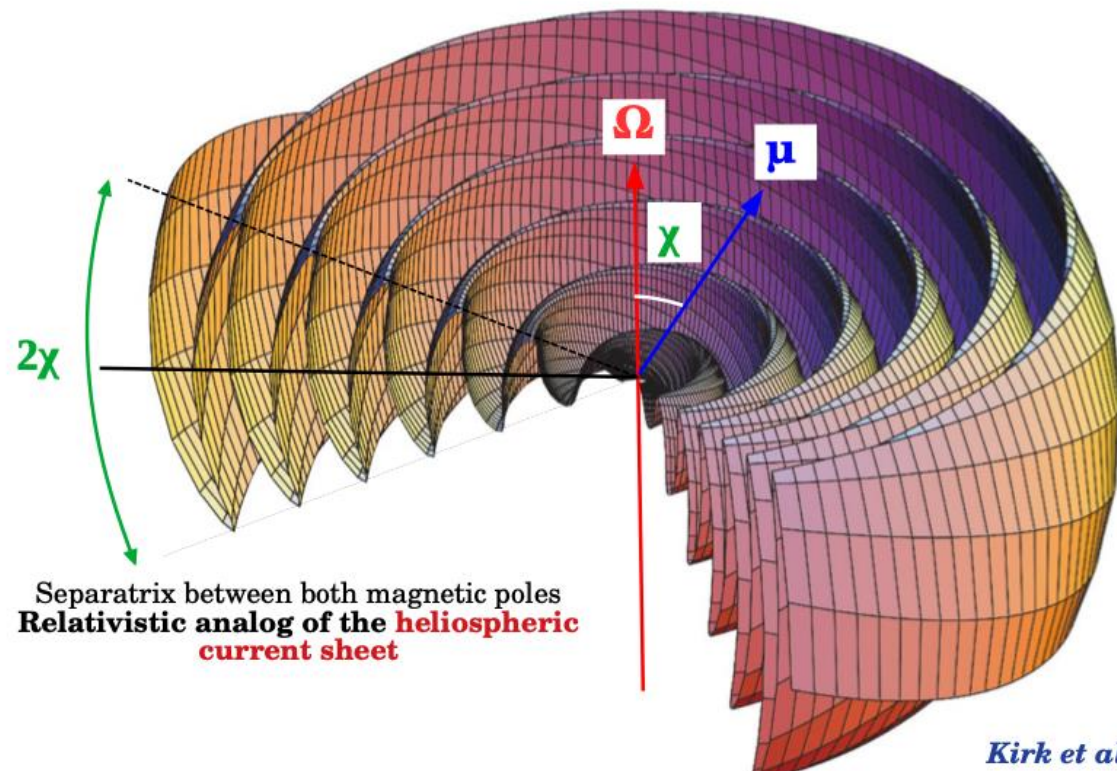
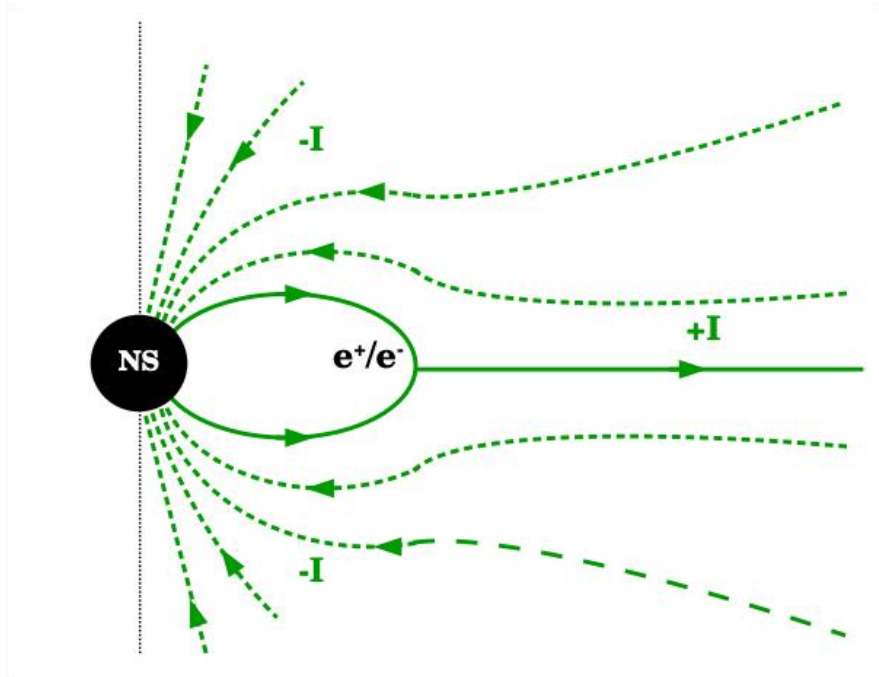


Coherent radio
pulsar emission

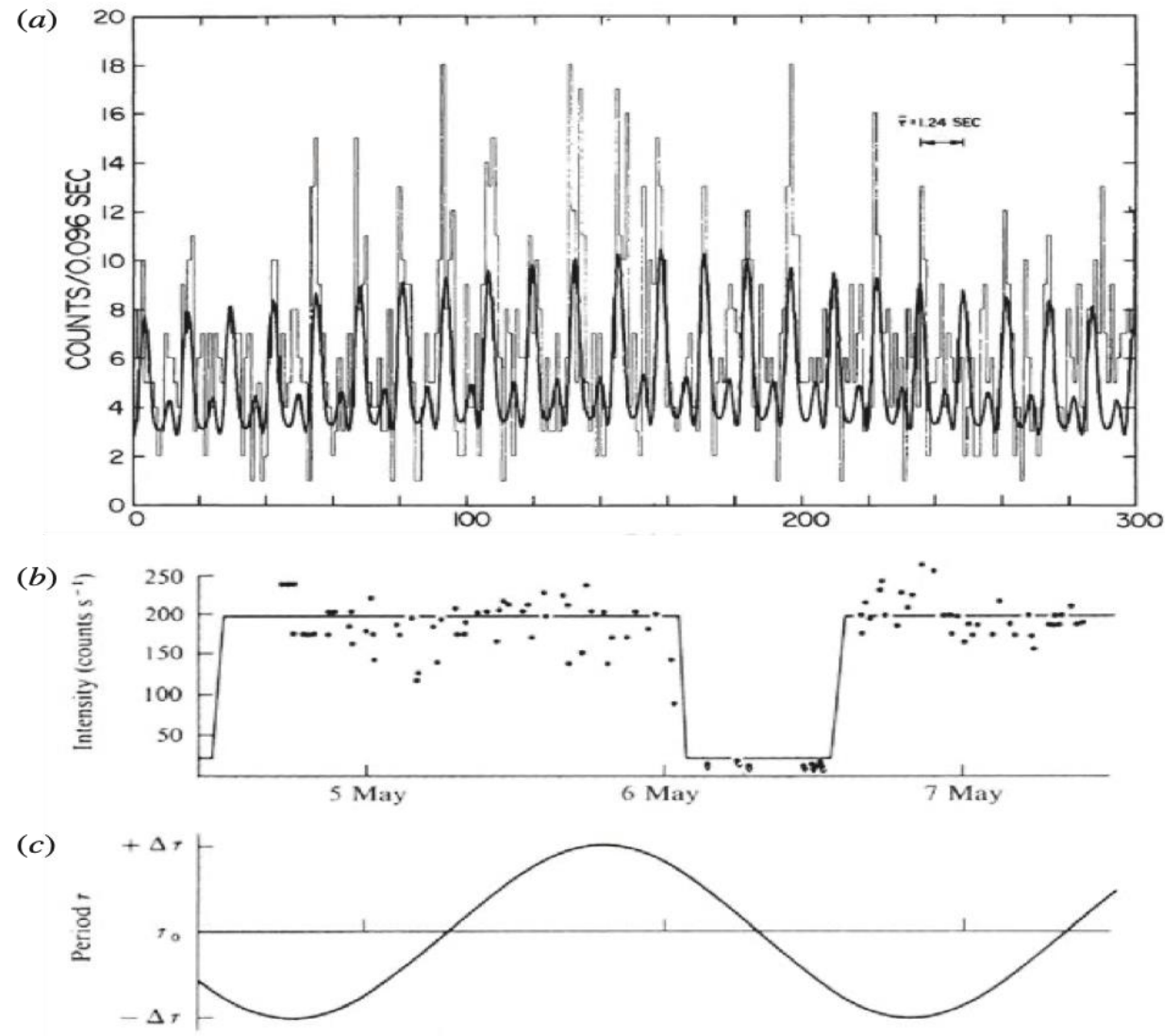
FIG. 1.—Schematic representation of a pair photon cascade initiated by an accelerated electron above the polar cap of a pulsar. Four generations of photons, including both curvature radiation photons, γ_{CR} , and synchrotron radiation photons, γ_{SR} , are shown.

Exercise: Show that $BP^2 \geq 10^7 \text{ Ts}^{-1}$ to form electron-photon cascades
Discuss difference between synchrotron and curvature radiation

Cerutti, HEAP 26

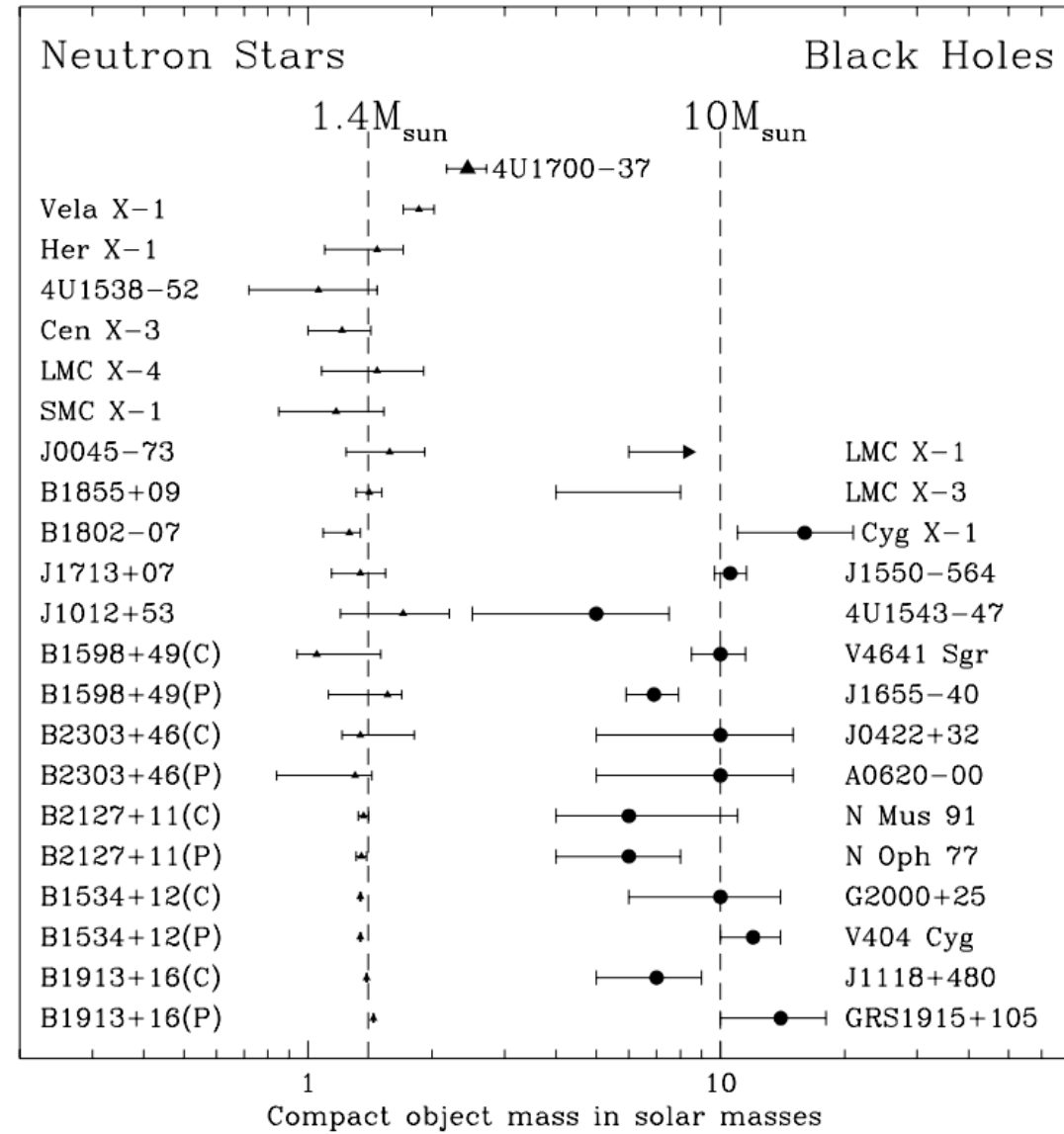


Accretion powered X ray pulsar



(a) The discovery record of the pulsating X-ray source Her X-1. The histogram shows the number of counts observed in successive 0.096 s bins. The continuous line shows the best-fitting harmonic curve to the observations, taking account of the varying sensitivity of the telescope as it swept over the source (Tananbaum *et al.*, 1972). (b) The rate of arrival of X-ray photons from Her X-1, showing the eclipse of the source by the primary star. The source is observed for about 34 hours and then is eclipsed for 6 hours. (c) Variations in the arrival time of pulses from Her X-1. The sinusoidal variation of the pulse arrival time is naturally attributed to the orbital motion of the X-ray source in a binary system.

$$f(M_X, M_0, i) = \frac{M_0^3 \sin^3 i}{(M_X + M_0)^2}$$

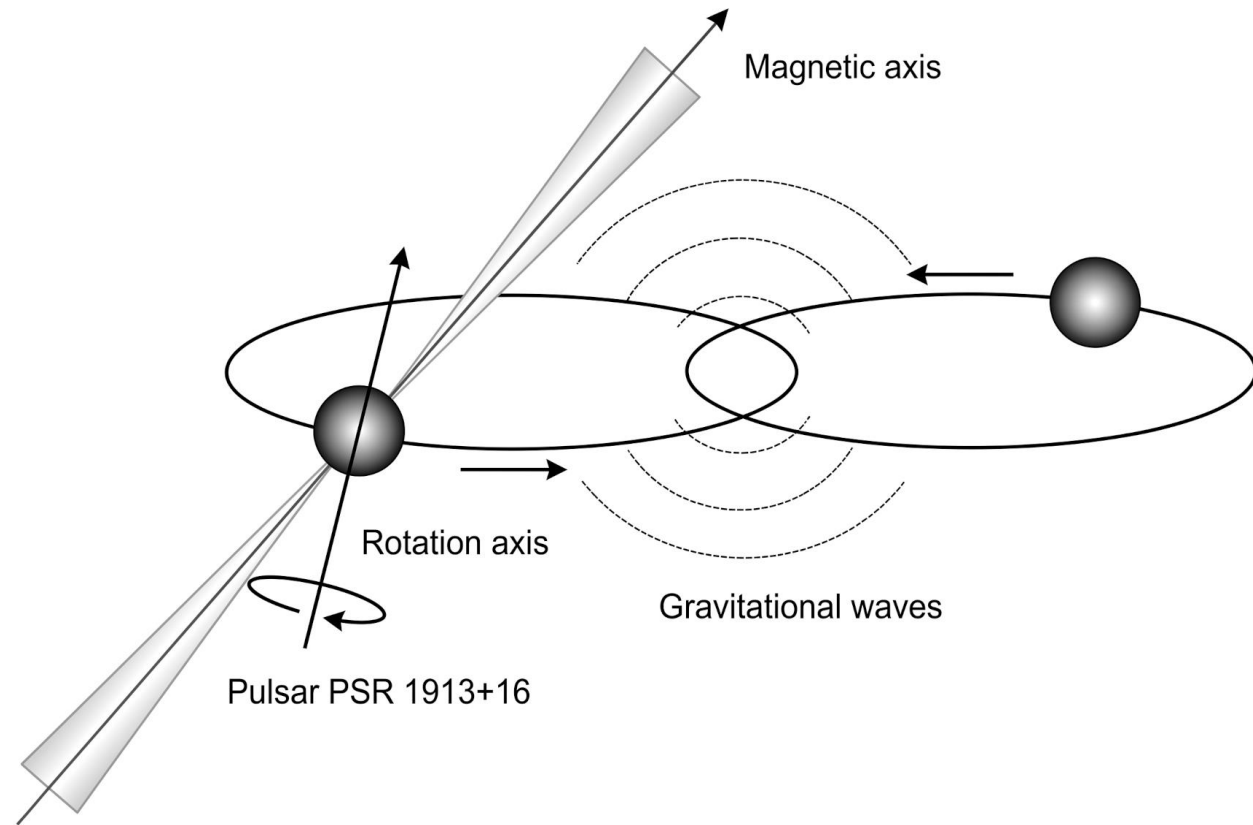


Examples of mass estimates for the neutron stars and black holes in X-ray binary systems and binary radio pulsars for which good mass determinations are available from their velocity curves and other information (Clark *et al.*, 2002).

