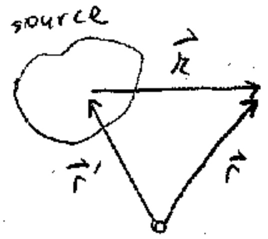
PHYS 334 LECTURE NOTES

Feb19



$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$\hat{\mathscr{r}} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = 0$$

Divergence thm:

$$\vec{v} \equiv \frac{\hat{r}}{r^2}: \quad \underbrace{\int_V \vec{\nabla} \cdot \vec{v}\, d\tau}_{0\,?} = \int_S \vec{v} \cdot d\vec{a}$$

over a sphere of radius R

$$d\vec{a} = R^2 \sin\theta\, d\theta\, d\varphi\, \hat{r}$$

$$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$\int_S \vec{v} \cdot d\vec{a} = \int_S \frac{\hat{r}}{R^2} \cdot \hat{r} R^2 \sin\theta\, d\theta\, d\varphi = \int_S \sin\theta\, d\theta\, d\varphi = 4\pi$$

Dirac Delta Fn:

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \qquad f(x)\delta(x) = f(0)\delta(x)$$

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases} \qquad f(x)\delta(x-a) = f(a)\delta(x-a)$$

Prop.s:

$$\delta(x-a) = \delta(a-x) \qquad \delta(x) = \delta(-x)$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \qquad \int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\, dx = f(a)$$

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$$ where $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$, 3D dirac delta fn

$$\vec{\nabla}\frac{1}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \vec{\nabla}^2 \frac{1}{\mathscr{r}} = -4\pi\delta^3(\vec{\mathscr{r}})$$

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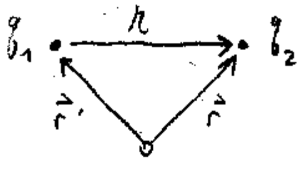
ELECTROSTATICS

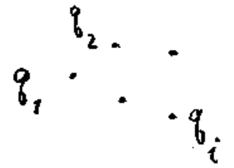
Electric field, $\vec{E}(\vec{r})$

Coulomb's law: $\vec{F} = k q_1 q_2 \frac{\hat{r}}{r^2}$ $\qquad q_1 \longrightarrow q_2$

$$k \equiv \frac{1}{4\pi\varepsilon_0} \qquad \varepsilon_0 : \text{permittivity of free space}$$



$$\vec{F} = k q_1 q_2 \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$



$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \dots \qquad \text{Superposition principle}$$

$$q_1 \to Q: \quad \vec{E} = \frac{1}{Q}\vec{F} = kq\frac{\hat{r}}{r^2}$$

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \dots$$

$$\vec{E}(\vec{r}) = k\int \frac{\hat{\imath}}{\imath^2}\,dq, \quad dq \sim \lambda dl \sim \sigma da \sim \rho d\tau$$

Vol charge:

$$\vec{E}(\vec{r}) = k\int_V \frac{\hat{\imath}}{\imath^2}\,\rho(\vec{r}')d\tau'$$

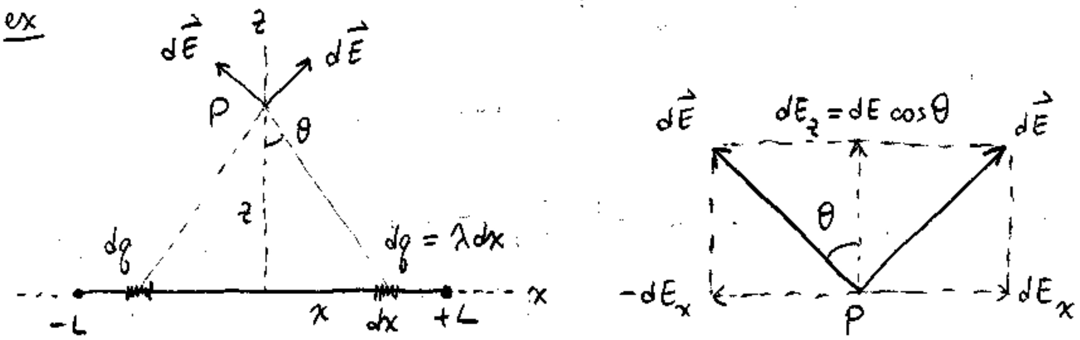
Surface charge:

$$\vec{E}(\vec{r}) = k\int_S \frac{\hat{\imath}}{\imath^2}\,\sigma(\vec{r}')da'$$

