4.3 Transforms

Transforms often provide us convenient ways with regard to certain mathematical manipulations.

The **transform**, in other words, the **moment generating function** (MGF), of a r.v. X is defined as

$$M_X(s) = E[e^{sX}].$$

Discrete case MGF: $M_X(s) = \sum_x e^{sx} p_X(x)$
Continuous case MGF: $M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$

Ex: MGF of a Poisson r.v.

Ex: MGF of an exponential r.v.

Ex: MGF of a linear function of a r.v.

Ex: MGF of a Gaussian random variable with mean μ and variance σ^2

Ex: Using the MGF of the Gaussian, and the property we just proved about the transform of a linear function of a random variable, show that a linear function of a Gaussian r.v. is also Gaussian.

4.3.1 Computation of Moments from Transforms

The name moment generating function follows from the following property.

$$E[X^n] = \frac{d^n}{ds^n} M_X(s) \,|_{s=0}$$

Proof:

Ex: Mean and variance of an exponential r.v.

Note: The transform $M_X(s)$ of a r.v. X uniquely determines the PDF of X. That is, one can always find $f_X(x)$ from $M_X(s)$.

4.3.2 Mixture of two distributions

Ex: The length in KBytes of IP packets received at a switch are, with 80 % probability, exponentially distributed with mean 10, and with 20 % probability, exponentially distributed with mean 100. Determine the MGF of the length of a randomly (uniformly) selected packet.

4.3.3 Sums of Independent R.V.s

When $X_1, X_2, \ldots, X_k$ are independent r.v.s, the MGF of their sum $Y = \sum_{i=1}^{k} X_i$ has a simple form. In deriving it, one may interpret e^{sX_i} as a function of X_i.

Ex: Sum of independent Poisson r.v.s

Ex: The sum of two independent Gaussian random variables is Gaussian. Let $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$.