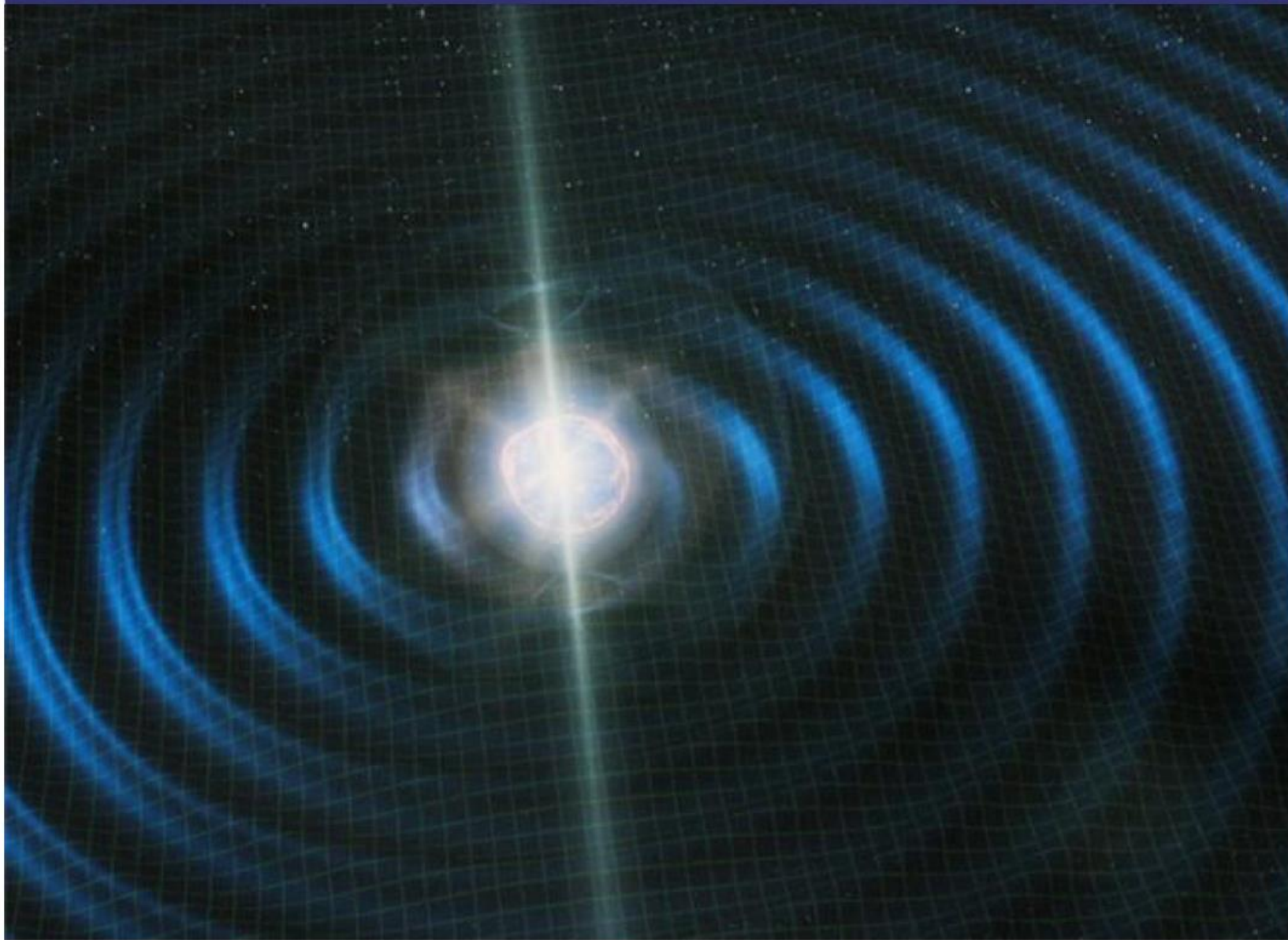
Binary Pulsars

$$-d\Omega/dt \propto \Omega^5$$

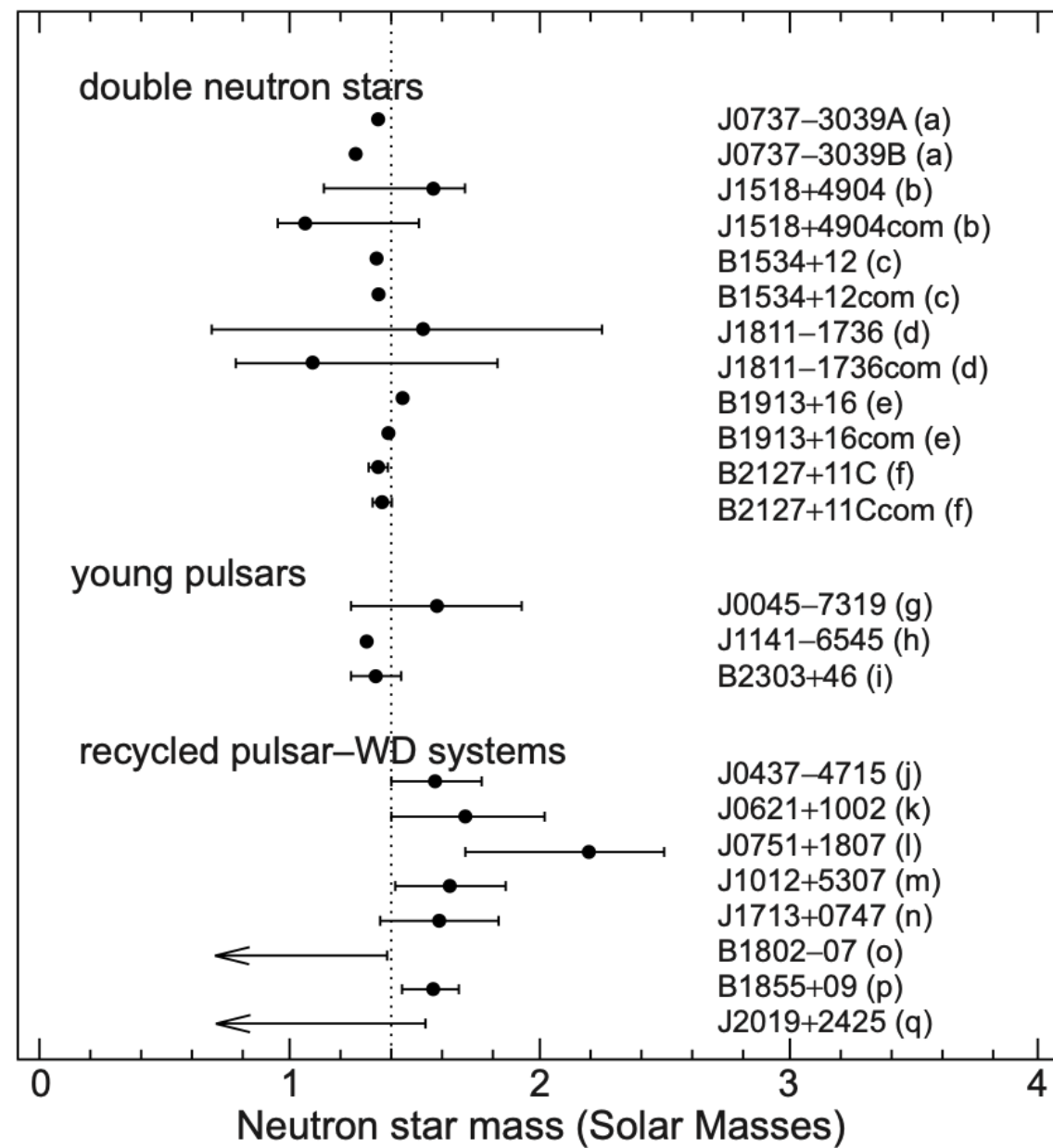
Exercise: Show this relation



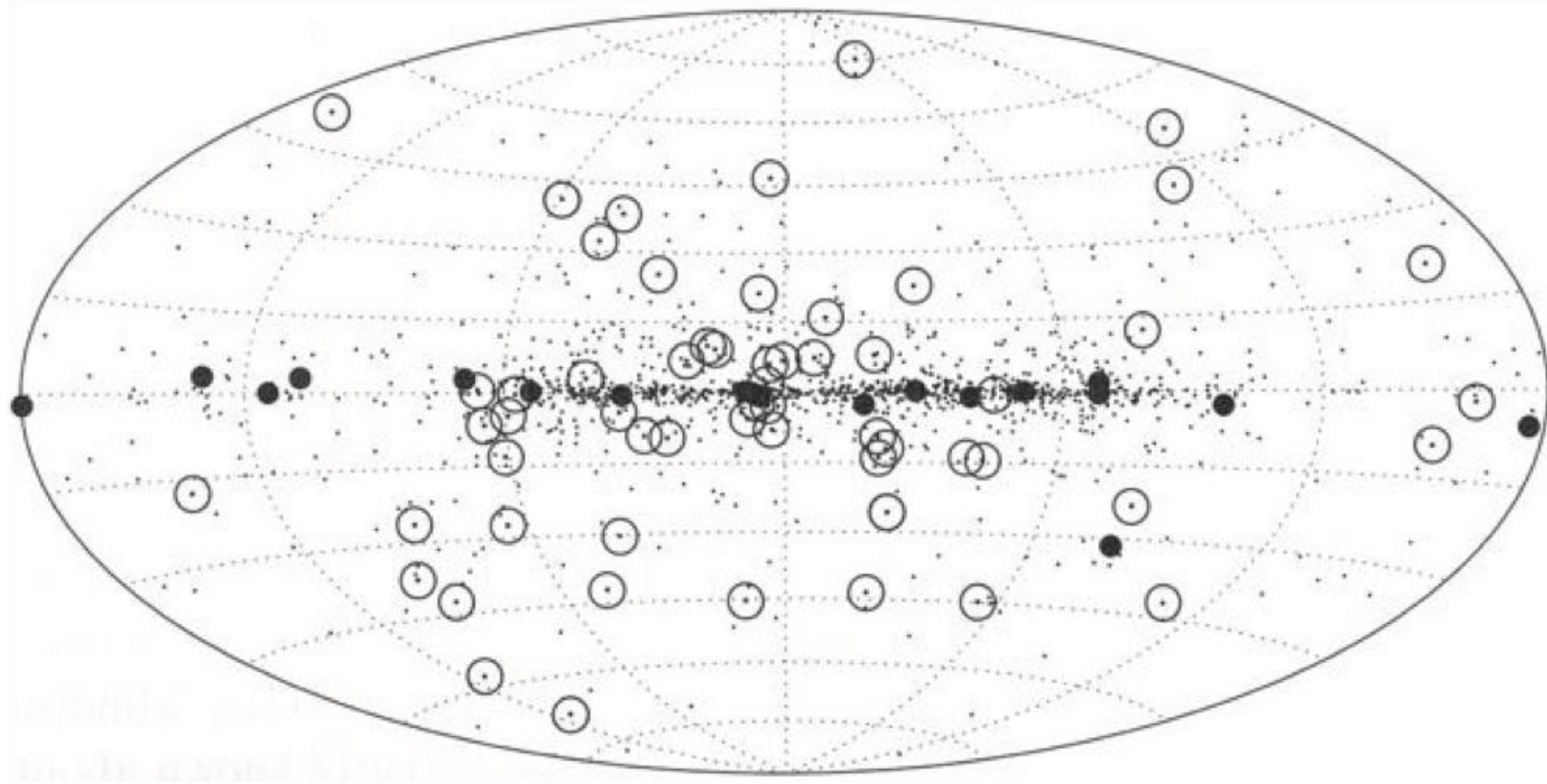
A schematic diagram showing the binary pulsar PSR B1913+16. As a result of the ability to measure precisely many parameters of the binary orbit by ultra-precise pulsar timing, the masses of the two neutron stars have been measured with very high precision.



This artist's impression shows the final stage of the merger of two neutron stars and the gravitational waves produced by the collision. Astronomers identified such a merger in 2017 using gravitational wave detectors, as well as with optical and X-ray telescopes.



The masses of neutron stars which are members of binary systems. The vertical dotted line indicates a mass of $1.4 M_{\odot}$ (Stairs, 2004; Lorimer and Kramer, 2005).



The Galactic distribution of pulsars in a Hammer–Aitoff projection. The dots are normal radio pulsars. Pulsar-supernova associations are shown as larger filled circles. Millisecond pulsars are indicated by dots with open circles (Lorimer and Kramer, 2005).

Black Holes

Mass, spin, charge, no hair theorem

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Sean Carroll; Ray D'Inverno & Vickers; Shapiro & Teukolsky

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

$$\kappa = \frac{8\pi G}{c^4} \approx 2.07665 \times 10^{-43} \text{ N}^{-1}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Einstein field equation

$$ds^2 = g_{ij}(x^k) dx^i dx^j$$

$$\frac{d^2 x^i}{ds^2} = -\Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} ,$$

Geodesic equation

where

$$\Gamma_{jk}^i = \frac{1}{2} g^{i\ell} (\partial_j g_{\ell k} + \partial_k g_{\ell j} - \partial_\ell g_{jk}) .$$

The quantity Γ_{jk}^i is called the affine connection.

$$ds^2 = dt^2 - \frac{1}{c^2} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]$$

Minkowski metric

$$ds^2 = \alpha dt^2 - \frac{1}{c^2} [\alpha^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] , \quad \text{where} \quad \alpha = \left(1 - \frac{2GM}{rc^2}\right)$$

Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GMr}{\rho c^2}\right) dt^2 - \frac{1}{c^2} \left[\frac{4GMr a \sin^2 \theta}{\rho c} dt d\phi + \frac{\rho}{\Delta} dr^2 + \rho d\theta^2 + \left(r^2 + a^2 + \frac{2GMr a^2 \sin^2 \theta}{\rho c^2}\right) \sin^2 \theta d\phi^2 \right] .$$

Kerr metric

$$\Delta = r^2 - \left(\frac{2GMr}{c^2}\right) + a^2 = 0 \quad \rho = r^2 + a^2 \cos^2 \theta. \quad a = (J/Mc)$$

$$r_+ = \frac{GM}{c^2} + \left[\left(\frac{GM}{c^2}\right)^2 - \left(\frac{J}{Mc}\right)^2 \right]^{1/2}$$

$$r_{\text{stat}} = \frac{GM}{c^2} + \left[\left(\frac{GM}{c^2}\right)^2 - \left(\frac{J}{Mc}\right)^2 \cos^2 \theta \right]^{1/2}$$

$$dx = \alpha^{-1/2} dr$$

$$dt' = \alpha^{1/2} dt$$

Schwarzschild metric

$$ds^2 = \alpha dt^2 - \frac{1}{c^2} [\alpha^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] , \quad \text{where} \quad \alpha = \left(1 - \frac{2GM}{rc^2}\right)$$

$$ds^2 = dt^2 - \frac{1}{c^2} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]$$

$$ds = d\tau = dt \left[\alpha - \frac{1}{c^2 \alpha} \left(\frac{dr}{dt} \right)^2 \right]^{1/2} = dt' \left(1 - \frac{1}{c^2} \frac{dx^2}{dt'^2} \right)^{1/2}$$

$$d\tau = dt' \left(1 - \frac{v^2}{c^2} \right)^{1/2} = \frac{dt'}{\gamma} \quad \text{proper time}$$

$$\text{Lorentz factor } \gamma = (1 - v^2/c^2)^{-1/2}$$

$$\Delta t' = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \Delta t$$



emitted

observed

$$\nu_{\infty} = \nu_{\text{em}} \left(1 - \frac{2GM}{rc^2}\right)^{1/2}$$

$$r \leq r_g = 2GM/c^2$$



$r \rightarrow r_g \Rightarrow \nu_{\infty} \rightarrow 0$ no radiation escape Schwarzschild radius

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv_\infty^2$$



$$v^2 = \dot{r}^2 + (r\dot{\theta})^2$$

$$\dot{r}^2 + (r\dot{\theta})^2 - \frac{2GM}{r} = v_\infty^2$$



$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2GM}{r} = \dot{r}_\infty^2$$



repulsive

Newtonian expressions

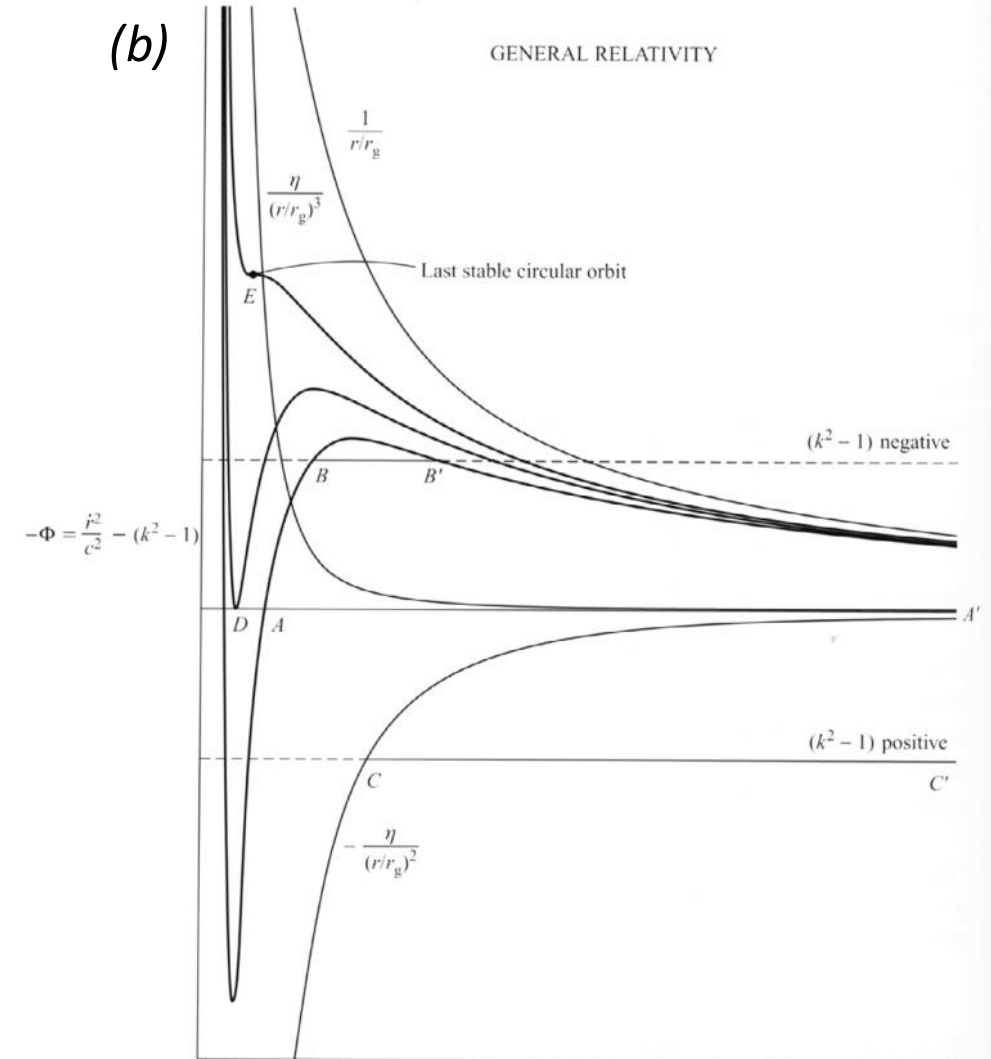
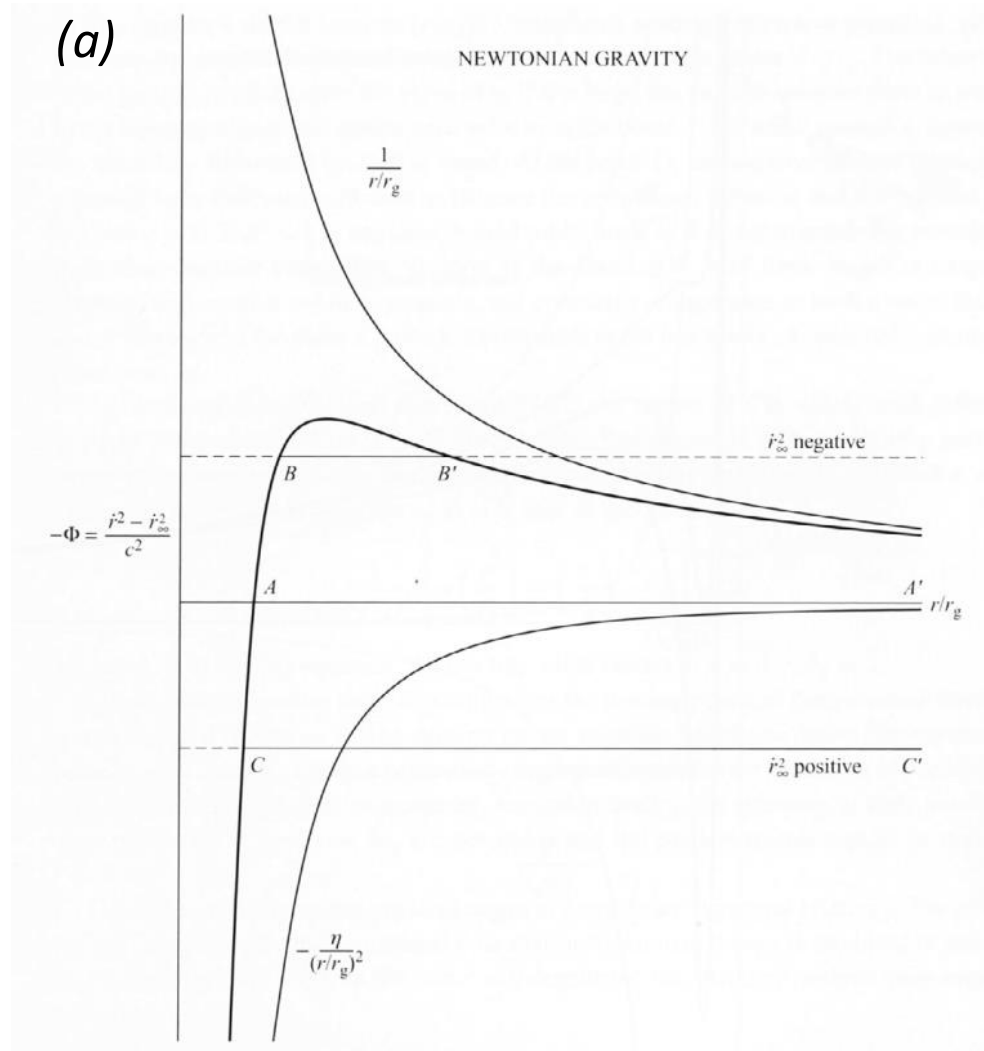
$h = \dot{\theta}r^2$ ← Angular momentum per unit mass which is constant

$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2GM}{r} - \frac{2GMh^2}{r^3 c^2} = (k^2 - 1)c^2 ,$$



Schwarzschild metric additional attractive term

$$r_g = \frac{2GM}{c^2} = 3 \left(\frac{M}{M_\odot} \right) \text{ km}$$



(a) Illustrating the variation of the potential $-\Phi = (\dot{r}^2 - \dot{r}_\infty^2)/c^2$ according to Newtonian gravity. There are two contributions to the potential, the gravitational potential varying as r^{-1} and the negative centrifugal potential varying as r^{-2} . (b) Illustrating the variation of the potential $-\Phi = (\dot{r}^2/c^2) - (k^2 - 1)$ according to general relativity. There is in addition an attractive gravitational potential varying as r^{-3} . The various loci illustrate the origin of black holes and the last stable orbit according to general relativity (Longair, 2003).