Line charge:

$$\vec{E}(\vec{r}) = k\int_L \frac{\hat{\imath}}{\imath^2}\,\lambda(\vec{r}')dl'$$

<u>ex</u>



$$\vec{E} = \int dE\cos\theta\,\hat{z}, \quad \cos\theta = \frac{z}{\sqrt{x^2+z^2}}, \quad dE = k\frac{1}{r^2}dq, \quad r^2 = x^2+z^2$$

$$\vec{E} = \hat{z}\,2\lambda kz\int_0^L \frac{dx}{(x^2+z^2)^{3/2}} = \hat{z}\,2\lambda kz\,\frac{x}{z^2\sqrt{z^2+x^2}}\Bigg|_0^L$$

$$\vec{E} = \hat{z}\,2\lambda k\lambda\,\frac{L}{z\sqrt{z^2+L^2}}$$

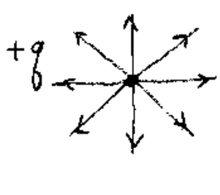
(i) $L \gg z$:

$$\vec{E} = \hat{z}\,2k\lambda\frac{1}{z}$$

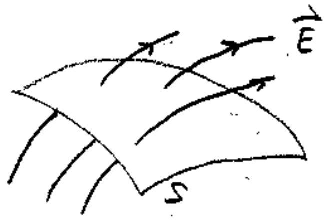
(ii) $L \ll z$:

$$\vec{E} = \hat{z}\,2k\lambda\frac{L}{z^2} = \hat{z}\,k\frac{Q}{z^2}, \quad Q: \text{total charge}$$

Divergence and Curl of $\vec{E}$:



$$\vec{\nabla}\cdot\vec{E} \neq 0$$
$$\vec{\nabla}\times\vec{E} = 0$$



$$\Phi_E = \int_S \vec{E}\cdot d\vec{a}$$

closed surface: $\oint \vec{E} \cdot d\vec{a}$

Point charge at origin:

$$\vec{E} = kq \frac{\hat{r}}{r^2}$$

over spherical surface:

$$d\vec{a} = r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$$

$$\vec{E} \cdot d\vec{a} = kq \sin\theta \, d\theta \, d\varphi$$

$$\oint_S \vec{E} \cdot d\vec{a} = \int_S kq \sin\theta \, d\theta \, d\varphi = kq \, 4\pi = \frac{1}{\varepsilon_0} q$$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} q_{enc}$$

Gauss' thm:

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} \, d\tau = \int_V \frac{1}{\varepsilon_0} \rho(\vec{r}) \, d\tau$$

Differential form of Gauss' law:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$$

$$\vec{E} = k \int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(\vec{r}') \, d\tau'$$

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = k \int \vec{\nabla} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(\vec{r}') \, d\tau'$$

$$= k \int 4\pi \delta^3(\vec{\mathscr{r}}) \rho(\vec{r}') \, d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \int 4\pi \delta^3(\vec{r} - \vec{r}') \rho(\vec{r}') \, d\tau'$$

$$= \frac{1}{\varepsilon_0} \rho(\vec{r})$$

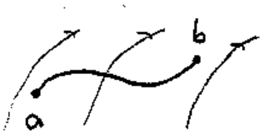
Gauss' law:

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

To use Gauss' law to find E-field, there needs to be a symmetry in charge distribution:

- spherical
- cylindrical
- planar
- straightly linear

$$\int_a^b \vec{E} \cdot d\vec{\ell}$$



for point charge,

$$\vec{E} = kq \frac{\hat{r}}{r^2}$$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

$$\vec{E}\cdot d\vec{l} = kq\,\frac{1}{r^2}\,dr$$

$$\int_a^b \vec{E}\cdot d\vec{l} = kq\int_a^b \frac{1}{r^2}\,dr$$

$$= kq\,\frac{-1}{r}\Big|_a^b = kq\left[-\left(\frac{1}{b}-\frac{1}{a}\right)\right]$$

If $b=a$,

$$\oint \vec{E}\cdot d\vec{l} = 0$$

Stokes' thm:

$$\oint_C \vec{E}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \iff \vec{E} = -\vec{\nabla}V,$$ "$-$" being conventional

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$$\vec{E} = kq\,\frac{\hat{r}}{r^2}$$

$$\int \vec{E}\cdot d\vec{l} \to \oint \vec{E}\cdot d\vec{l} = 0$$

$$\oint_C \vec{E}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a}$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{E} = \vec{\nabla}\Psi$$ for some scalar potential fn Ψ

