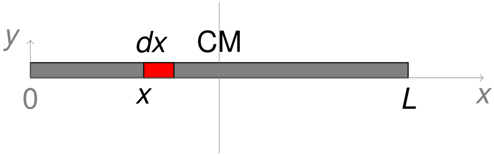
Example-Calculating Moments of Inertia

Moment of Inertia Of a Rigid Rod Rotating Around CM



$$dm = \frac{M}{L}dx$$

$$I = \sum_i m_i R_i^2 = \sum_i \frac{M}{L} dx \left(\frac{L}{2} - x\right)^2$$

$$= \frac{M}{L}\int_0^L \left(\frac{L}{2} - x\right)^2 = \frac{M}{3L}\left(\frac{L}{2} - x\right)^3 \Bigg|_{x=0}^{x=L}$$

$$= \frac{M}{3L}\left(\frac{L}{2}\right)^3 - \frac{M}{3L}\left(-\frac{L}{2}\right)^3 = \frac{M}{12}L^2$$

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Parallel Axis Theorem

Proof

- Let I_{CM} be the moment of inertia around an axes going through the CM.
- Choose z axis to be along the this axes.
- Choose a second axes that goes through the point (x_0, y_0).
- The distance of point with coordinates (x_i, y_i, z_i) from the second axis is $R^2 = (x_i - x_0)^2 + (y_i - y_0)^2$
- The moment of inertial with respect to the second axis is

$$I = \sum_i m_i \left[(x_I - x_0)^2 + (y_i - y_0)^2 \right] \tag{125}$$

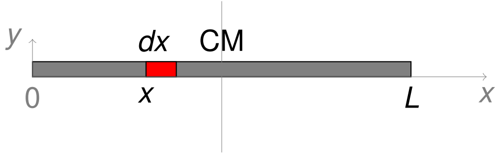
$$= \sum_i m_i (x_i^2 + y_i^2) + \sum_i m_i (x_0^2 + y_0^2) - 2 \sum_i m_i (x_i x_0 + y_i y_0) \tag{126}$$

$$= I_{CM} + Md^2 \tag{127}$$

where $d^2 = x_0^2 + y_0^2$ and $\sum_i m_i x_i = \sum_i m_i y_i = 0$

Parallel Axis Theorem

Example

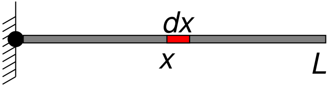


$$dm = \frac{M}{L}dx,\ I_{CM} = \frac{1}{12}ML^2$$

Moment of inertia of a thin rod around one end:

$$I = I_{CM} + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2$$

Example: Thin Rod Rotating Around One End



- The net torque on the rod around the fixed axes:

$$\vec{\tau} = \sum_i \vec{r}_i \times (m_i \vec{g}) = \left(\sum_i m_i \vec{r}_i \right) \times \vec{g} = (M\vec{r}_{CM}) \times \vec{g} = \vec{r}_{CM} \times \vec{w} \tag{125}$$

  Hence the center of gravity for an object in uniform gravitational field is its CM.
- $\tau = \frac{MgL}{2}$
- $\alpha = \frac{\tau}{I} = \frac{\frac{MgL}{2}}{\frac{1}{3}ML^2} = \frac{3}{2}\frac{g}{L}$
- If the rod is initially at rest: $a^{CM} = a^{CM}_{tan} = \alpha \frac{L}{2} = \frac{3}{4} g < g$
- $a_{CM} = \frac{F^{ext}}{M}$. Which other force is acting on the rod?

Perpendicular Axis Theorem

Proof

- Consider a very thin object in the xy plane.
- For any point in the object $z_i \simeq 0$
- $I_x = \sum_i m_i(y_i^2 + z_i^2) \simeq \sum_i m_i y_i^2$
- Similarly $I_y = \sum_i m_i(x_i^2 + z_i^2) \simeq \sum_i m_i x_i^2$
- Then $I_z = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 = I_x + I_y$

Perpendicular Axis Theorem

Example

- The moment of inertia of a loop of mass M and radius R for rotations around an axis that is perpendicular to its plane and going through its CM: $I_z = \sum_i m_i R^2 = MR^2$
- Its moment of inertia around any axis that goes through its CM and is in its plane:
  - If x and y axes are the two axes in the plane of the loop, due to symmetry $I_x = I_y$
  - By perpendicular axis theorem: $I_z = I_x + I_y = 2I_x$
  - Hence $I_x = \frac{1}{2}MR^2$.
- Moment of inertia for rotation around an axis that goes through the edge and is in the plane of the loop:
  $I' = I_{CM} + Md^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$

