Momentum

- Rewrite Newton's second law as:

$$\vec{F} = m\frac{d\vec{v}}{dt} \equiv \frac{d(m\vec{v})}{dt}$$

- Define the ***momentum*** of the particle as $\vec{p} = m\vec{v}$:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

  i.e. force is the rate of change of momentum

- Can also be written as

$$d\vec{p} = \vec{F}dt$$

  i.e., if there is force, the change in momentum in dt time is $\vec{F}dt$
  (Note: Compare the form of this equation with $d\vec{v} = \vec{a}dt$)

Momentum Conservation

- $d\vec{p} = \vec{F}dt$
- Consider two masses m_1 and m_2 exerting forces $\vec{F}_{12}$ and $\vec{F}_{21}$ on each other.
- $d\vec{p}_1 = \vec{F}_{12}dt$ and $d\vec{p}_2 = \vec{F}_{21}dt$
- $d(\vec{p}_1 + \vec{p}_2) = (\vec{F}_{12} + \vec{F}_{21})dt$
- Newton's third law: $\vec{F}_{12} = -\vec{F}_{21}$, i.e. $\vec{F}_{12} + \vec{F}_{21} = 0$
- $d(\vec{p}_1 + \vec{p}_2) = 0$, i.e. $\vec{p}_1 + \vec{p}_2$ is constant.
- Newton's third law is valid for any system under any condition. Hence momentum conservation is valid under any condition
- Energy conservation for a system is valid only under the absence of dissipative forces.

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- Consider two masses m_1 and m_2 moving along a line with velocities v_1 and v_2 (since they are moving along a line, these are the components along the line)
- Initial total momentum of the system: $P_i = m_1 v_1 + m_2 v_2$
- Two masses collide and after collision they move with velocities v_1' and v_2'.
- Final momentum of the system is $P_f = m_1 v_1' + m_2 v_2'$
- Momentum conservation:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

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- ... v_1'
-
- I...

NOTE

If a quantity is conserved, it is conserved independent of how complicated the interactions are in the intermediate stages!

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

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- $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$
- Unknowns: v_1', v_2': two unknowns but a single equation
- We need one more condition to determine v_1' and v_2'
- **Elastic collision**: energy is conserved: $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$
- **Completely inelastic collision:** The two masses stick together: $v_1' = v_2'$
- **Partially inelastic collision:** Coefficient of restitution:

$$C_R = \frac{\text{relative speed after collision}}{\text{relative speed before collision}} = \frac{|v_1' - v_2'|}{|v_1 - v_2|} \tag{114}$$

1D Collision of Two Masses

- Consider an elastic collision: $\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$
- Energy conservation can be re written as

$$m_1(v_1 - v_1')(v_1 + v_1') = m_2(v_2' - v_2)(v_2' + v_2)$$

- Momentum conservation: $m_1 v_1 - m_1 v_1' = m_2 v_2' - m_2 v_2$
- Combined $v_1 + v_1' = v_2 + v_2'$, or $v_1 - v_2 = -(v_1' - v_2')$
- Consider the special case: $v_2 = 0 \implies v_2' = v_1 + v_1'$
- $m_1 v_1 - m_1 v_1' = m_2(v_1 + v_1')$
- $v_1' = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_1$
- $v_2' = \frac{2m_1}{m_1 + m_2} v_1$

1D Collision of Two Masses

- $v_1' = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_1$
- $v_2' = \frac{2m_1}{m_1 + m_2} v_1$
- Special case: $m_1 = m_2$: $v_1' = 0$ and $v_2' = v_1$
- Special case: $m_2 \gg m_1$: $v_1' = -v_1$ and $v_2 = 0$
- Special case: $m_1 \gg m_2$: $v_1' = v_1$, $v_2 = 2v_1$ (discuss also in the reference frame in which $v_1 = 0$)

General Momentum Conservation

- System of point particles of mass m_i.
- Let $\vec{F}_{ij}^{int}$ be the force acting on mass m_i due to the mass m_j (can be gravitational attraction, EM attraction, push, pull, etc)
- Let $\vec{F}_i^{ext}$ be the force acting on mass m_i due to objects that are not part of my system.
- Newtons's second law: $\frac{d\vec{p}_i}{d_t} = \sum_{j\neq i} \vec{F}_{ij}^{int} + \vec{F}_i^{ext}$
- Sum over all i

$$\sum_i \frac{d\vec{p}_i}{dt} = \sum_{i,j\neq i} F_{ij}^{int} + \sum_i \vec{F}_i^{ext} \tag{115}$$

$$\frac{d}{dt}\sum_i p_i = \vec{0} + \vec{F}_T^{ext} \tag{116}$$

$$\frac{d\vec{P}_T}{dt} = \vec{F}_T^{ext} \tag{117}$$

General Momentum Conservation

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- Note that for any $\vec{F}_{ij}$, there is a $\vec{F}_{ji} = -\vec{F}_{ij}$ because of Newton's third law. Hence the first sum is zero.
- $\vec{P}_T = \sum_i \vec{p}_i$ is that total momentum and $F_T^{ext} = \sum_i \vec{F}_i^{ext}$ is the total force acting on the system