Define V:

$$V \equiv \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\,\rho\,d\tau$$

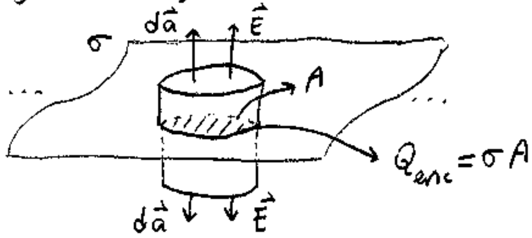
$$\int_\infty^r \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\,q\int_\infty^r \frac{1}{r^2}\,dr = \frac{q}{4\pi\varepsilon_0}\,\frac{-1}{r}$$

$$V(\vec{r}) = -\int_\infty^r \vec{E}\cdot d\vec{l}$$

$$\vec{E} = -\vec{\nabla}V$$

$$\begin{tikzcd}
 & \rho \arrow[dl, bend left, "V = k\int \frac{\rho\,d\tau}{\mathscr{r}}"'] \arrow[dr, bend left, "\vec{E} = k\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho\,d\tau"] & \\
V \arrow[ur, bend left=50, "\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}"] \arrow[rr, bend right, "V = -\int \vec{E}\cdot d\vec{l}"'] & & \vec{E} \arrow[ul, bend right, "\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}"'] \arrow[ll, bend right, "\vec{E} = -\vec{\nabla}V"']
\end{tikzcd}$$

<u>ex</u> (Gauss' law) Use Gauss' law to find $\vec{E}$-field of an ∞-long sheet of charge density σ.

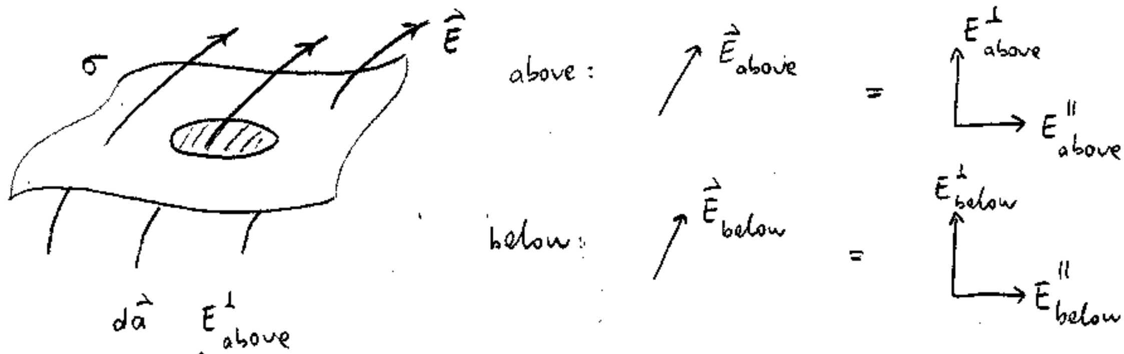


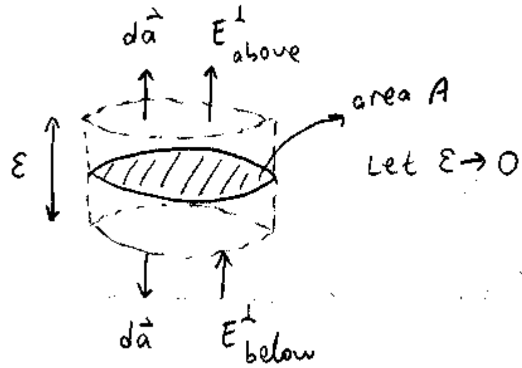
$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc}$$

$$EA + EA = \frac{1}{\varepsilon_0}\sigma A$$

$$E = \frac{\sigma}{2\varepsilon_0} \qquad \vec{E} = E\hat{n}$$

Continuity of E-field while crossing some charge distribution:



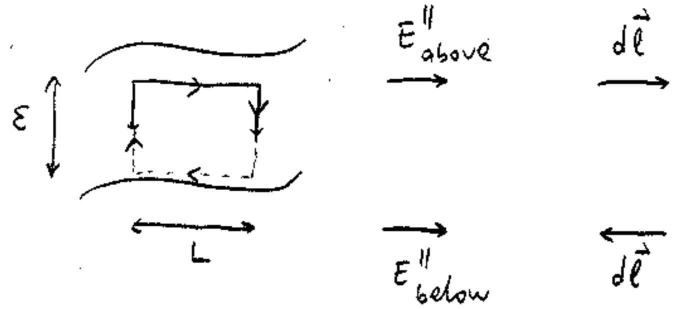


$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc} = \frac{1}{\varepsilon_0} \sigma A$$

$$E^{\perp}_{above} A - E^{\perp}_{below} A = \frac{1}{\varepsilon_0} \sigma A$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{1}{\varepsilon_0} \sigma$$ $\Rightarrow$ $\perp$-comp. is discont. at surface.