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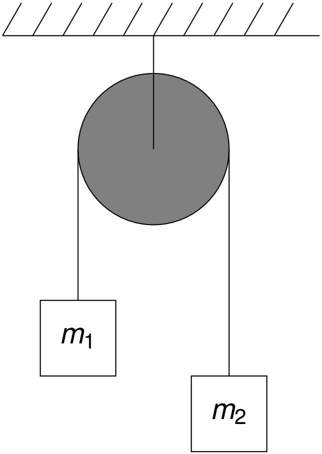
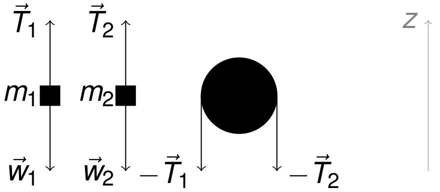
Heavy Pulley

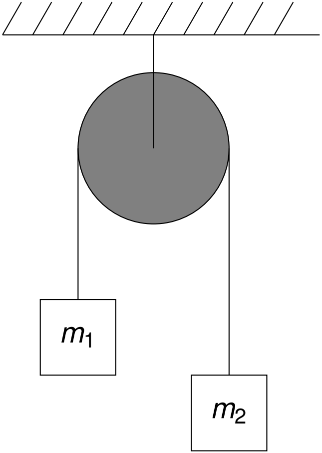


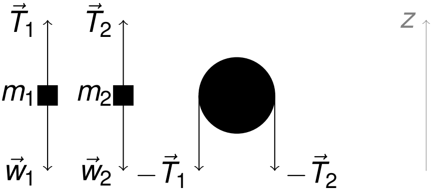
- Since the pulley has a mass, the tensions at each end of the pulley will be different.



- Let x axis be out of the screen

Heavy Pulley



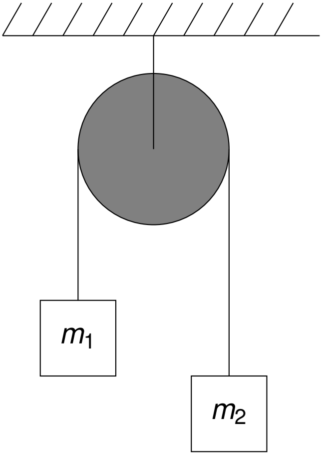


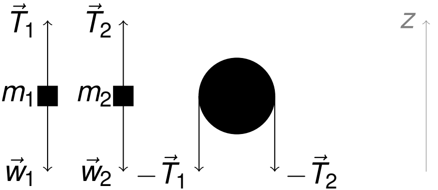
- The net forces acting on mass m_1 and m_2 are:

$$\vec{F}_{1T} = (T_1 - m_1 g)\hat{z} \tag{126}$$

$$\vec{F}_{2T} = (T_2 - m_2 g)\hat{z} \tag{127}$$

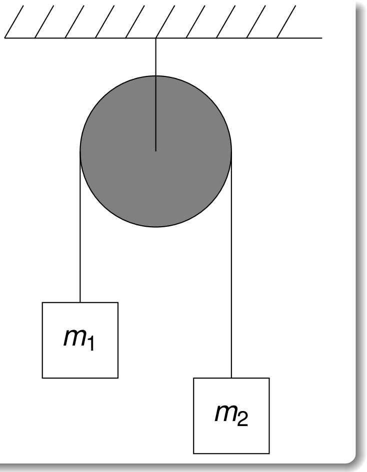
Heavy Pulley





- The torque acting on the pulley is $\vec{\tau} = R(T_1 - T_2)\hat{x}$

Heavy Pulley



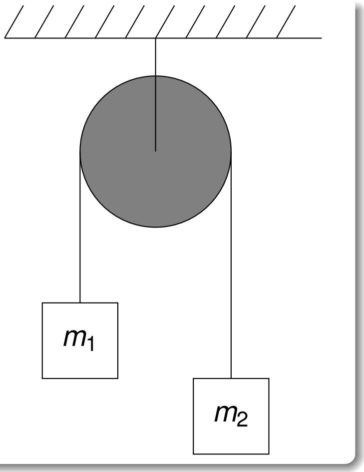
- Let $\vec{a}_1 = a_i\hat{z}$ and $\vec{\alpha} = \alpha\vec{x}$. Then

$$T_1 - m_1 g = m_1 a_1 \tag{126}$$
$$T_2 - m_2 g = m_2 a_2 \tag{127}$$
$$R(T_1 - T_2) = I\alpha \tag{128}$$

- Unkowns: T_1, T_2, a_1, a_2, and α: 5 unknowns
- The remaining two eqns are:

$$a_1 = -a_2 \tag{129}$$
$$\alpha R = -a_1 \tag{130}$$

Heavy Pulley



- The solutions of these equations are:

$$a_1 = -a_2 = \frac{(m_2 - m_1)R^2}{I + (m_1 + m_2)R^2} g \tag{126}$$

$$\alpha = \frac{(m_1 - m_2)R}{I + (m_1 + m_2)R^2} g \tag{127}$$

$$T_1 = \frac{m_1 g(I + 2m_2 R^2)}{I + (m_1 + m_2)R^2} \tag{128}$$

$$T_2 = \frac{m_2 g(I + 2m_1 R^2)}{I + (m_1 + m_2)R^2} \tag{129}$$