1.6 Independence

Definition: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

- If $\mathrm{P}(A) > 0$, independence implies $\mathrm{P}(B|A) = \mathrm{P}(B)$.
- Symmetrically, if $\mathrm{P}(B) > 0$, independence implies $\mathrm{P}(A|B) = \mathrm{P}(A)$.
- Show that, if A and B are independent, so are A and B^c. (If A is independent of B, the occurrence (or non-occurrence) of B does not convey any information about A.)

- Show that, if A and B are disjoint, they are always *dependent*.

Ex: Consider two independent rolls of a tetrahedral die.

(a) Let $A_i = \{\text{the first outcome is } i\}$. Let $B_j = \{\text{the second outcome is } j\}$. "Independent rolls" implies A_i and B_j are independent for any i and j. Find $P(A_i, B_j)$.

(b) Let $A = \{\text{the max of the two rolls is 2}\}$. Let $B = \{\text{the min of the two rolls is 2}\}$. Are A and B independent?

(c) Note that independence can be counter-intuitive. For example, let $A_2 = \{\text{the first roll is } 2\}$. Let $S_5 = \{\text{the sum of the two rolls is } 5\}$. Show that A_2, S_5 are independent, although the sum of the two rolls and the first roll are dependent in general (try A_2, S_6 as a counterexample.)

1.6.1 Conditional Independence

Recall that conditional probabilities form a legitimate probability law. So, A and B are independent, conditional on C, if

$$\mathrm{P}(A \cap B|C) = \mathrm{P}(A|C)\mathrm{P}(B|C)$$

Show that this implies

$$\mathrm{P}(A|B \cap C) = \mathrm{P}(A|C)$$

(assuming $\mathrm{P}(B|C) \neq 0$, $\mathrm{P}(C) \neq 0$.)

Conditioning may affect independence.

Ex: Assume A and B are independent, but $A \cap B \cap C = \emptyset$. If we are told that C occurred, are A and B independent? (draw Venn Diagram exhibiting a counterexample.)

Ex: Two unfair coins, A and B.

$$\mathrm{P}(H|\mathrm{coinA}) = 0.9,\ \mathrm{P}(\mathrm{H}|\mathrm{coinB}) = 0.1$$

Choose either coin with equal probability.

- Once we know it is coin A, are future tosses independent.
- If we don't know which coin it is, are future tosses independent?
- Compare

  $$\mathrm{P}(\text{5th toss is a T})$$

  and P(5th toss is a T|first 4 tosses are T).

Independence of a collection of events: Information on some of the events tells us nothing about the occurrence of the others.

- Events A_i, $i = 1, 2, \ldots, n$ are independent iff $\mathrm{P}(\cap_{i \in S} A_i) = \pi_{i \in S} \mathrm{P}(A_i)$ for any $S \subset \{1, 2, \ldots, n\}$
- Note that

$$\mathrm{P}(A_5 \cup A_2 \cap (A_1 \cup A_4)^c | A_3 \cup A_6^c) = \mathrm{P}(A_5 \cup A_2 \cap (A_1 \cup A_4)^c)$$

- Pairwise independence does not imply independence! (Checking $\mathrm{P}(A_i \cap A_j) = \mathrm{P}(A_i)\mathrm{P}(A_j)$ for all i and j is not sufficient for confirming independence.)
- For three events, checking $\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)$ is not enough for confirming independence.

Ex: Consider two independent tosses of a fair coin. $A =$ First toss is H.
$B =$ Second toss is H.
$C =$ First and second toss have the same outcome.
Are these events pairwise independent?

$$\begin{aligned} \mathrm{P}(C) &= \\ \mathrm{P}(C \cap A) &= \\ \mathrm{P}(C \cap A \cap B) &= \\ \mathrm{P}(C | B \cap A) &= \end{aligned}$$

Ex: Network Connectivity: In the electrical network in Fig. 1.2, each circuit element is "on" with probability p, independently of all others. What is the probability that there is a connection between points A and B?

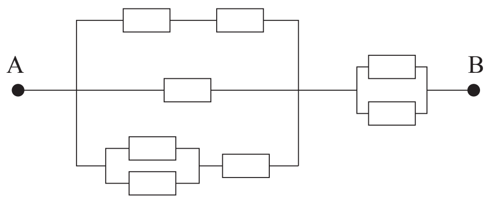

Figure 1.1: Electrical network with randomly operational elements.