$$\oint \vec{E} \cdot d\vec{\ell} = 0$$



$$E^{\parallel}_{above} L - E^{\parallel}_{below} L = 0$$

$$E^{\parallel}_{above} = E^{\parallel}_{below}$$ $\Rightarrow$ $\parallel$-comp. is cont. at surface.

General:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0} \hat{n}, \quad \hat{n} \parallel d\vec{a}$$

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{\ell}$$

$$\int_a^b \vec{\nabla} V \cdot d\vec{\ell} = -\int_a^b \vec{E} \cdot d\vec{\ell}$$

$$\vec{E} = -\vec{\nabla} V$$

ex Solid sphere, R, const ρ. $V=?$ inside.

$$V=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

$$\vec{\mathscr{r}}=\vec{r}-\vec{r}\,'$$

$$\vec{r}=(0,0,z),\qquad \vec{r}\,'=r'\hat{r}$$

$$\mathscr{r}=\left[(z\hat{z}-r'\hat{r})\cdot(z\hat{z}-r'\hat{r})\right]^{1/2}=\left[z^2+r'^2-2zr'\cos\theta\right]^{1/2}$$

$$d\tau'=r'^2\sin\theta'\,dr'\,d\theta'\,d\varphi'$$

Omit the primes:

$$V(z)=\frac{\rho}{4\pi\varepsilon_0}\left[\int_0^{z\le r}r^2dr\int_0^{\pi}\frac{\sin\theta}{(z^2+r^2-2zr\cos\theta)^{1/2}}\,d\theta\int_0^{2\pi}d\varphi\right.$$

$$\left.+\int_z^{R}r^2dr\int_0^{\pi}\frac{\sin\theta}{(z^2+r^2-2zr\cos\theta)^{1/2}}\,d\theta\int_0^{2\pi}d\varphi\right]$$

$$V=\frac{q}{8\pi\varepsilon_0R^3}(3R^2-r^2),\quad r<R$$

$$V=kq\frac{1}{R},\quad r=R$$

$$V=kq\frac{1}{r},\quad r>R$$

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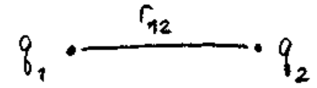
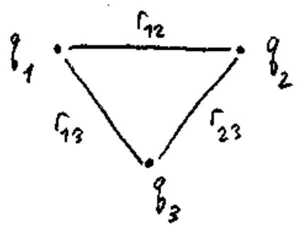
Recitation



$$W=\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}}$$



$$W=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}}+\frac{q_1q_3}{r_{13}}+\frac{q_2q_3}{r_{23}}\right)$$

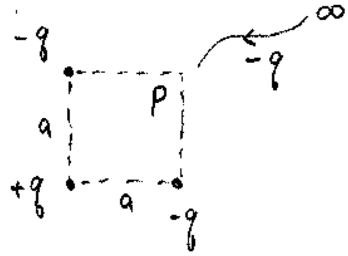
$$W=\frac{1}{2}\left[\frac{1}{4\pi\varepsilon_0}\left(\frac{q_2}{r_{12}}+\frac{q_3}{r_{13}}\right)q_1+\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{12}}+\frac{q_3}{r_{23}}\right)q_2+\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{13}}+\frac{q_2}{r_{23}}\right)q_3\right]$$

$$=\frac{1}{2}\left[V_1(\text{other})q_1+V_2(\text{other})q_2+V_3(\text{other})q_3\right]$$

$$W=\frac{1}{2}\sum_{i=1}^{N}q_i\sum_{\substack{j=1\\ j\ne i}}^{N}\frac{kq_j}{r_{ij}}$$

$$W=\frac{1}{2}\int\rho V\,d\tau$$

ex

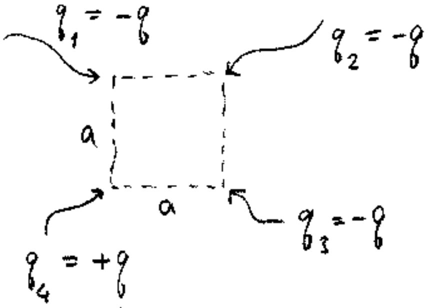


a) Bring $-q$ from ∞:

$$V_P = \frac{1}{4\pi\varepsilon_0} q \left(\frac{-2}{a} + \frac{1}{a\sqrt{2}} \right)$$

$$W = -q V_P$$

b) Bring them all one by one from ∞.