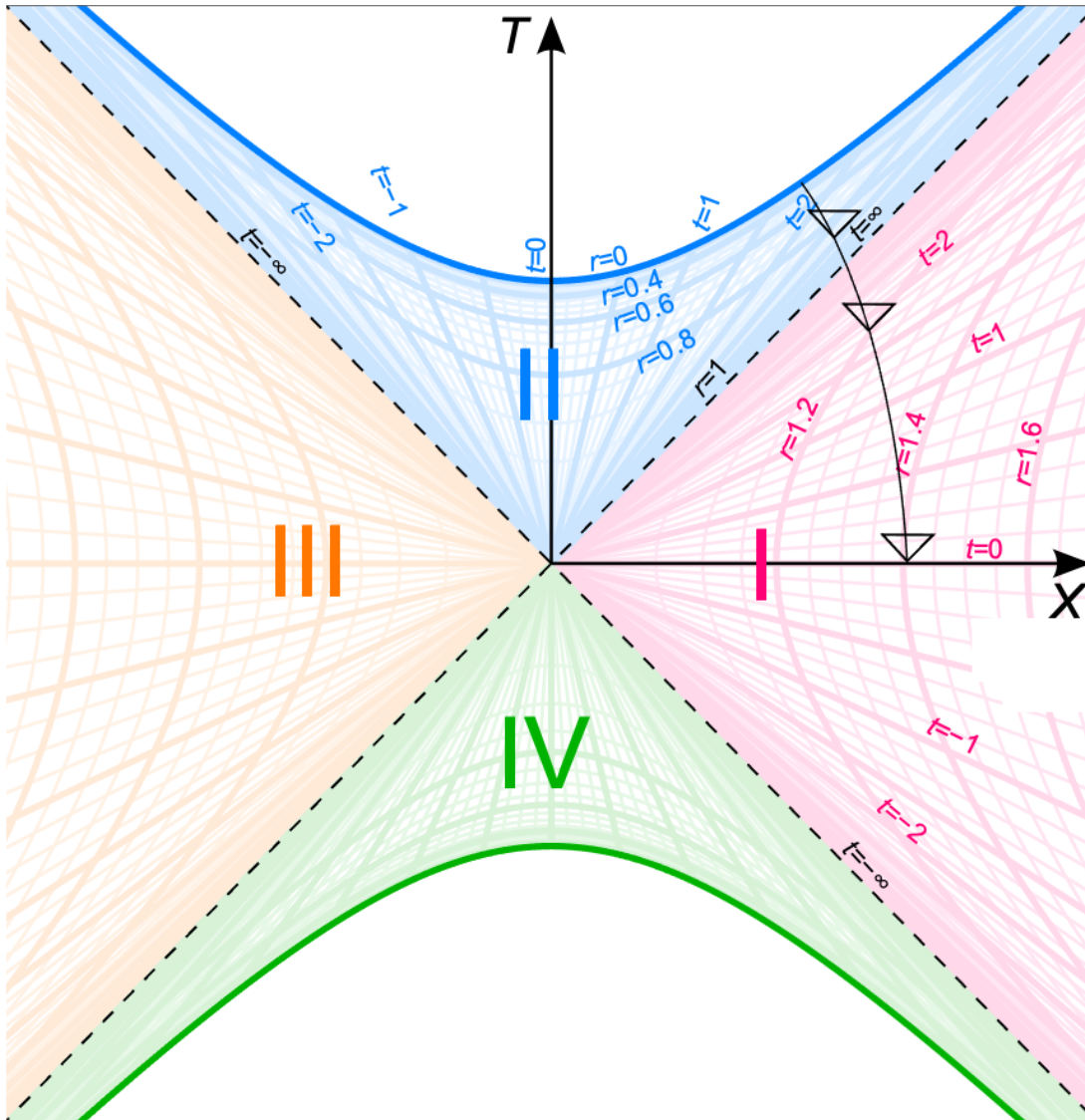
$r = 0$ real singularity

Penrose and Hawking singularity theorems

$$R_g = 2GM/c^2$$

Schwarzschild radius

Singularity in Schwarzschild metric disappears if Kruskal or Finkelstein coordinates are used



wikipedia

Kruskal–Szekeres diagram, illustrated for $2GM = 1$. The quadrants are the black hole interior (II), the white hole interior (IV) and the two exterior regions (I and III). The dotted 45° lines, which separate these four regions, are the [event horizons](#). The darker hyperbolas which bound the top and bottom of the diagram are the physical singularities. The paler hyperbolas represent [contours](#) of the Schwarzschild r coordinate, and the straight lines through the origin represent contours of the Schwarzschild t coordinate.

$$T = \left(\frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \sinh \left(\frac{t}{4GM} \right)$$

$$X = \left(\frac{r}{2GM} - 1 \right)^{1/2} e^{r/4GM} \cosh \left(\frac{t}{4GM} \right)$$

for the exterior region $r > 2GM$ outside the [event horizon](#)

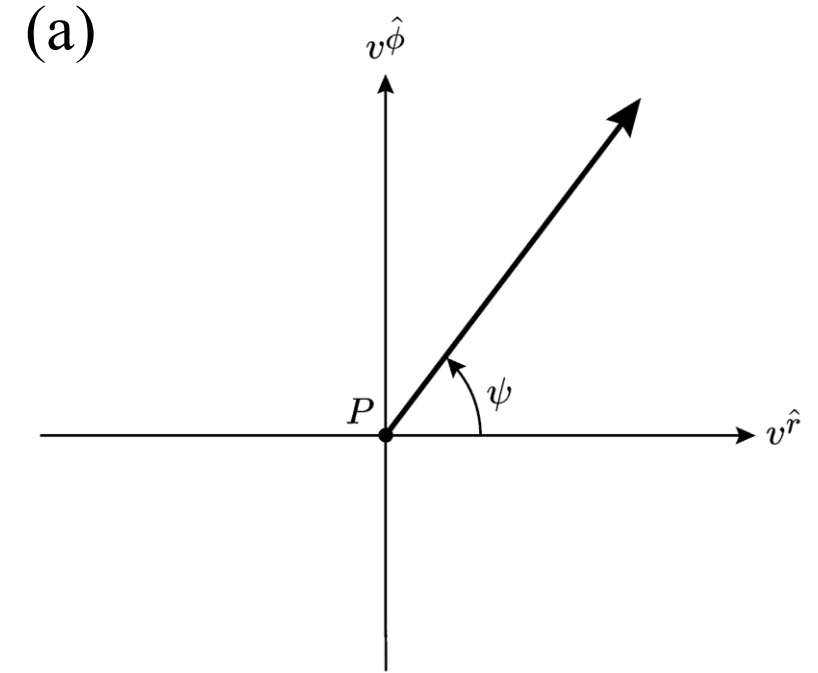
$$T = \left(1 - \frac{r}{2GM} \right)^{1/2} e^{r/4GM} \cosh \left(\frac{t}{4GM} \right)$$

$$X = \left(1 - \frac{r}{2GM} \right)^{1/2} e^{r/4GM} \sinh \left(\frac{t}{4GM} \right)$$

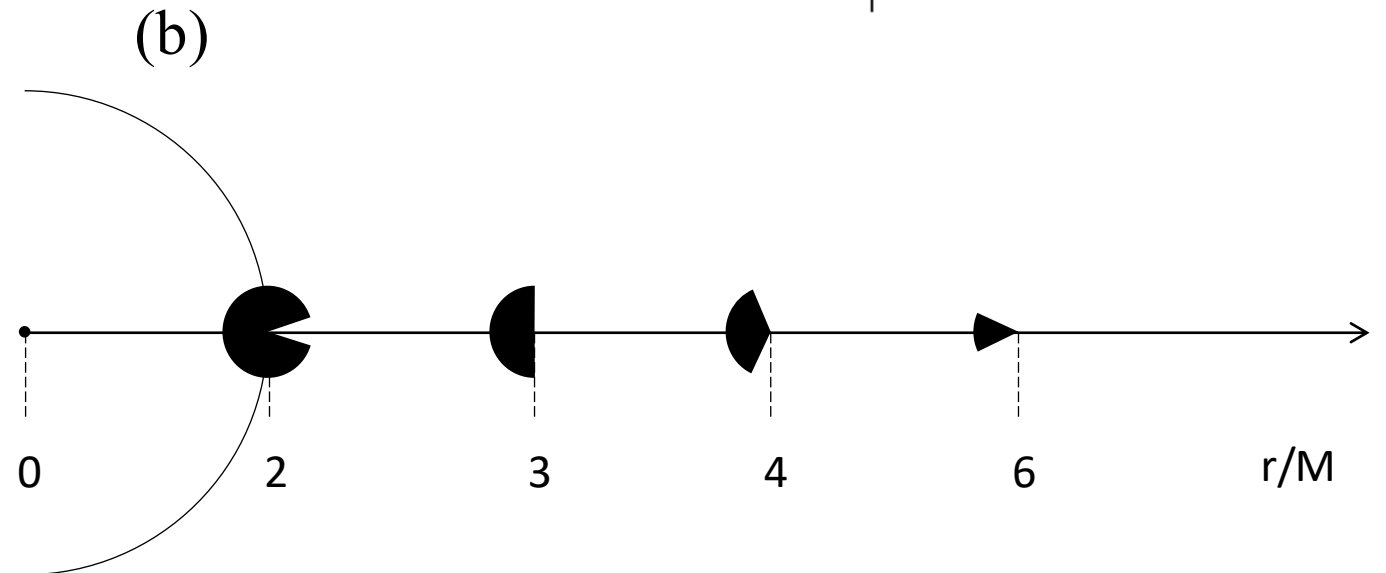
for the interior region $0 < r < 2GM$

Figure 12.6

- (a) The angle ψ between the propagation direction of a photon and radial direction at a given point p .
- (b) Gravitational capture of radiation by a Schwarzschild black hole. Rays emitted from each other into the interior of the shaded conical cavity are captured. The indicated capture cavities are those measured in the orthonormal frame of a local static observer.



Shapiro & Teukolsky



$$ds^2 = \left(1 - \frac{2GM}{\rho c^2}\right) dt^2 - \frac{1}{c^2} \left[\frac{4GMra \sin^2 \theta}{\rho c} dt d\phi \right. \\ \left. + \frac{\rho}{\Delta} dr^2 + \rho d\theta^2 + \left(r^2 + a^2 + \frac{2GMra^2 \sin^2 \theta}{\rho c^2}\right) \sin^2 \theta d\phi^2 \right]$$

$$\Delta = r^2 - \left(\frac{2GM}{c^2}\right)r + a^2 = 0$$

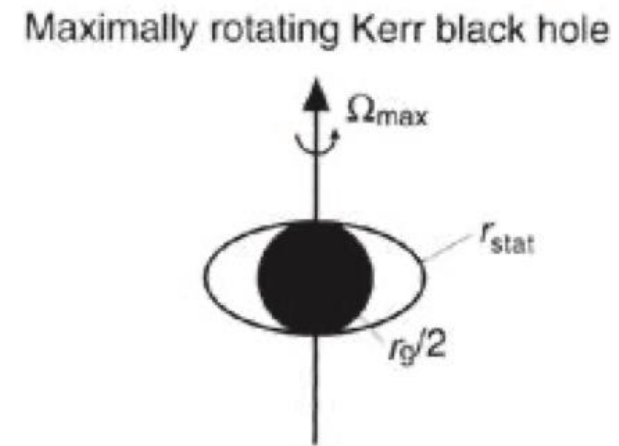
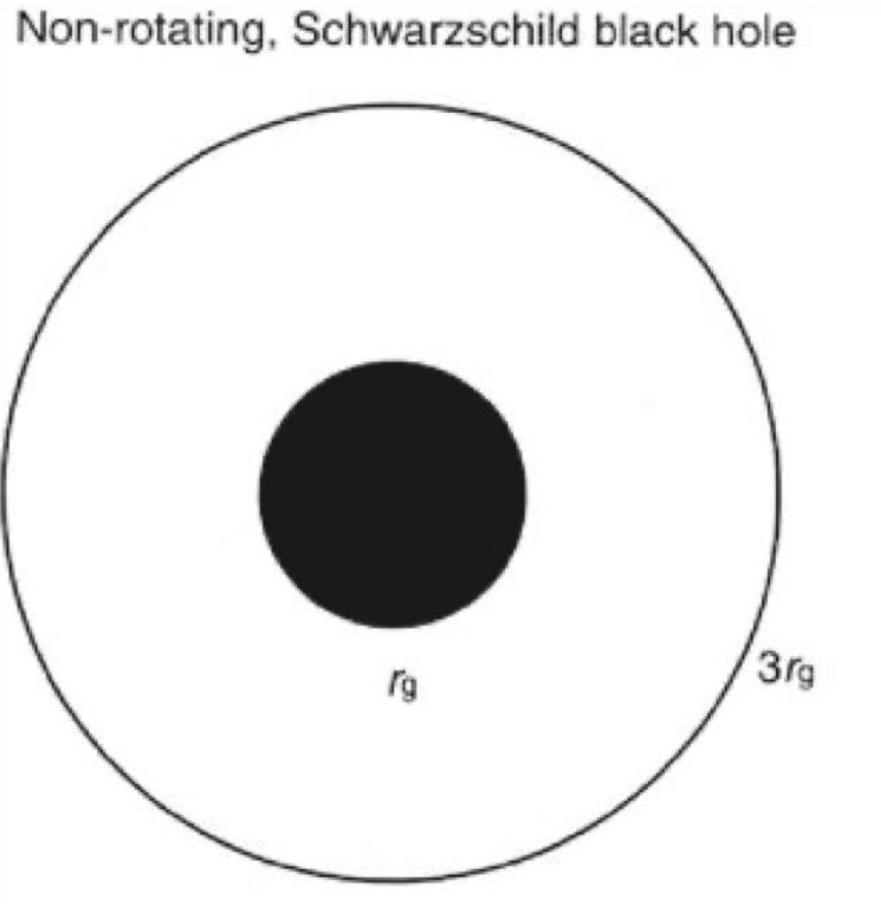
Positive
root

$$J = GM^2/c$$

Maximum angular momentum

$$r_+ = \frac{GM}{c^2} + \left[\left(\frac{GM}{c^2}\right)^2 - \left(\frac{J}{Mc}\right)^2 \right]^{1/2} \longrightarrow r_+ = \frac{GM}{c^2} \text{ where } r_g = \frac{2GM}{c^2}$$

$$r_{\text{stat}} = \frac{GM}{c^2} + \left[\left(\frac{GM}{c^2}\right)^2 - \left(\frac{J}{Mc}\right)^2 \cos^2 \theta \right]^{1/2} \longrightarrow \text{Static limit}$$



A schematic diagram illustrating the various regions about a maximally rotating black hole compared with the corresponding dimensions for a non-rotating black hole of the same mass.

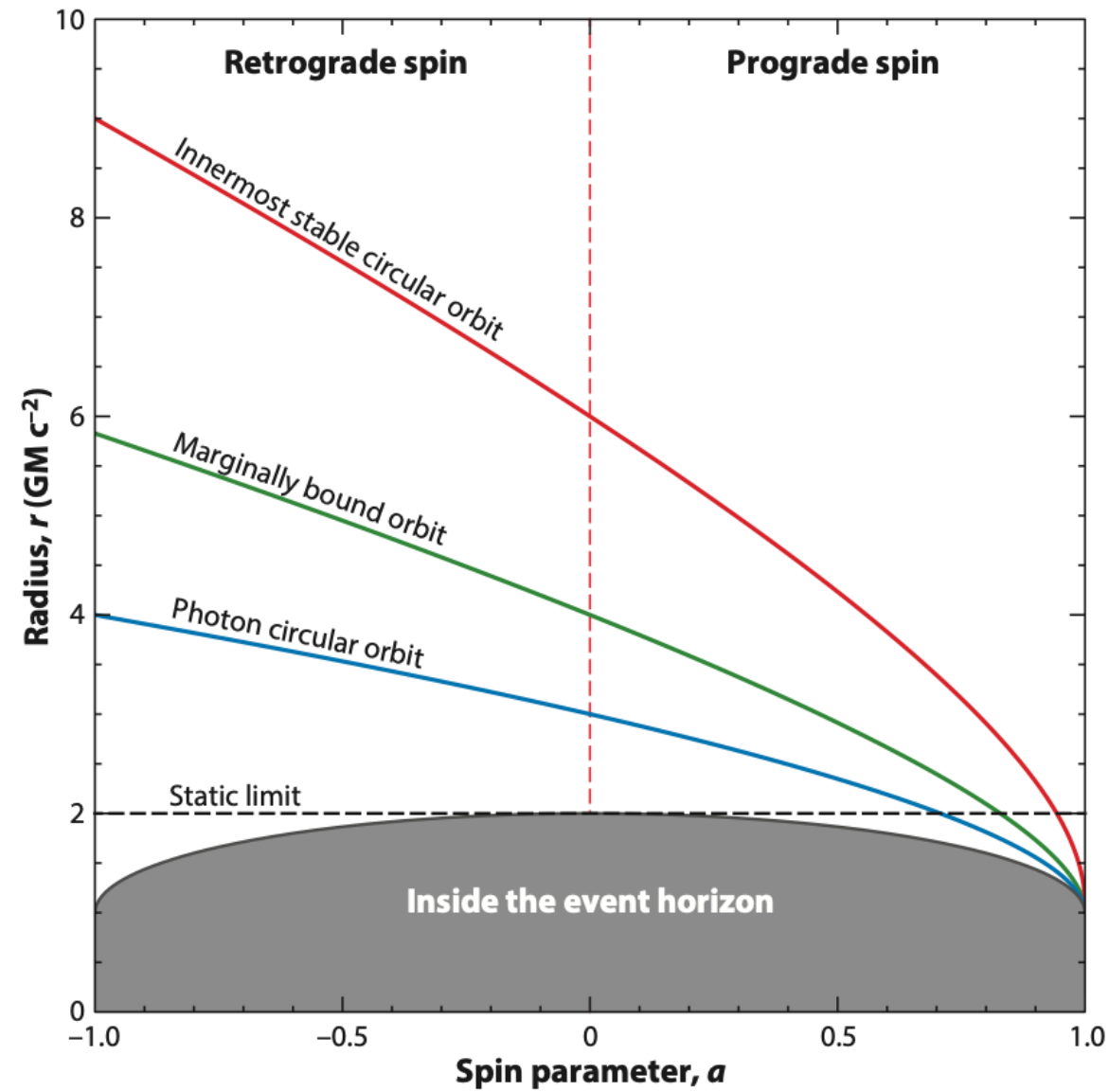


Figure 1

Boyer–Lindquist radius of special orbits in the equatorial plane of a Kerr black hole as a function of dimensionless spin parameter a . See Section 2 for a detailed discussion.