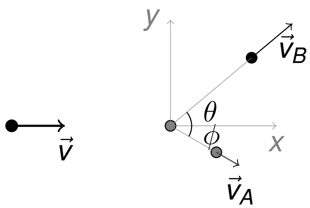
- $$\vec{F}_T^{ext} = \frac{d\vec{P}_T}{dt}$$
- Newton's second law is also valid for extended objects if the momentum is taken as the total momentum of the system, and the force is taken as the total force acting on the system.
- Define $\vec{v}_{CM}$ as $\vec{P}_T = M\vec{v}_{CM}$ where $M = \sum_i m_i$ and

$$\vec{v}_{CM} = \frac{1}{M}\sum_i \vec{p}_i = \frac{1}{M}\sum_i m_i\vec{v}_i \tag{118}$$

$$= \frac{1}{M}\sum_i m_i\frac{d\vec{r}_i}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum_i m_i\vec{r}_i\right) \equiv \frac{d\vec{r}_{CM}}{dt} \tag{119}$$

- $\vec{r}_{CM} = \frac{1}{M}\sum_i m_i\vec{r}_i$ is the position of the ***center of mass*** of the system.
- A system of particles behaves like a point mass of mass M and position $\vec{r}_{CM}$ with regards to translational motion.

Collisions in 3D



$\theta > 0$ and $\phi < 0$

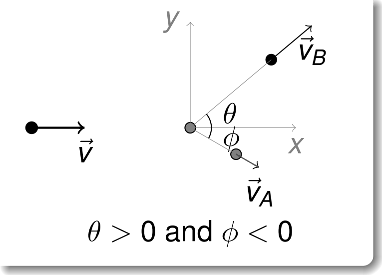
- Initial state: m_B is at rest, m_A moves with velocity $\vec{v}$
- Final state: the velocities of m_A and m_B are $\vec{v}_A$ and $\vec{v}_B$.
- Initial momentum:

$$\vec{P}_i = m_A \vec{v} = m_A v \hat{x} \tag{120}$$

- Final momentum:

$$\begin{aligned} \vec{P}_f &= m_A \vec{v}_A + m_B \vec{v}_B \\ &= (m_A v_A \cos\phi + m_B v_B \cos\theta)\,\hat{x} \\ &+ (m_A v_A \sin\phi + m_B v_B \sin\theta)\,\hat{y} \end{aligned}$$

Collisions in 3D



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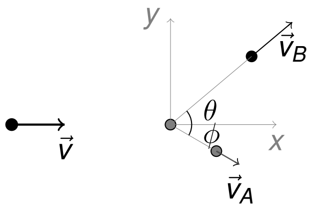
- Unknowns v_A, v_B, ϕ, θ are related through:

$$m_a v = m_a v_A \cos\phi + m_B v_B \cos\theta \tag{121}$$

$$0 = m_A v_A \sin\phi + m_B v_B \sin\theta \tag{122}$$

- We need two more equations

Collisions in 3D



$\theta > 0$ and $\phi < 0$

- In elastic collisions: $\frac{1}{2}m_A v^2 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$
- One more equation involving v_A and v_B.
- Still we need to measure one of the unknowns to determine the other three
- In completely inelastic collisions $\vec{v}_A = \vec{v}_B$, hence $v_A = v_B$ and $\theta = \phi$

- Unknowns v_A, v_B, ϕ, θ are related through:

$$m_a v = m_a v_A \cos\phi + m_B v_B \cos\theta \tag{120}$$

$$0 = m_A v_A \sin\phi + m_B v_B \sin\theta \tag{121}$$

- We need two more equations

System of Two Masses

- In terms of $\vec{v}_{CM}$ and $\vec{v} = \vec{v}_1 - \vec{v}_2$, $\vec{v}_1 = \vec{v}_{CM} + \frac{m_2}{m_1+m_2}\vec{v}$ and $\vec{v}_2 = \vec{v}_{CM} - \frac{m_1}{m_1+m_2}\vec{v}$
- Total kinetic energy:

$$
\begin{aligned}
K &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\
&= \frac{1}{2}m_1\left(\vec{v}_{CM} + \frac{m_2}{m_1+m_2}\vec{v}\right)^2 + \frac{1}{2}m_2\left(\vec{v}_{CM} - \frac{m_1}{m_1+m_2}\vec{v}\right)^2 \\
&= \frac{1}{2}(m_1+m_2)v_{CM}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}v^2 \\
&\equiv \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\mu v^2 \qquad (122)
\end{aligned}
$$

- The first term is the kinetic energy of the whole system
- The second term is the internal energy of the whole system.
- In elastic collisions, only the direction of $\vec{v}$ can change