$$W_1 = 0$$

$$W_2 = q_2 V_2 = q_2 \frac{q_1}{4\pi\varepsilon_0 a}$$

$$W_3 = q_3 V_3 = q_3 \left(\frac{q_1}{4\pi\varepsilon_0 a} + \frac{q_2}{4\pi\varepsilon_0 a\sqrt{2}} \right)$$

$$W_4 = q_4 V_4 = q_4 \left(\frac{q_1}{4\pi\varepsilon_0 a\sqrt{2}} + \frac{q_2}{4\pi\varepsilon_0 a} + \frac{q_3}{4\pi\varepsilon_0 a} \right)$$

$$W = \sum_{i=1}^{4} W_i$$

ex Insulating solid sphere, R, $\rho(r) = 5\varepsilon_0 k r^2$

a) Total charge, Q?

$$Q = \int dq = \int \rho \, d\tau = \int 5\varepsilon_0 k r^2 \, r^2 \sin\theta \, dr \, d\theta \, d\varphi = 5\varepsilon_0 k \, 4\pi \left. \frac{r^5}{5} \right|_0^R$$

$$Q = 4\pi\varepsilon_0 k R^5$$

b) $\vec{E}_{in}$, $\vec{E}_{out}$?

Gauss' law:

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

$r > R$:

$$E 4\pi r^2 = \frac{1}{\varepsilon_0} Q \rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}, \quad r > R, \quad Q \text{ from part (a)}$$

$r < R$:

$$E 4\pi r^2 = \frac{1}{\varepsilon_0} \int \rho(r) \, d\tau = \frac{1}{\varepsilon_0} \int_0^r 5\varepsilon_0 k r'^2 \, r'^2 \sin\theta \, dr' \, d\theta \, d\varphi$$

$$E r^2 = 5k \left. \frac{r'^5}{5} \right|_0^r = k r^5$$

$$E = k r^3, \quad \vec{E} = E\hat{r}, \quad k = \frac{Q}{4\pi\varepsilon_0 R^5}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^5} r^3 \hat{r}$$

c) V_{in}, V_{out} ? Take ∞ as reference.

$$V = -\int_{ref}^{r} \vec{E} \cdot d\vec{\ell}$$

$$V_{out} = -\int_{\infty}^{r} \vec{E}_{out} \cdot d\vec{\ell} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r}$$

$$V_{in} = -\int_{\infty}^{R} \vec{E}_{out} \cdot d\vec{l} - \int_{R}^{r} \vec{E}_{in} \cdot d\vec{l}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{R} - \frac{Q}{4\pi\varepsilon_0 R^5}\frac{1}{4}(r^4 - R^4)$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}\left(1 - \frac{r^4 - R^4}{4R^4}\right) = \frac{Q}{16\pi\varepsilon_0}\frac{5R^4 - r^4}{R^5}$$

d) Energy of sphere? Use $W = \frac{1}{2}\int_V \rho V d\tau$.

ρ known, $V = V_{in}$

(Answer:

$$W = \frac{Q^2}{4\pi\varepsilon_0}\frac{5}{9}\frac{1}{R}$$)

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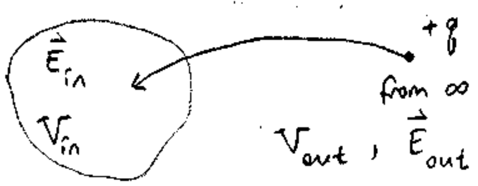
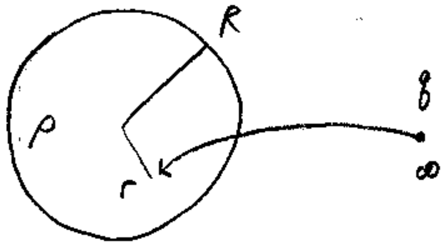
$W = \frac{1}{2}\int_V \rho V d\tau$, energy/work to assemble a volume charge

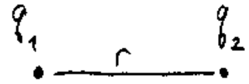
(V: where the charge is)