Sean Carroll,
Spacetime and Geometry, 2014

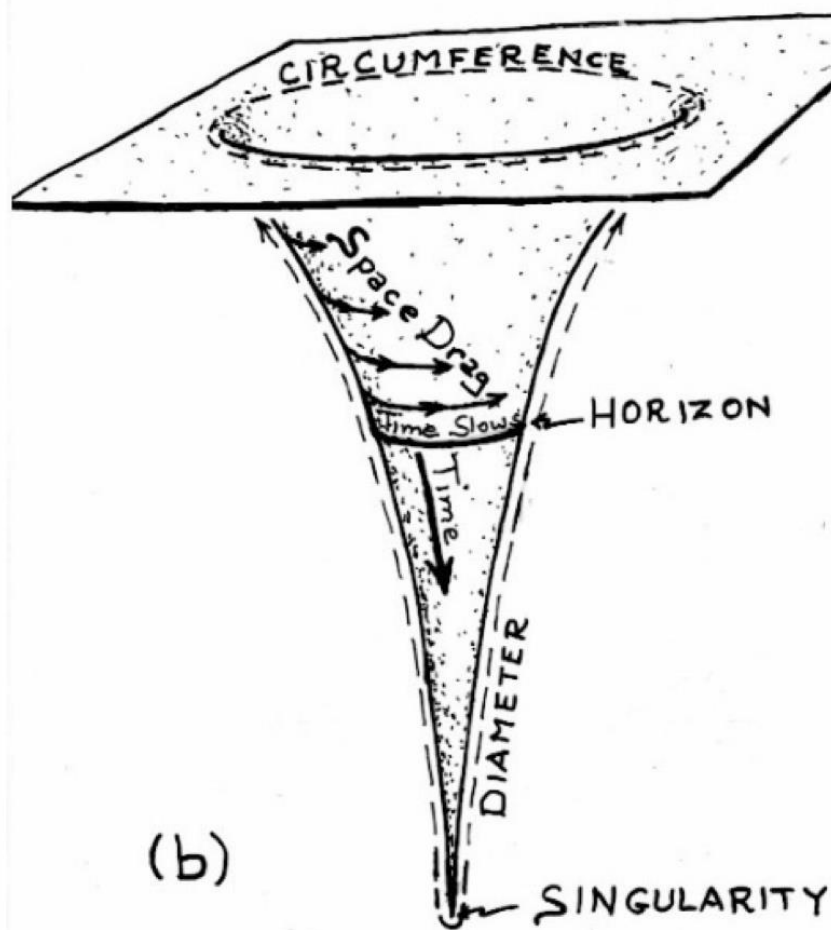
- $\Omega_{BH} = ac^2/(2GM r_{horizon})$
- Black hole spin: $10^5 / \left(\frac{M}{M_{sun}} \right) \frac{a}{1+\sqrt{1-a^2}}$ rad/sec
- If $M = 100 M_{sun}$ and $a = 0.5 \rightarrow 27$ rad/sec (4 rotations per second)
 $a = 0.9 \rightarrow 626$ rad/sec (100 rotation per second)

Stellar-mass black holes

Cygnus X-1	$a \sim 0.998$
GRS 1915+105	$a \sim 0.98$
A0620-00	$a \sim 0.12$
LMC X-3	$a \sim 0.25$

Supermassive black holes

M87*	$a \sim 0.9$
Sagittarius A*	$a \sim 0.5$



The whirl of spacetime is produced by the hole's huge rotational angular momentum (that resides in the spacetime whirl itself). The warpage's energy and angular momentum create the warpage, according to Einstein's equations. Warpage begets warpage. K. Thorne

<http://www.its.caltech.edu/~kip/index.html/PubScans/BlackHoles-Thorne-Starmus.pdf>

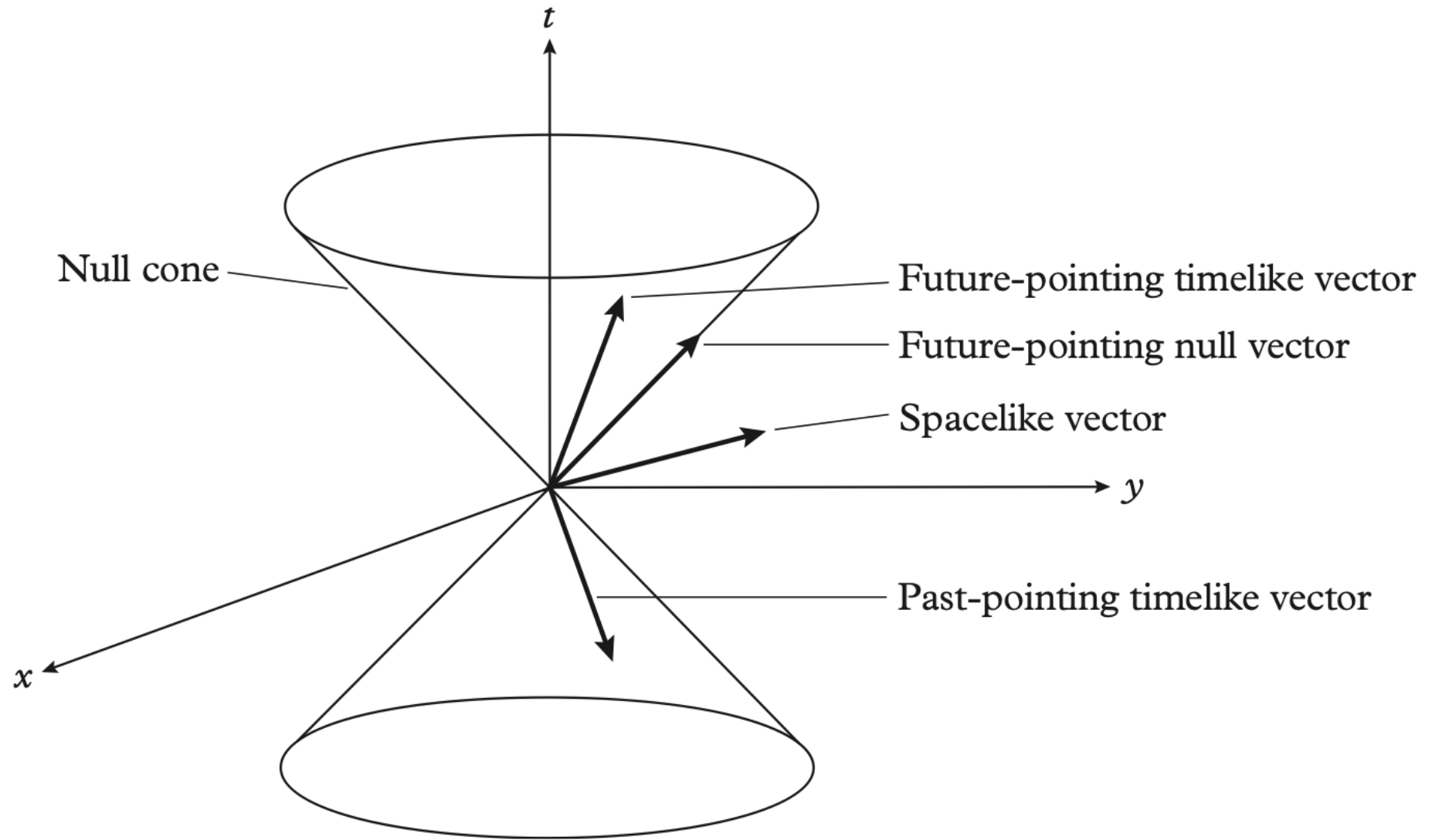


Fig. 8.1 The null cone with one dimension (the z -direction) suppressed.

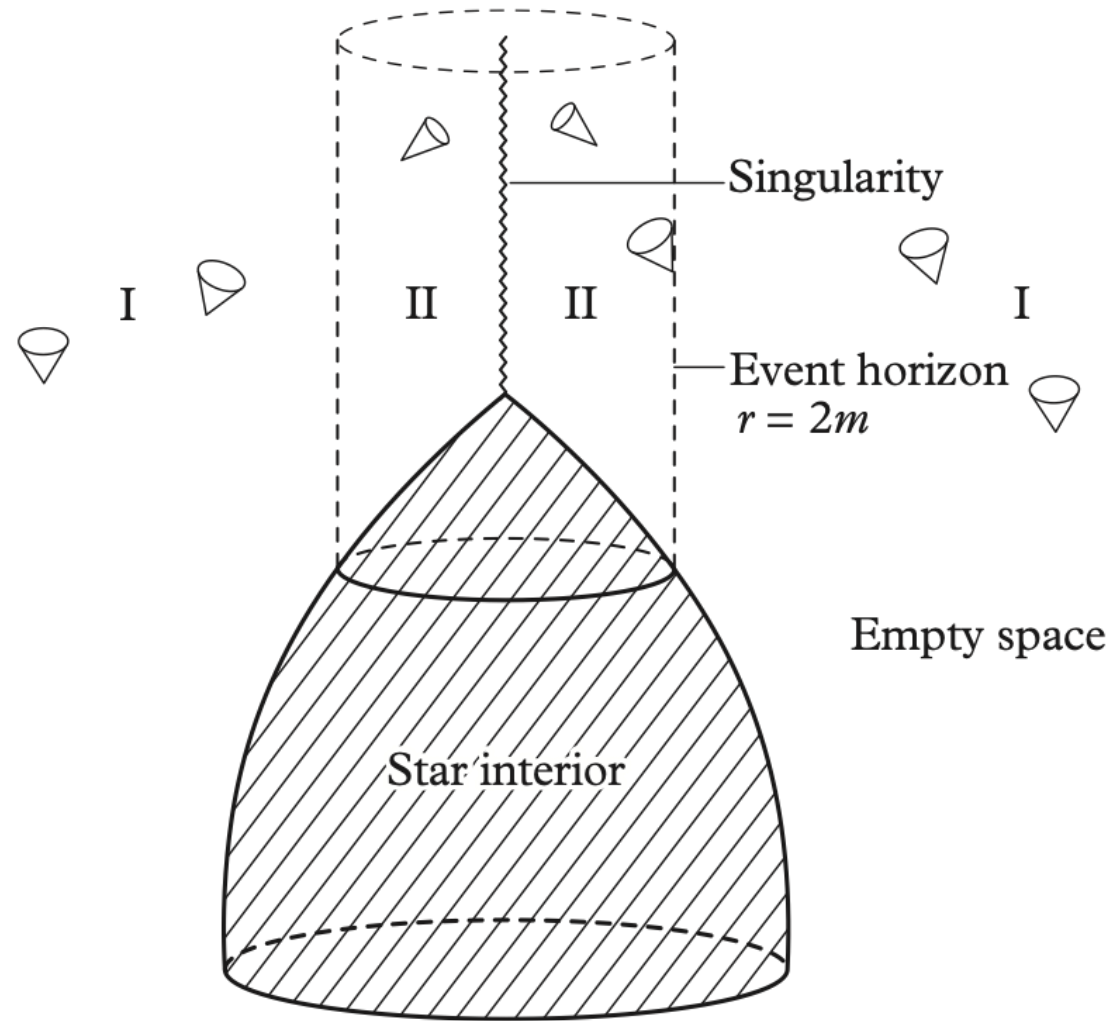


Fig. 17.14 Gravitational collapse (one spatial dimension suppressed).

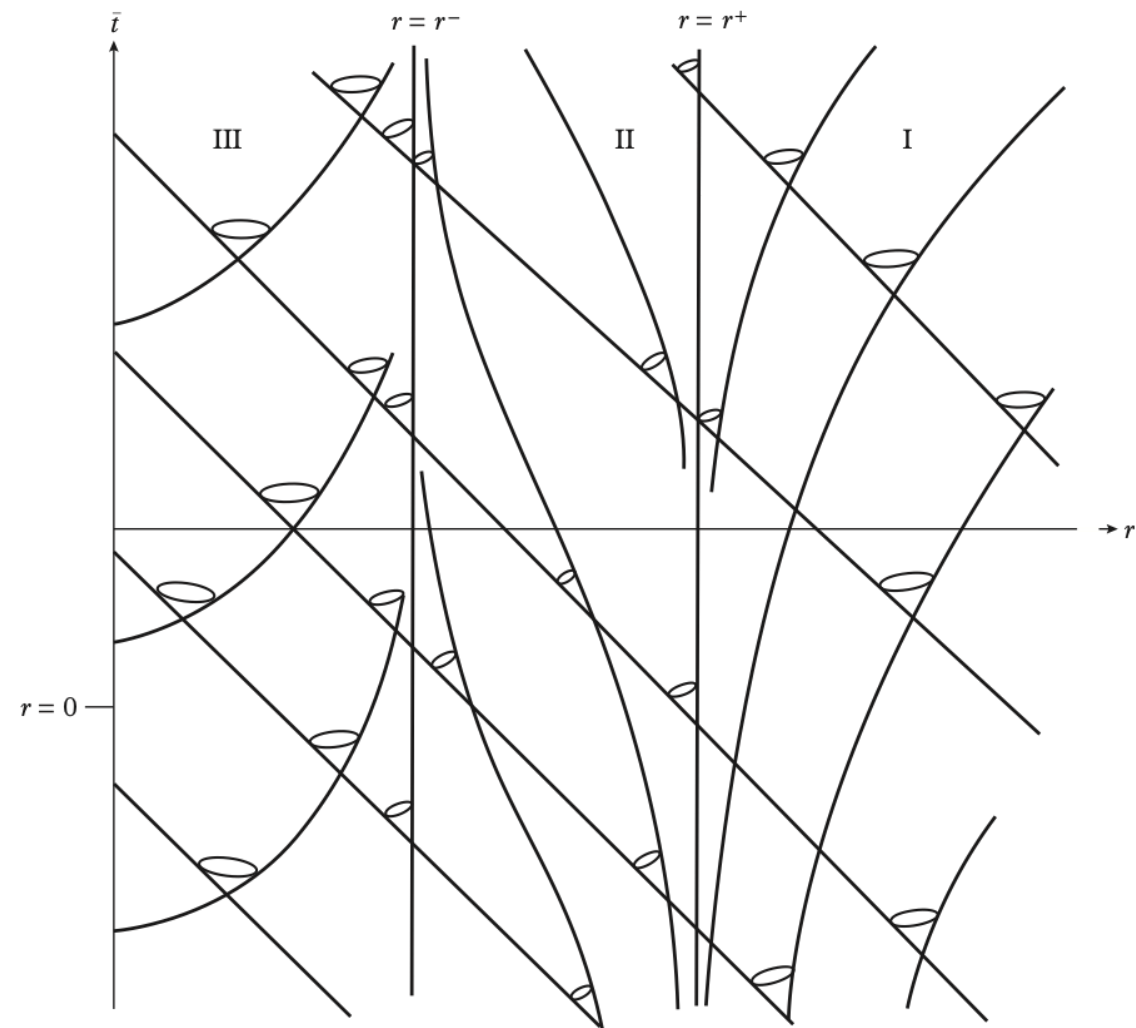


Fig. 20.4 Kerr solution ($a^2 < m^2$) in advanced Eddington–Finkelstein type coordinates.

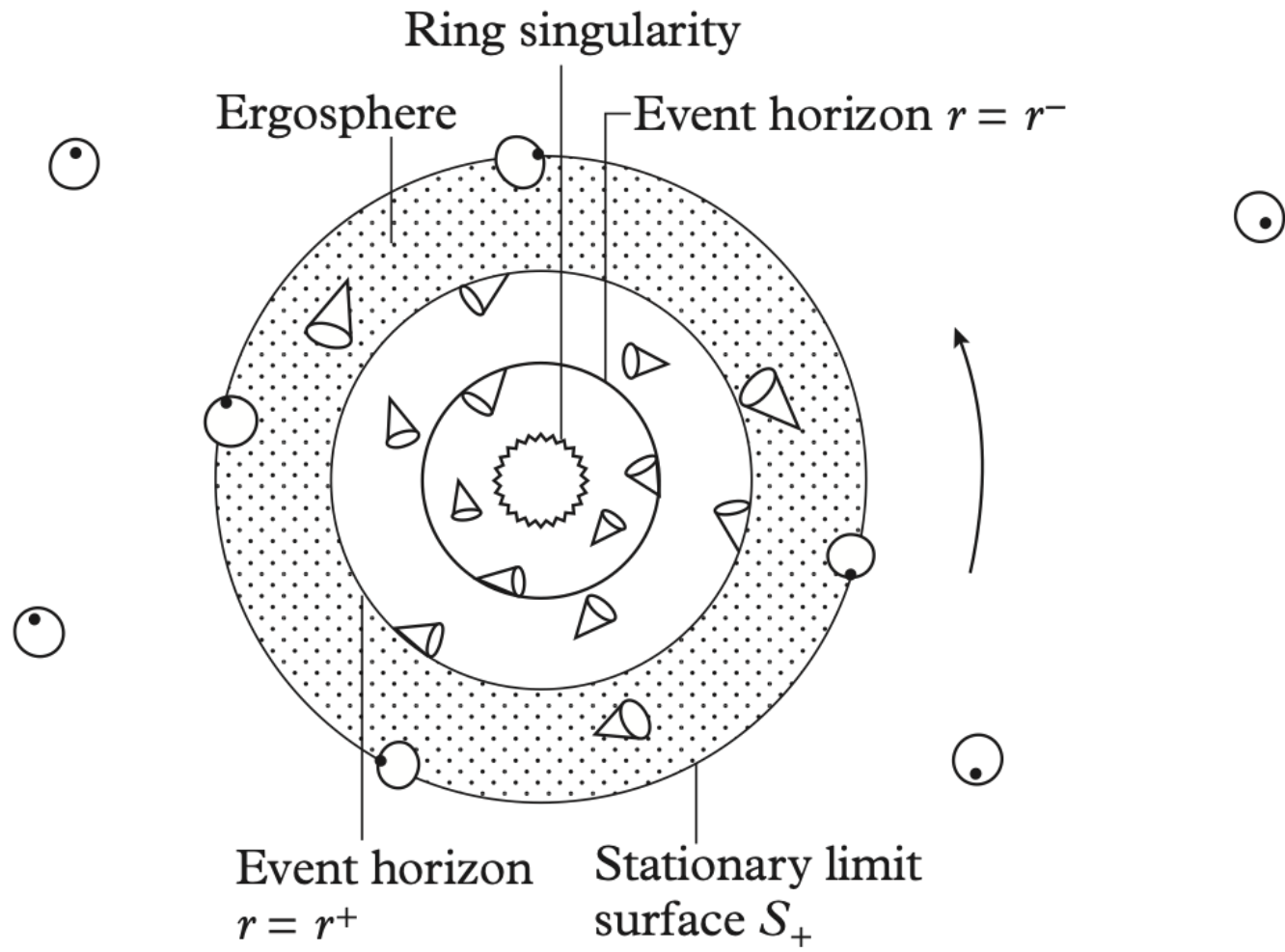
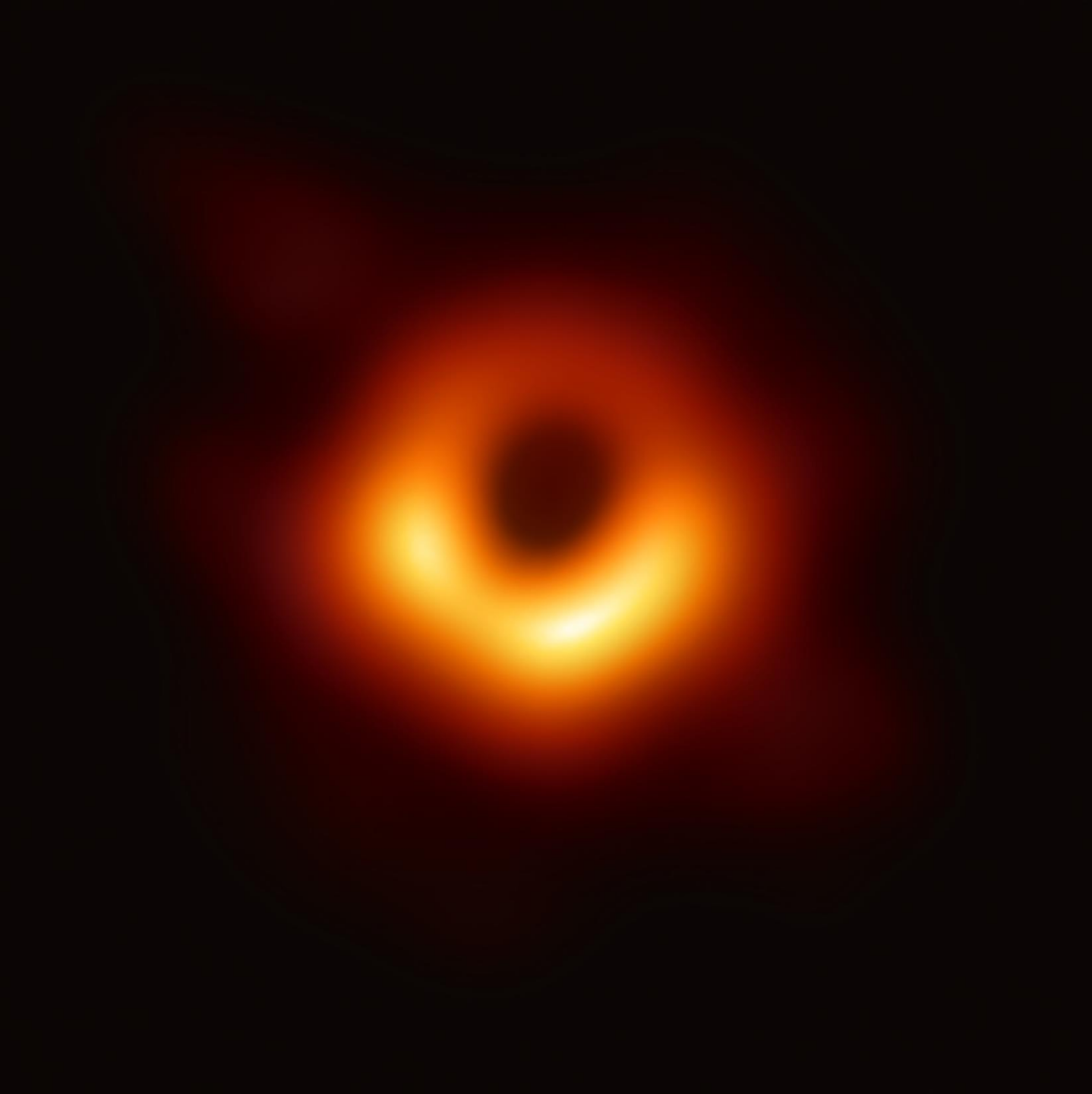


Fig. 20.5 Spatial diagram of the Kerr solution ($\alpha^2 < m^2$) in the equatorial plane.

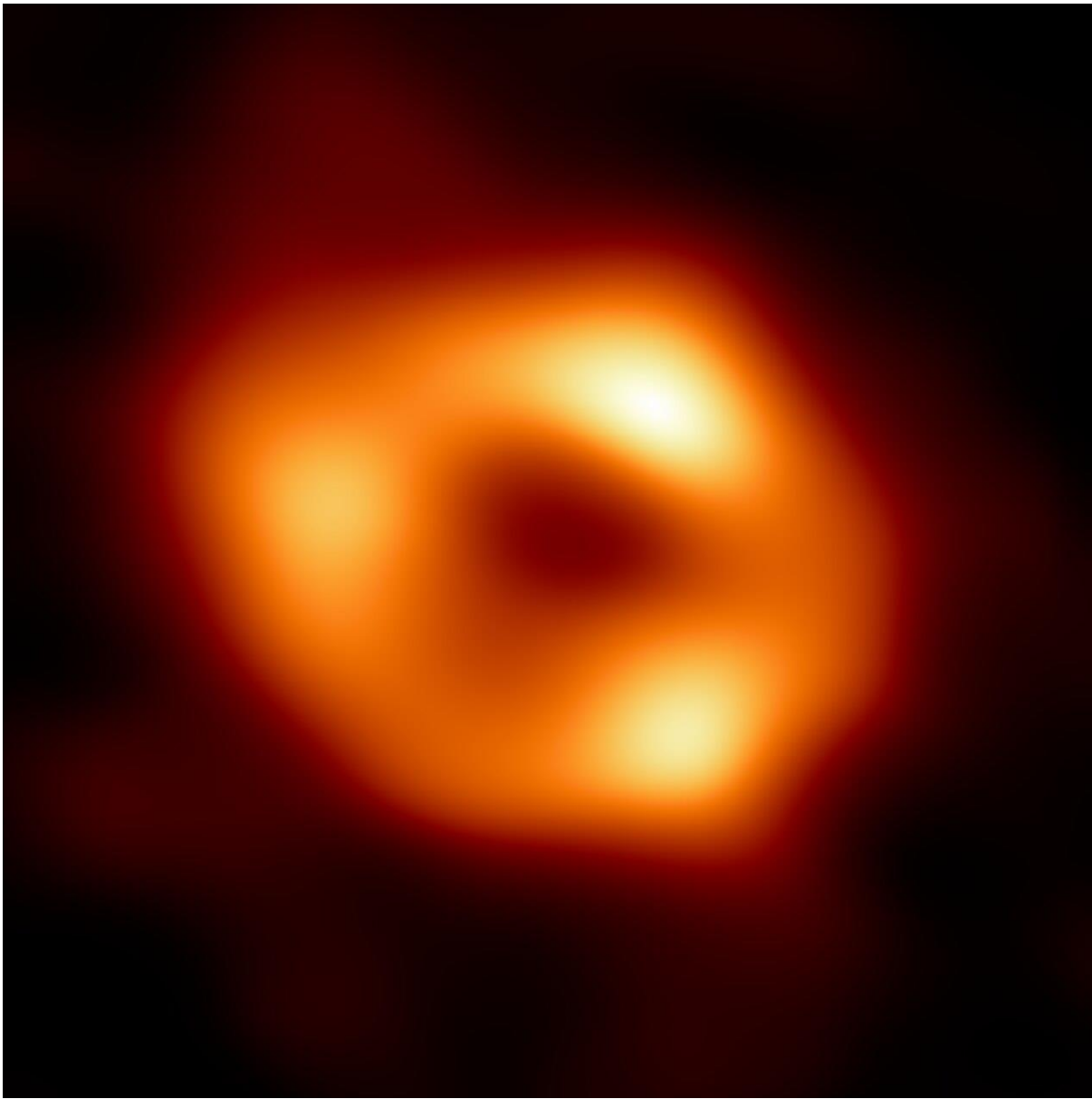


M87*

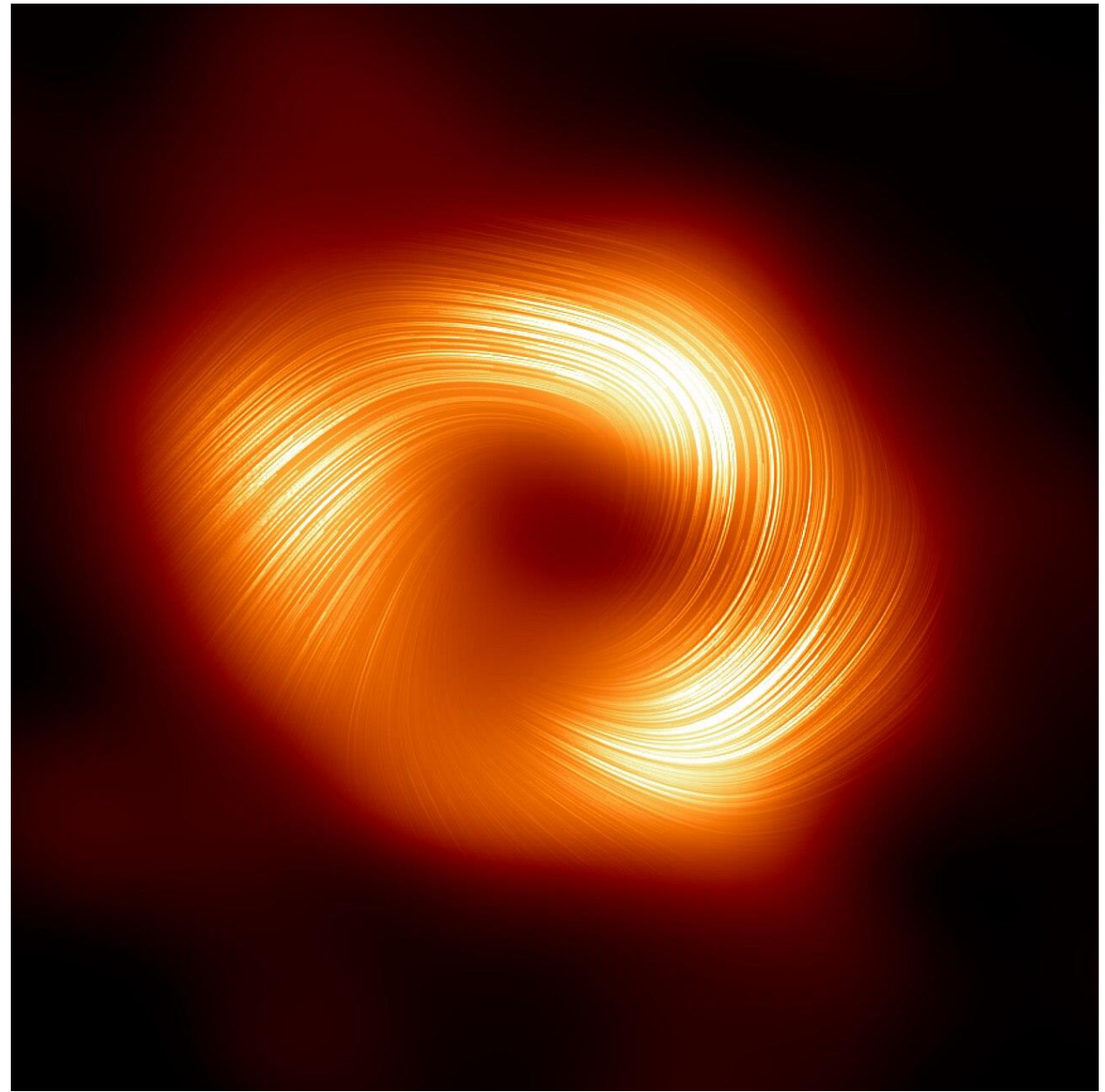
Event Horizon
Telescope image

$$M \sim 7 \times 10^9 M_{Sun}$$

Wikipedia



Sagittarius A* ($M \sim 4 \times 10^6 M_{Sun}$)



left: 2022, right: polarized

Wikipedia



EHT array

NASA/GSFC simulation

Image of the disk's far side

The black hole's gravitational field alters the path of light from the far side of the disk, producing this part of the image.

Photon ring

A ring of light composed of multiple distorted images of the disk. The light making up these images has orbited the black hole two, three or even more times before escaping to us. They become thinner and fainter closer to the black hole.

Black hole shadow

This is an area roughly twice the size of the event horizon — the black hole's point of no return — that is formed by its gravitational lensing and capture of light rays.

Doppler beaming

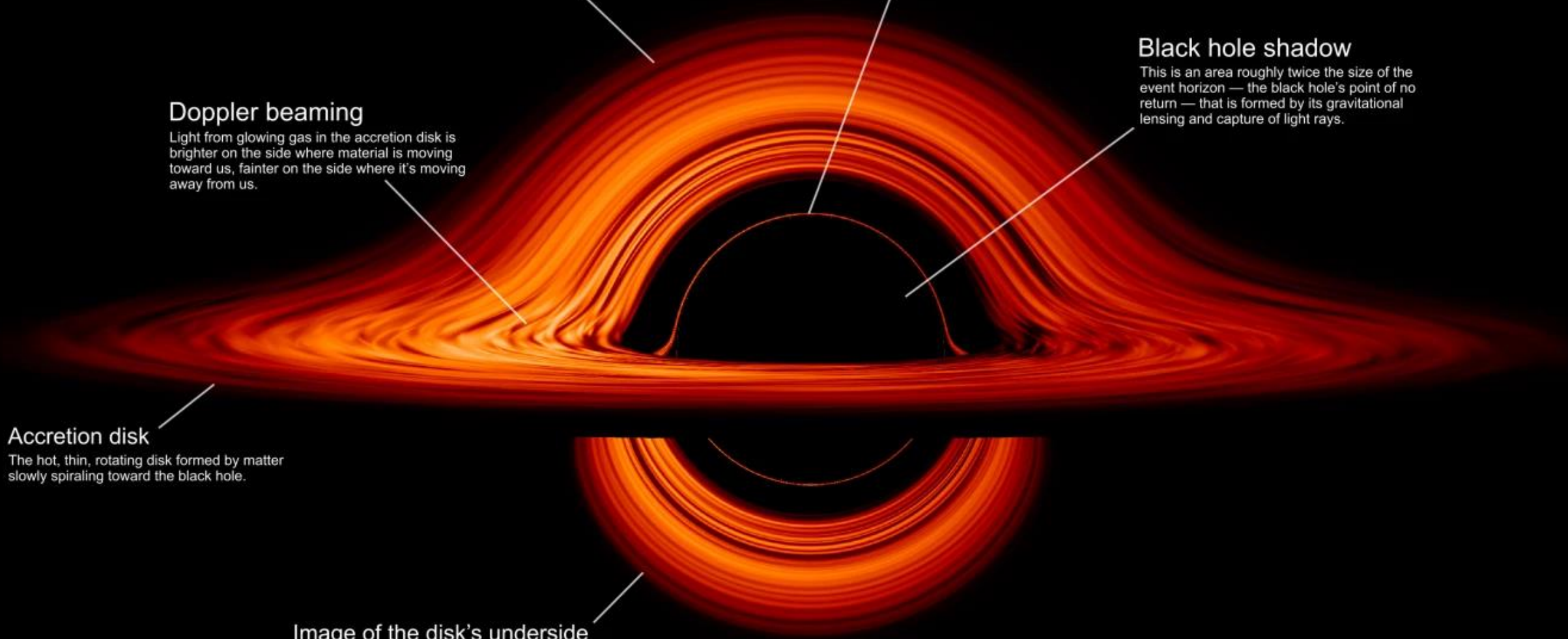
Light from glowing gas in the accretion disk is brighter on the side where material is moving toward us, fainter on the side where it's moving away from us.

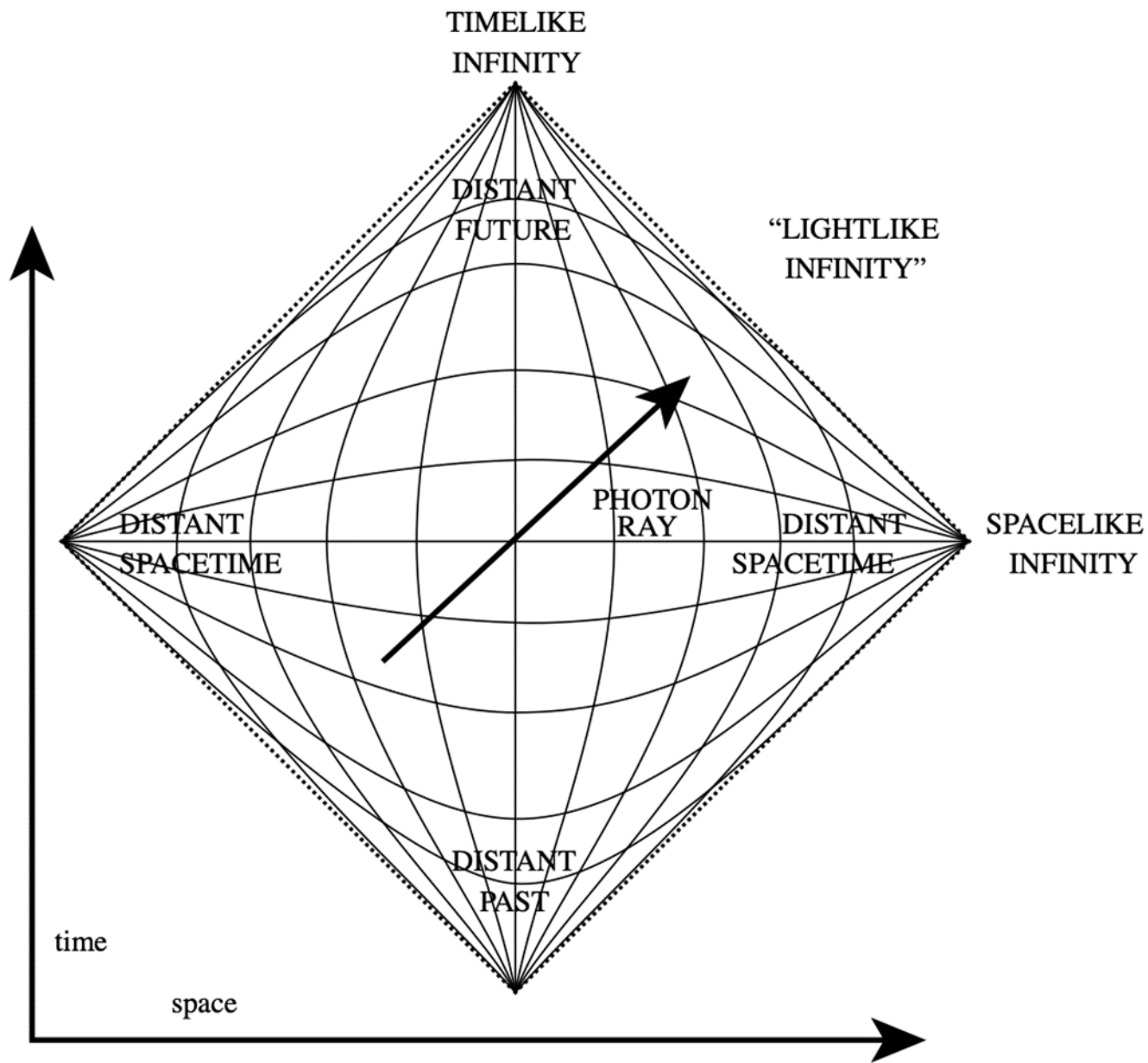
Accretion disk

The hot, thin, rotating disk formed by matter slowly spiraling toward the black hole.

Image of the disk's underside

Light rays from beneath the far side of the disk are gravitationally "lensed" to produce this part of the image.

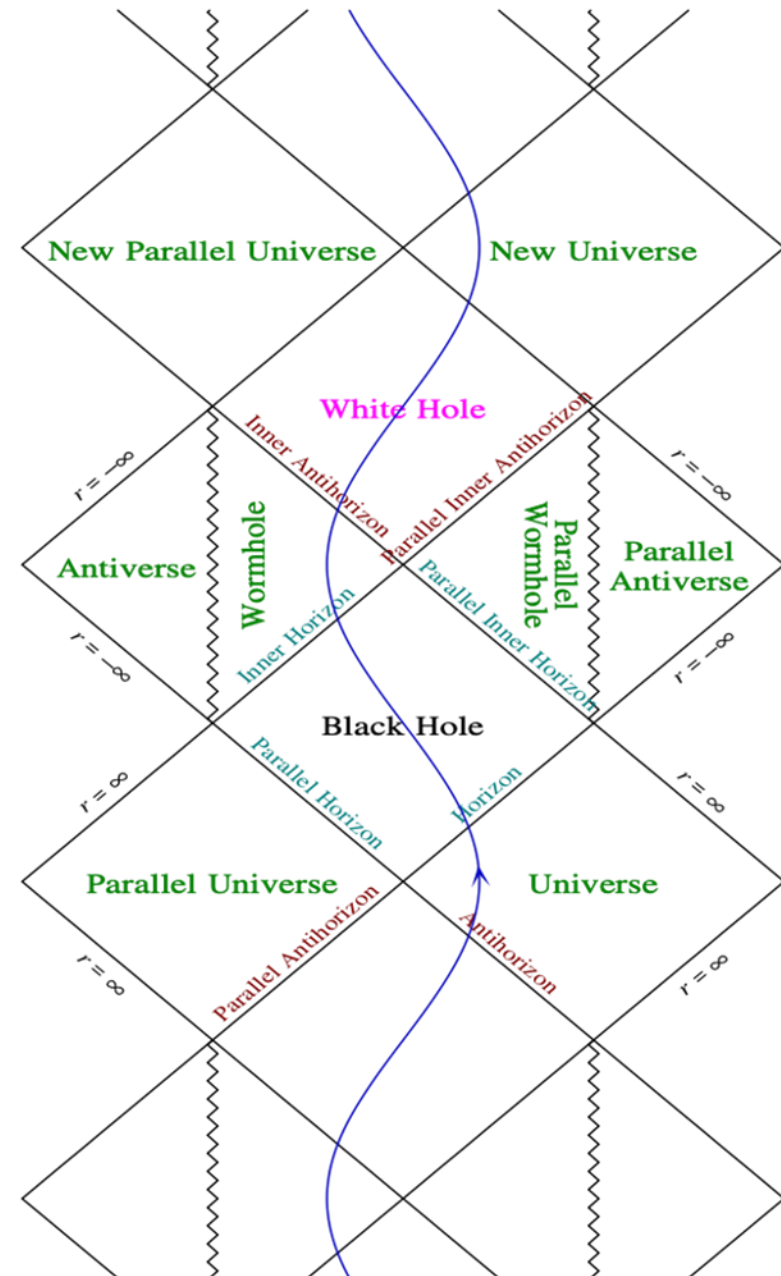
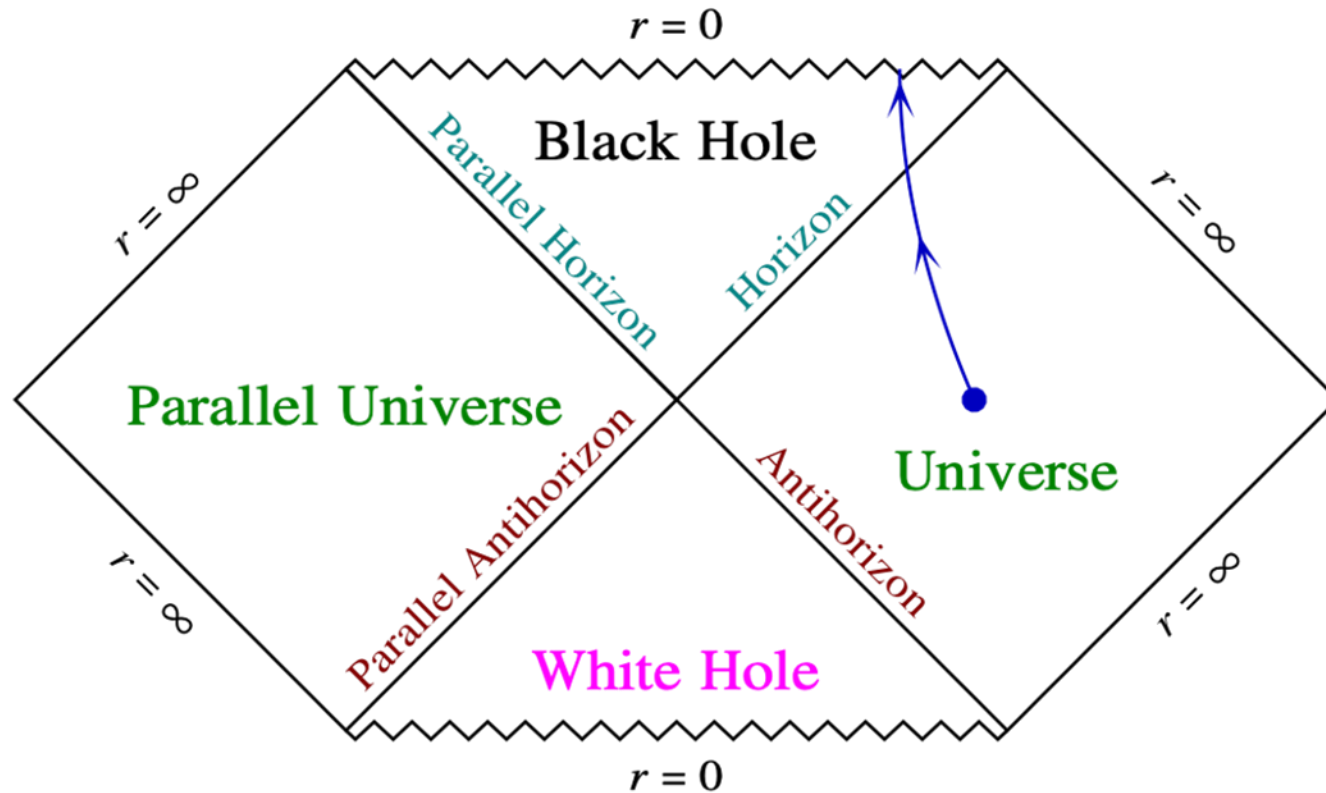




Penrose diagram, for Minkowski Space

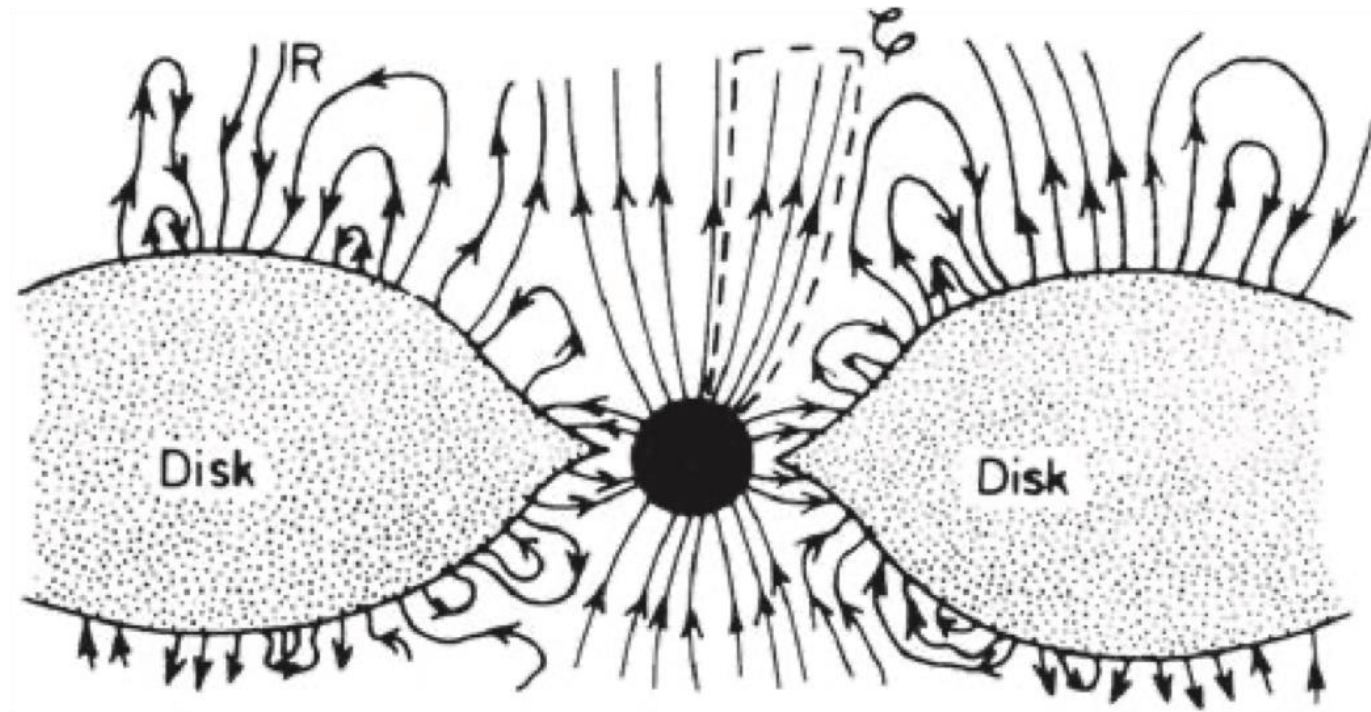
$$\tan(u \pm v) = x \pm t$$

Penrose Diagrams for Schwarzschild and Kerr Black Holes



As already stated, an isolated black hole cannot possess any properties other than mass, electric charge and angular momentum. There is no reason, however, why a magnetic field cannot be associated with a black hole provided it is 'tied to' the surrounding medium. This is illustrated schematically in Fig. 13.26 which shows the magnetic field lines in the vicinity of a black hole with a thick accretion disc. The electrodynamics of such configurations may well be important in understanding the origin of jets of high energy particles observed in active galactic nuclei. In their remarkable book *Black Holes – the Membrane Paradigm*, Thorne, Price and Macdonald (1986) show that the electrodynamics of all types of black holes can be understood if the horizon is replaced by a two-dimensional membrane with the resistivity of free space.

Thorne et al. 1986



A possible configuration for the magnetic field distribution about a black hole which is surrounded by a thick accretion disc. The magnetic field lines are linked to both the accretion disc and the surrounding medium (Thorne *et al.*, 1986).

Blandford–Znajek Process

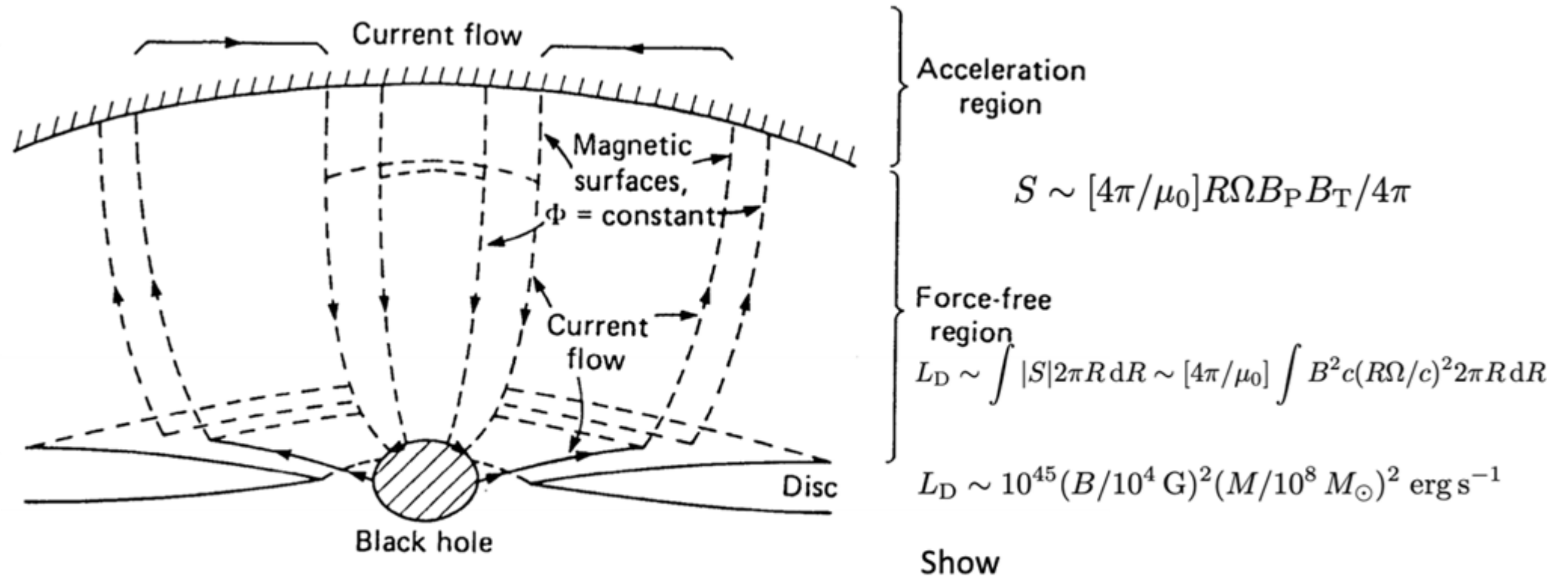


Fig. 9.5. Magnetic surface and current flows in the vicinity of the black hole and disc.
(Adapted from D. Macdonald & K.S. Thorne (1982) *M.N.R.A.S.* 179, 433.)

$$q\mathbf{E} + \mathbf{J} \times \mathbf{B} = 0$$

$$\mathbf{B}_T = (0, B_\phi, 0)$$

$$\mathbf{B}_P = (B_R, 0, B_z) = \mathbf{B} - \mathbf{B}_T.$$

$$\mathbf{E}_T = 0, \mathbf{E} = \mathbf{E}_P$$

$$\nabla \cdot \mathbf{B}_T = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{B}_P = 0$$

$$E_p + (\Omega_{field} \times r) \times B_p = 0$$

$$\nabla \times E_p = 0 \quad \text{due to time-independent magnetic field}$$

$$\nabla \cdot \mathbf{B}_P = 0$$

$$\mathbf{B}_P \cdot \nabla \Omega_F = 0 \quad \Omega_{field} \text{ does not change along poloidal field lines}$$

$$q = \nabla \cdot E = 2\Omega \cdot B / c \neq 0 \quad \begin{array}{l} \text{similar to the pulsar models:} \\ \text{in this case, disc is the source} \end{array}$$

$$B_T = \frac{\mu_0 I}{2\pi R}$$

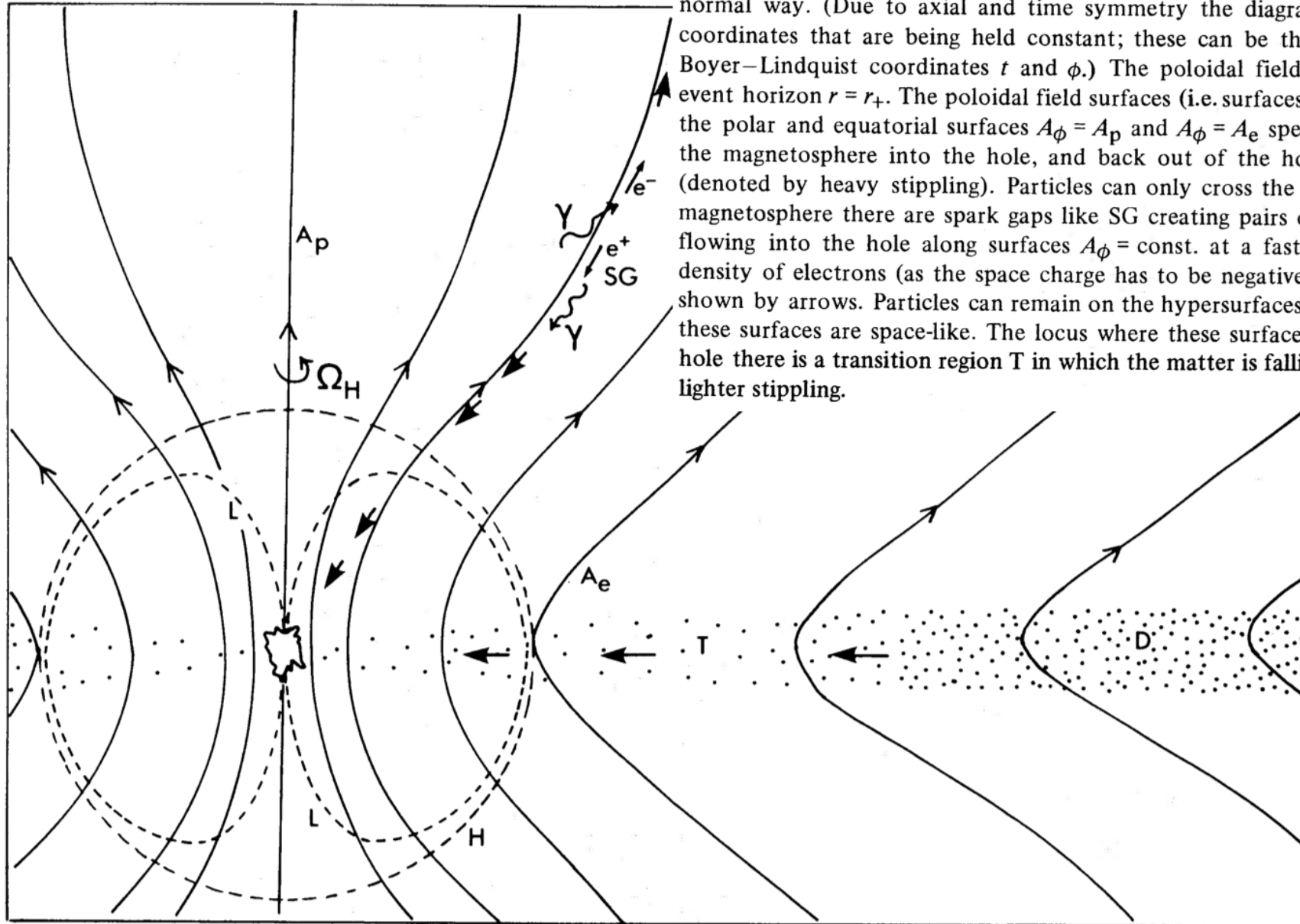
Magnetic field from Ampere's Law

$$E_p = -(\Omega \times R) \times B_p$$

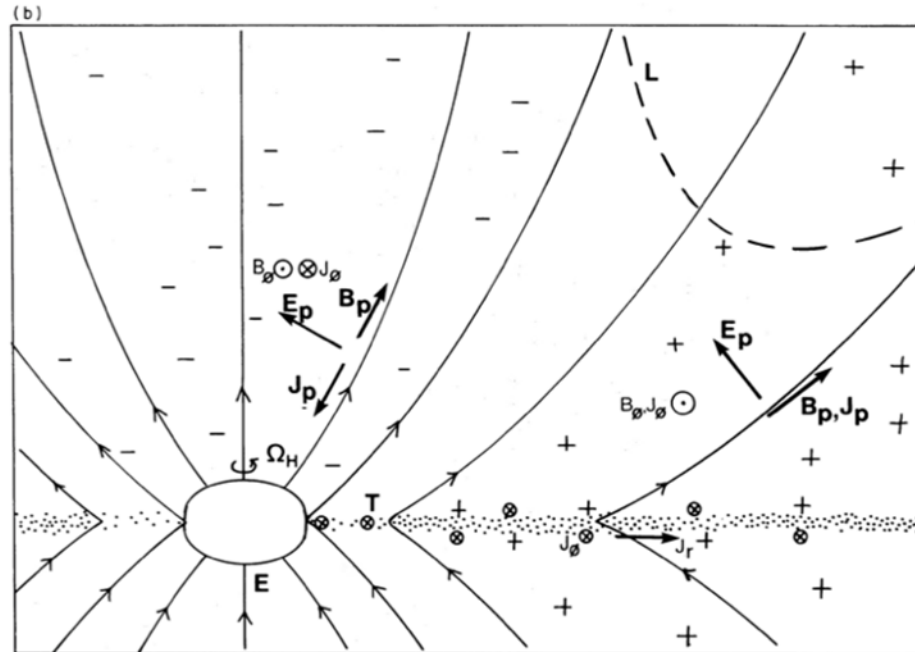
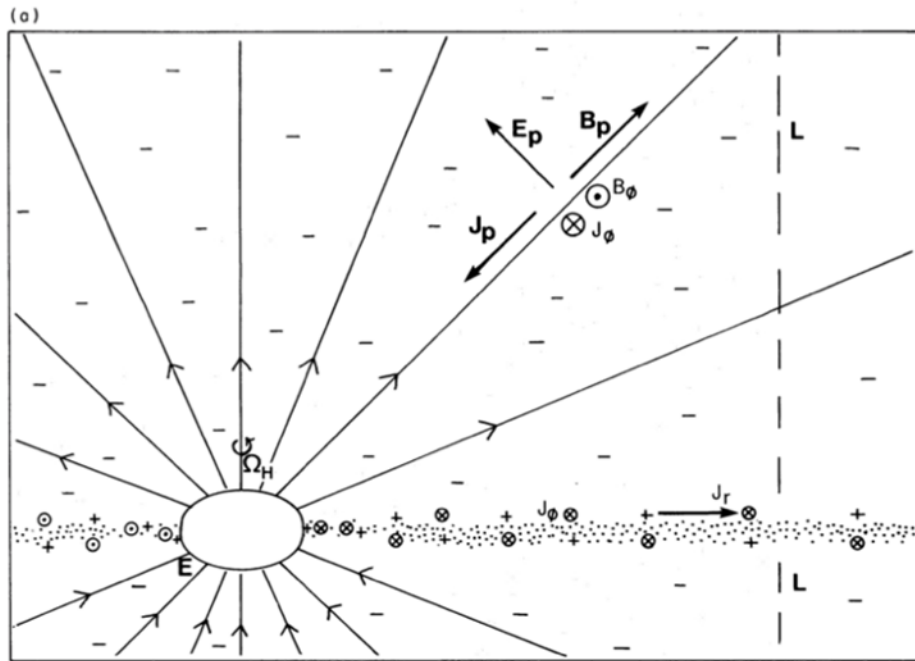
$$S = \frac{1}{\mu_0} E_p \times B_T$$

$$S \sim [4\pi/\mu_0] R \Omega B_P B_T / 4\pi$$

Figure 1. Schematic cross-section of black hole and magnetosphere, using r and θ coordinates in normal way. (Due to axial and time symmetry the diagram is independent of the azimuthal and time coordinates that are being held constant; these can be the Kerr coordinates v and $\tilde{\phi}$, or for $r > r_+$ the Boyer-Lindquist coordinates t and ϕ .) The poloidal field has been chosen so that $\Omega_H \cdot \mathbf{B} > 0$. H is the event horizon $r = r_+$. The poloidal field surfaces (i.e. surfaces of constant A_ϕ) are shown as solid lines, with the polar and equatorial surfaces $A_\phi = A_p$ and $A_\phi = A_e$ specifically labelled. A current I is flowing from the magnetosphere into the hole, and back out of the hole into the disc D lying in the $\theta = \pi/2$ plane (denoted by heavy stippling). Particles can only cross the event horizon one way, into the hole. In the magnetosphere there are spark gaps like SG creating pairs of positrons e^+ and electrons e^- . Positrons are flowing into the hole along surfaces $A_\phi = \text{const.}$ at a faster rate than electrons, and there is a higher density of electrons (as the space charge has to be negative). Projections of typical particle velocities are shown by arrows. Particles can remain on the hypersurfaces of constant A_ϕ only as long as the normals to these surfaces are space-like. The locus where these surfaces become null is L . Between the disc and the hole there is a transition region T in which the matter is falling from the disc to the hole. This is shown by lighter stippling.



Blanford & Znajek 1977



Blanford & Znajek 1977

$$S = E \times B \quad \text{Poynting vector}$$

$$E + (v \times B) = E + [(\Omega \times r) \times B] = 0 \quad \text{Show}$$

$$(\Omega_{\text{Field}} - \omega)$$

Figure 2. Electromagnetic structure of force-free magnetosphere with (a) radial and (b) paraboloidal magnetic fields. $\Omega_H \cdot B$ is taken to be positive. Space charge, currents and non-zero field components shown are as seen by a static observer outside the ergosphere E . Physical observers travelling round the hole at constant r and θ and angular velocity $d\phi/dt$ will see the electric field reverse direction on the surface $d\phi/dt = \omega$. Inside this surface they see a Poynting flux of energy going towards the hole. (For a system of observers with time-like worldlines $d\phi/dt \rightarrow \Omega_H$ on the event horizon and $d\phi/dt \rightarrow 0$ at infinity. Hence when $0 < \omega < \Omega_H$, i.e. when the hole is losing energy electromagnetically, this surface always exists.) The discs are assumed to be Keplerian; the electromagnetic structure of the magnetosphere of the transition region, T, in (b) cannot be determined without additional assumptions. Outside this region Blandford's (1976) Newtonian solution applies. Note the difference in the shape of the light surface, L, in the two cases. For a paraboloidal field the energy appears to be focussed along the rotation axis.

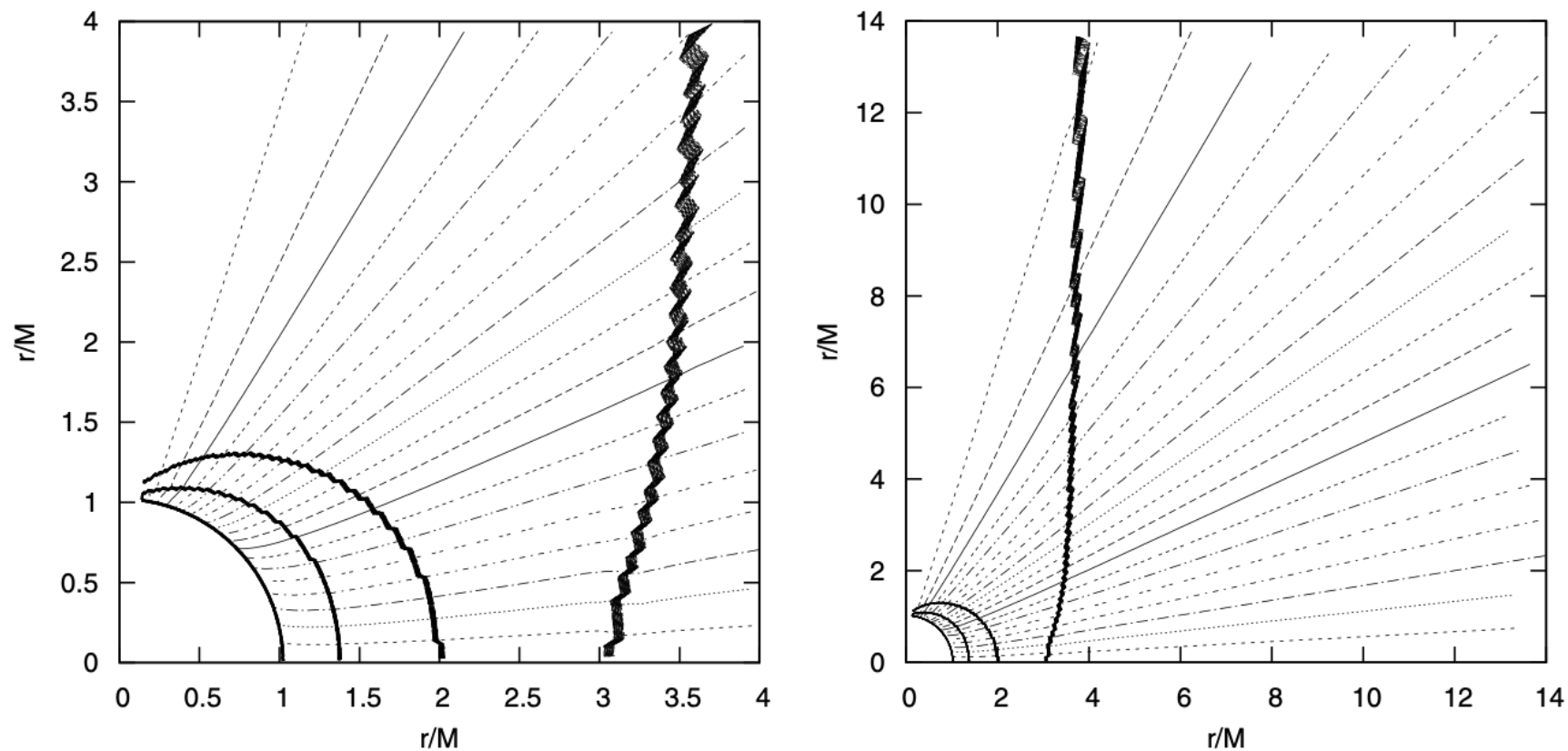
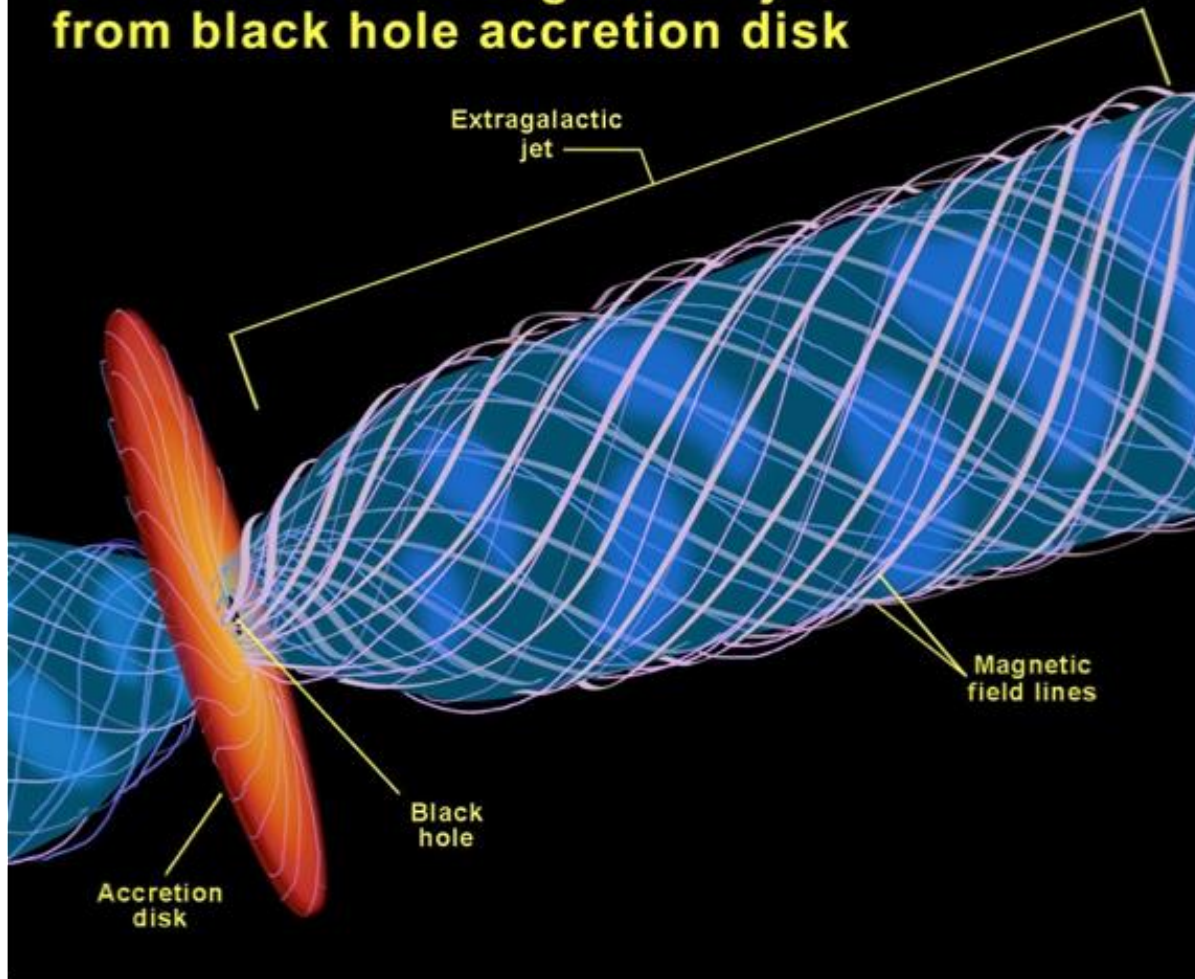
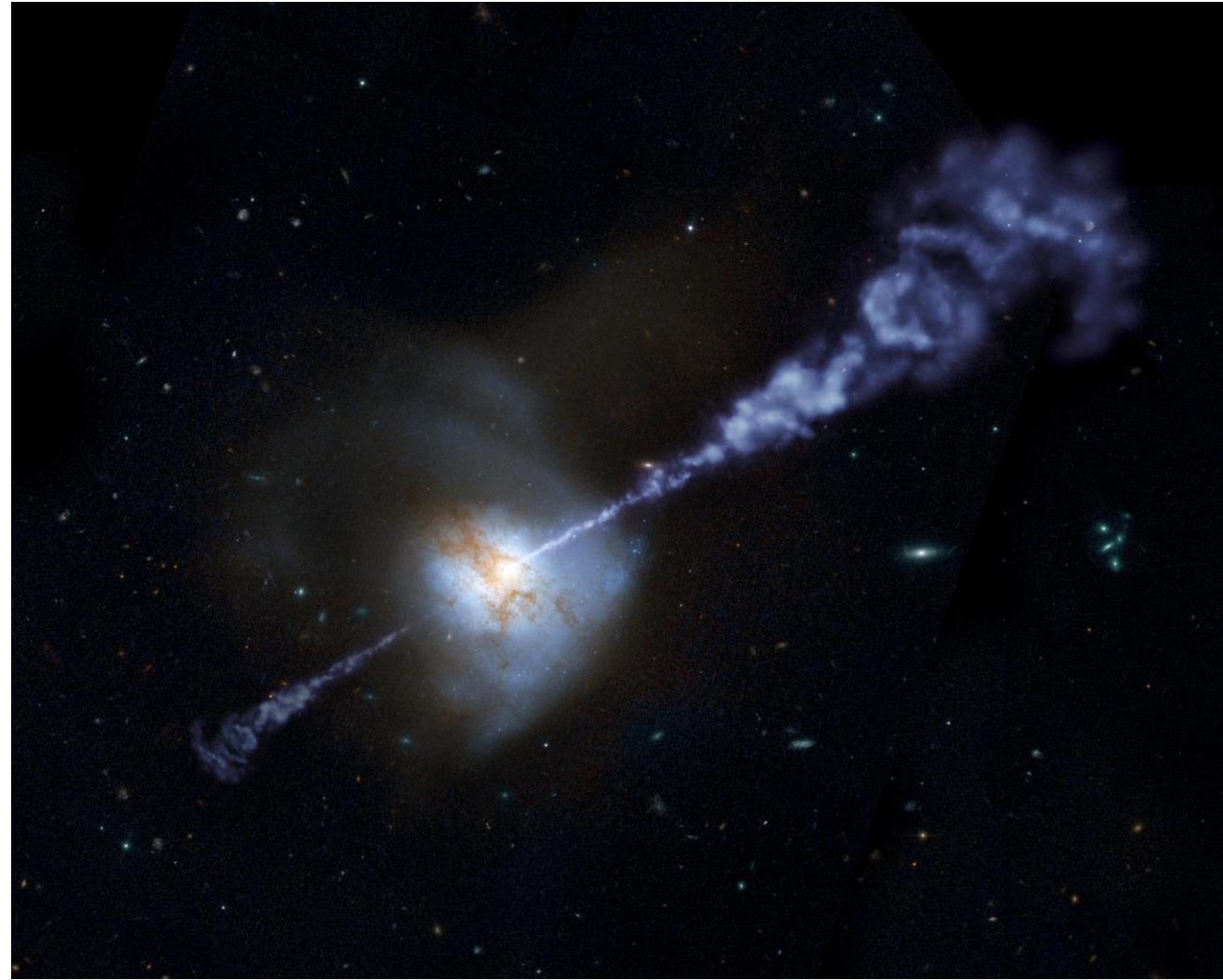


Figure 2. Poloidal magnetic field lines (lines of constant Ψ in 20 equal subdivisions of $\Psi_{\max}$) for $a/M = 0.9999$ close to the event horizon (left panel) and farther out (right panel). The thick lines are the event horizon, the inner LS, the outer boundary of the ergosphere, and the outer LS (from the center outward, respectively).

Formation of extragalactic jets from black hole accretion disk



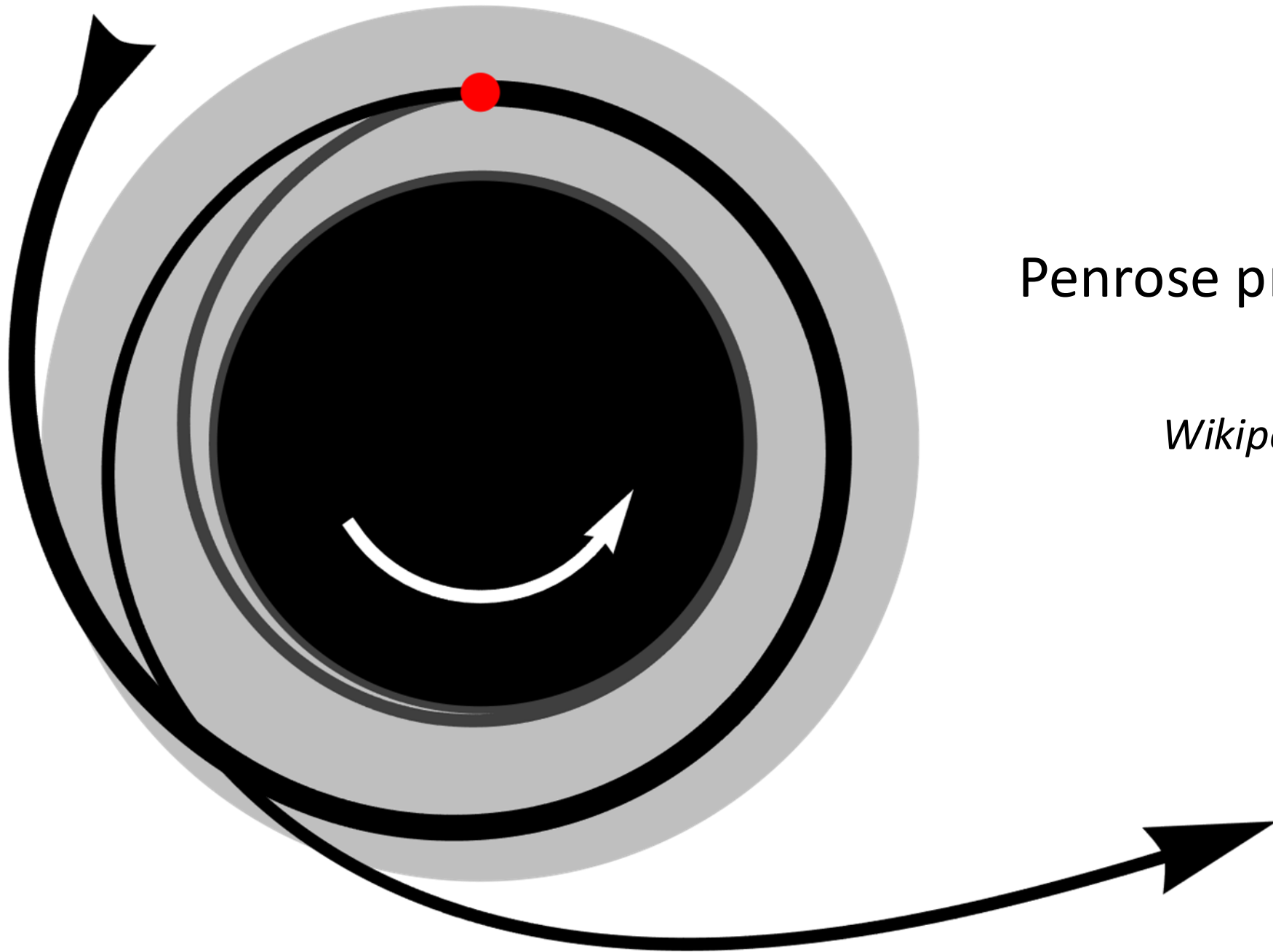
NASA/ESA and Ann Feild (Space Telescope Science Institute)



An active galactic nucleus with relativistic jets
(NASA)

Longair

The region of space-time between the radii r_+ and r_{stat} is referred to as the *ergosphere* and it has a number of important properties which may be of importance astrophysically. Although matter has to rotate in the same direction as the black hole within the ergosphere, particles can escape from it by the *Penrose process* which depends upon the existence of negative energy orbits within the region defined by the static limit. Penrose (1969) showed that, if a particle enters one of these negative energy orbits and splits into two, one of them can fall down the hole while the other escapes to infinity with greater energy than the original particle had when it fell in. The source of the extra energy is the rotational energy of the black hole. The reason that the volume between the horizon at r_+ and the static limit at r_{stat} is called the *ergosphere* is that this is the region from which energy can be extracted by the Penrose process.



Penrose process

Wikipedia

Thus, the rotational energy of the black hole can be tapped to provide energy to the exterior Universe. Penrose (1969) showed that the fraction of the rest-mass energy of a rotating black hole which can be made available to the exterior Universe is

$$1 - 2^{-1/2} [1 + [1 - (J/J_{\max})^2]^{1/2}]^{1/4} . \quad (13.67)$$

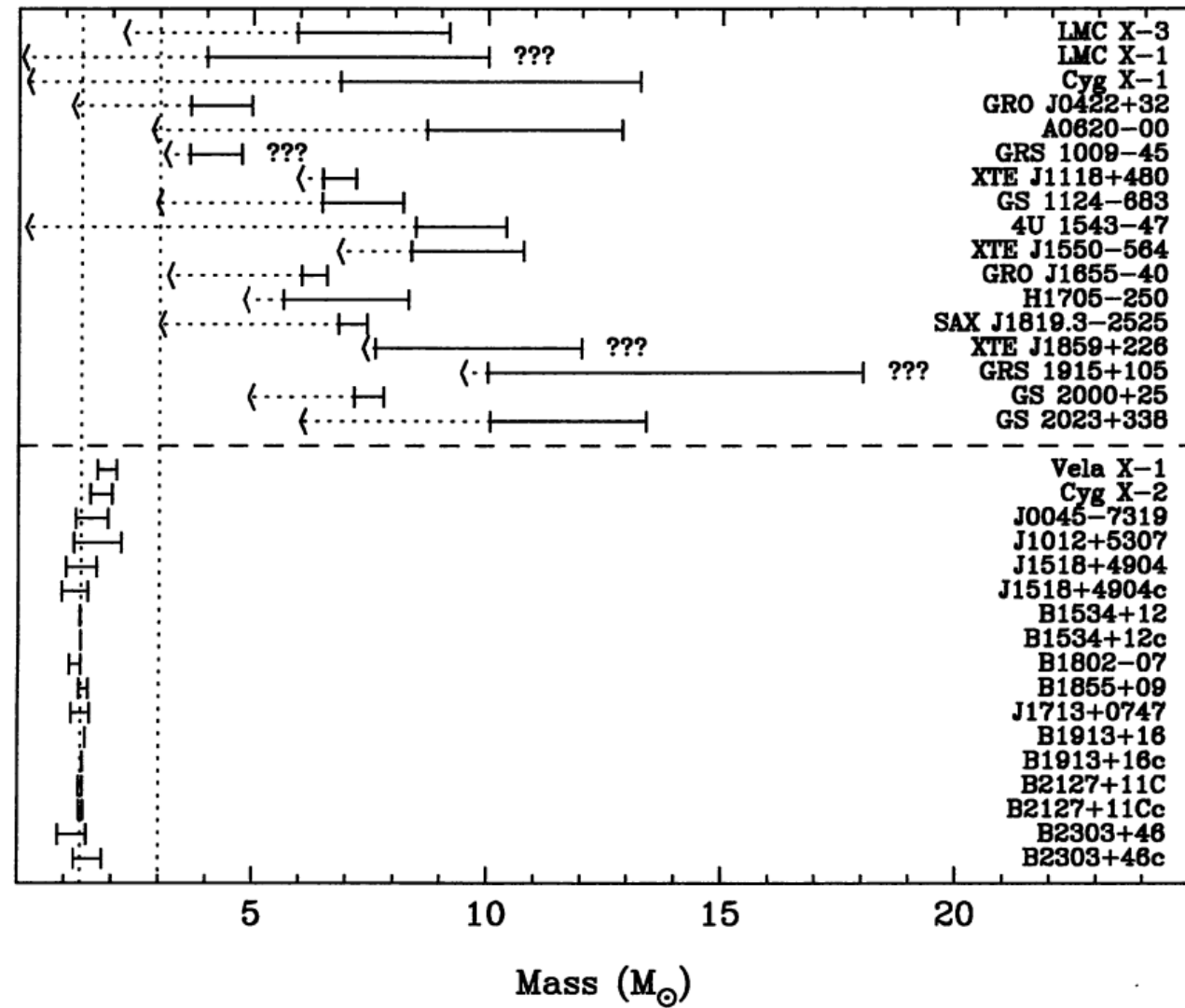
For maximally rotating black holes, this percentage amounts to 29%, again demonstrating their importance as energy sources for high energy astrophysical phenomena.

According to general relativity, nothing can escape from a black hole and therefore it cannot emit radiation. In a remarkable analysis, Hawking showed, however, that, when quantum electrodynamical processes are considered at the horizon of a black hole, there is a finite probability of particles escaping to infinity (Hawking, 1975).

$$T = \frac{\hbar c^3}{8\pi G M k} \approx \frac{10^{-7}}{(M/M_{\odot})} \text{ K}$$

Virtual photons are created from the vacuum at the horizon. Before they can annihilate, one falls into the black hole while the other escapes to infinity.

Exercise: Using the entropy $S = \hbar c^3 / 8\pi G k M$, find the Hawking temperature



A compilation of mass estimates for the 17 black holes, the arrows indicating the value of the mass function, which is a firm lower limit for the mass of the black hole (Orosz, 2007). The solid lines indicate the best estimates of the masses of black holes. In the lower half of the figure, the masses of neutron stars in binary neutron star systems are also shown.

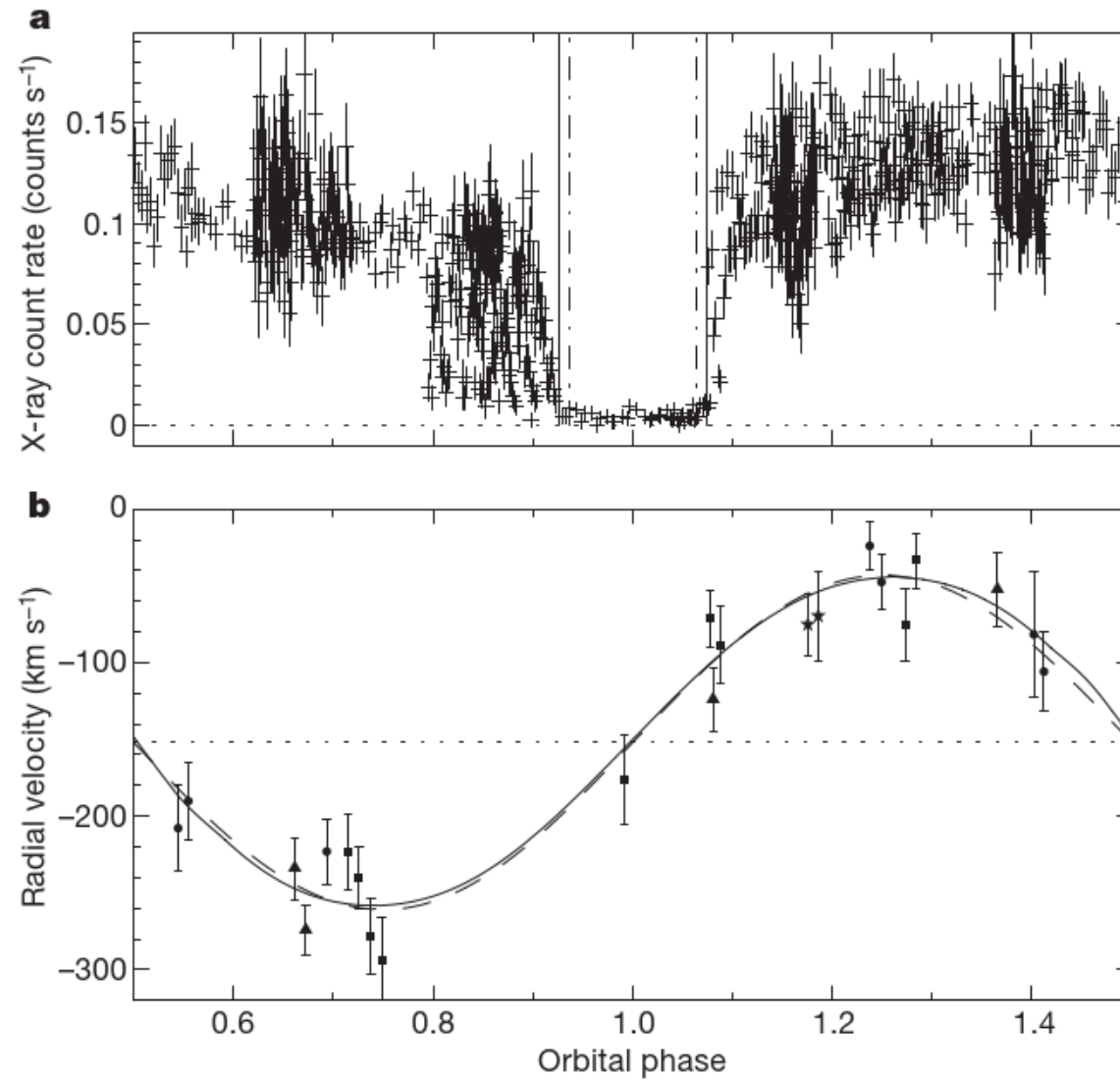
$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

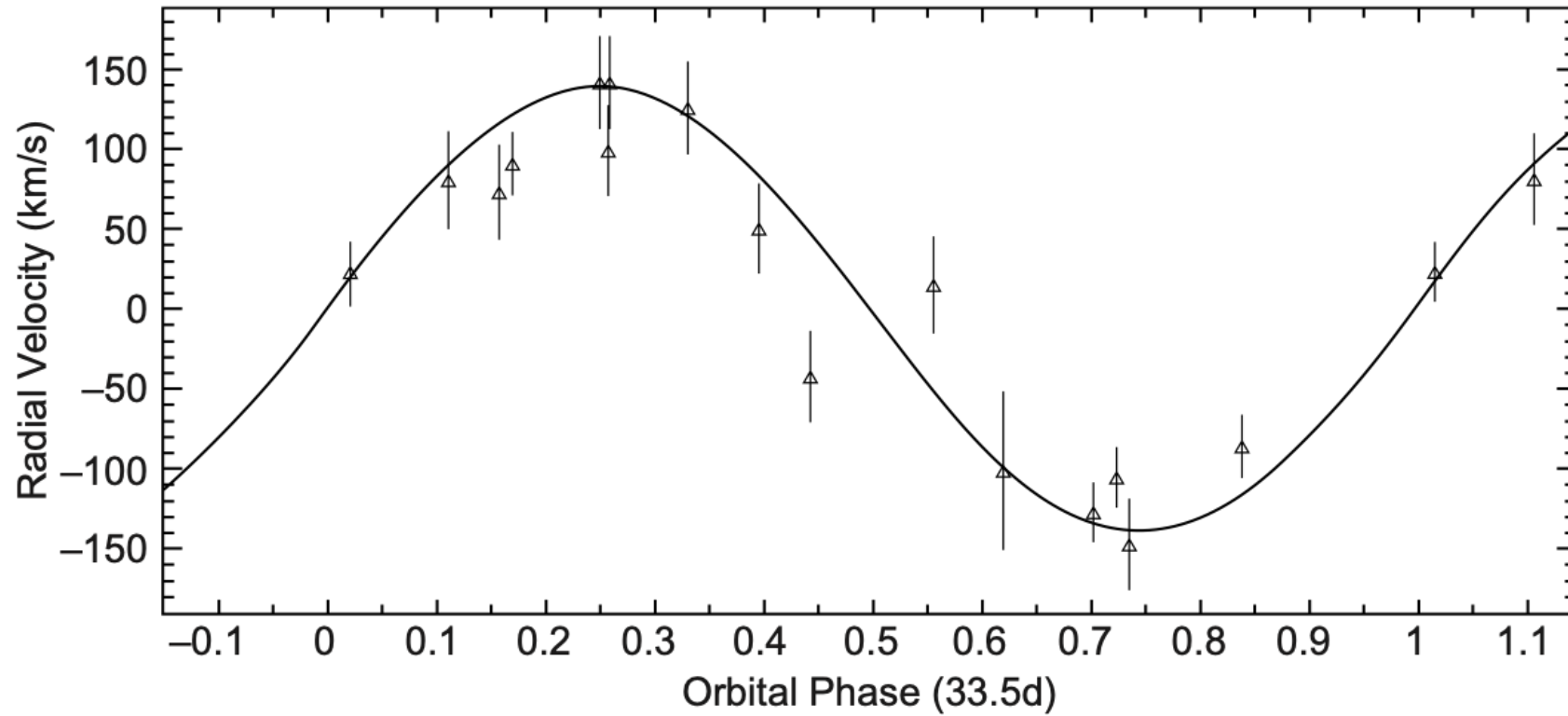
$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{Gm}{r^2} \rho \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi r^3 P}{mc^2}\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1}$$

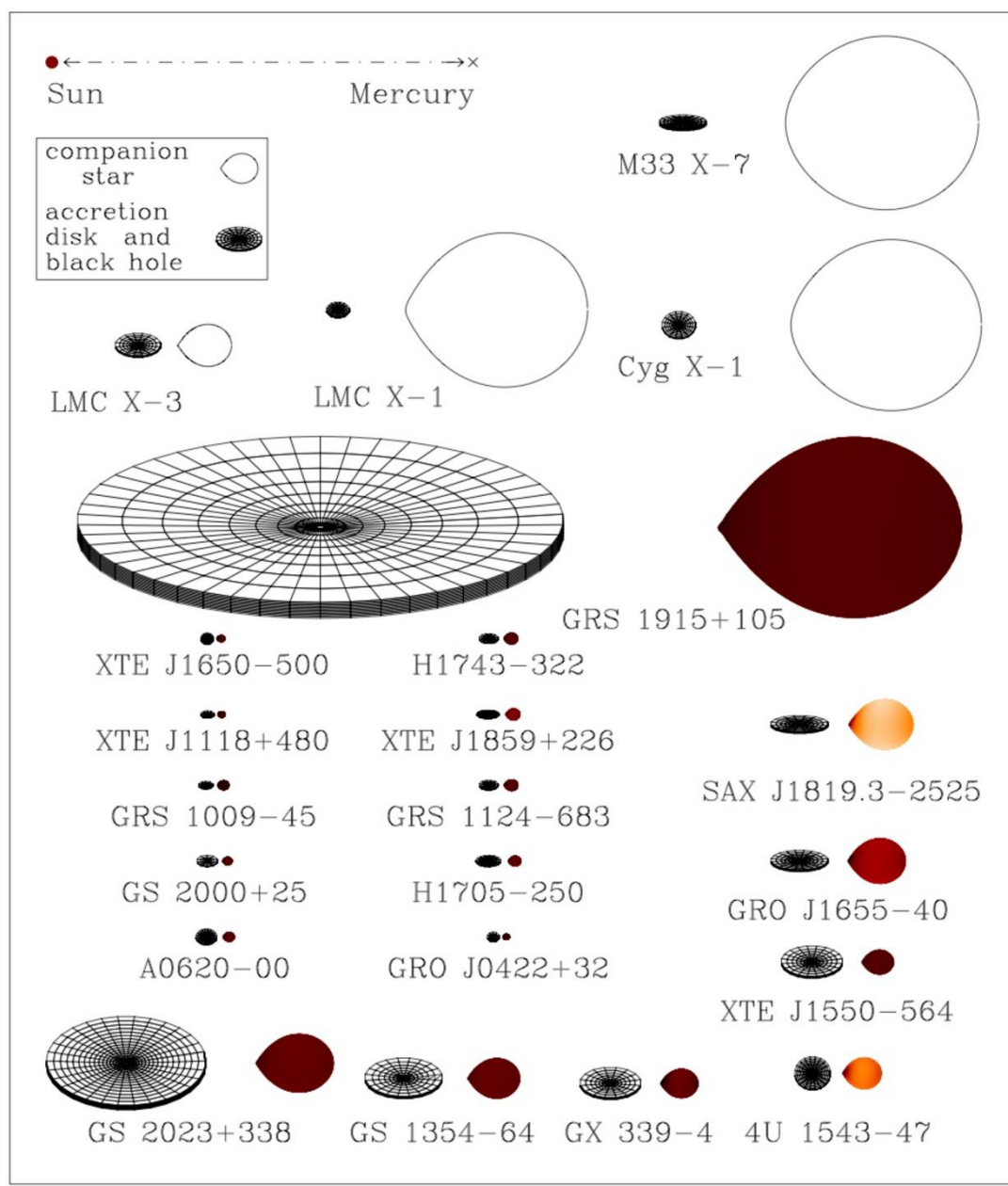
Tolman-Oppenheimer-Volkoff equation for neutron stars mass less than $3 M_{\text{Sun}}$, around $2.23 M_{\text{Sun}}$



(a) The X-ray light curve of the X-ray binary system M33 X-7 in the nearby spiral galaxy, showing the eclipse of the X-ray source by the luminous primary O star. (b) The velocity curve of the primary star was determined from observations with the Gemini North telescope (Orosz *et al.*, 2007).



The velocity curve of the microquasar GRS1915+105 as determined from observations with the ISAAC infrared spectrograph on the Antu telescope of the ESO VLT (Greiner *et al.*, 2001).



Schematic sketches, to scale, of plausible models for the 20 stellar mass black hole candidates and their associated accretion discs discussed by Orosz (2007).