Assume sphere for V:



$$V_{in} = -\int_{\infty}^{R} \vec{E}_{out} \cdot d\vec{l} - \int_{R}^{r} \vec{E}_{in} \cdot d\vec{l}$$



$$W = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

Energy stored: Gauss' law

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho \quad \text{or} \quad \rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$$

$$W = \frac{\varepsilon_0}{2}\int_V \vec{\nabla} \cdot \vec{E}\, V d\tau, \quad V\text{: where the charge is.}$$

$$\vec{\nabla} \cdot (\vec{E} V) = \vec{\nabla} \cdot \vec{E}\, V + \vec{E} \cdot \vec{\nabla} V = V \vec{\nabla} \cdot \vec{E} - \vec{E}^2$$

$$W = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} \vec{\nabla} \cdot (\vec{E} V) d\tau + \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} \vec{E}^2 d\tau$$

$$\int_{\substack{\text{all}\\ \text{space}}} \vec{\nabla} \cdot (\vec{E} V) d\tau = \oint_{S \equiv \partial V_{\text{all space}}} \vec{E} V \cdot d\vec{a} \to 0 \text{ since } V \propto \frac{1}{r},\ \vec{E} \propto \frac{1}{r^2}$$

(for very large volume)

so

$$W = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} \vec{E}^2 d\tau \text{, energy stored in field}$$

Linear superposition does not apply to W due to $\vec{E}^2$. For ex, for $\vec{E}_1$ and $\vec{E}_2$ total energy is

$$W = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} (\vec{E}_1 + \vec{E}_2)^2 d\tau = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} (\vec{E}_1^2 + \vec{E}_2^2 + 2\vec{E}_1\cdot\vec{E}_2) d\tau$$

$$= W_1 + W_2 + \varepsilon_0 \int_{\substack{\text{all}\\ \text{space}}} \vec{E}_1\cdot\vec{E}_2 d\tau$$

<u>ex</u> Shell, R, σ.

total charge: $q = \sigma 4\pi R^2$

pot. on surface: $V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$

$$W = \frac{1}{2}\frac{q}{4\pi\varepsilon_0}\frac{1}{R}\underbrace{\int_S \sigma da}_{q} = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 R}$$

or

$$E_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, \quad r>R; \quad E_{in} = 0, \quad r<R$$

$$W = \frac{\varepsilon_0}{2}\int_R^\infty \left(\frac{1}{4\pi\varepsilon_0}\right)^2 q^2 \frac{1}{r^4} r^2 \sin\theta dr\, d\theta\, d\varphi$$

$$= \frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 q^2 \int_R^\infty \frac{dr}{r^2}\, 4\pi$$

$$= \frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 q^2\, 4\pi \frac{1}{R}$$

$$= \frac{\varepsilon_0}{2}\frac{1}{(4\pi)^2\varepsilon_0^2} q^2\, 4\pi\frac{1}{R}$$

$$= \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 R}$$

<u>ex</u> Solid, $\rho = 5\varepsilon_0 k r^2$.

$$V_{in} = -\int_\infty^R \vec{E}_{out}\cdot d\vec{l} - \int_R^r \vec{E}_{in}\cdot d\vec{l} = \ldots$$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{Q^2}{4\pi\varepsilon_0}\frac{5}{9}\frac{1}{R}, \quad Q = 5\varepsilon_0 k\int_0^R r^2 r^2 4\pi dr = 4\pi k\varepsilon_0 R^5$$

$$W = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} \vec{E}^2 d\tau = \frac{\varepsilon_0}{2}\left[\int_{r<R} \vec{E}_{in}^2 d\tau + \int_{r>R} \vec{E}_{out}^2 d\tau\right]$$

$$E_{in} = \frac{Q}{4\pi\varepsilon_0}\frac{r^3}{R^5}, \quad r<R, \qquad \vec{E}_{in} = E_{in}\hat{r}$$

$$E_{out} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}, \quad r>R, \quad \vec{E}_{out} = E_{out}\hat{r}$$

[BTW, continuity of E-field: No surface charge $\Rightarrow$ continuous $\vec{E}$-field, or

$$E_{in}\Big|_{r=R} \overset{?}{=} E_{out}\Big|_{r=R}$$

$$\frac{Q}{4\pi\varepsilon_0}\frac{1}{R^2} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R^2} \quad \checkmark$$

$\therefore$ cont E-field at the surface]

$$W = \frac{\varepsilon_0}{2}\Big[\int_0^R \frac{Q^2}{(4\pi)^2\varepsilon_0^2}\frac{1}{R^{10}}r^6 r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

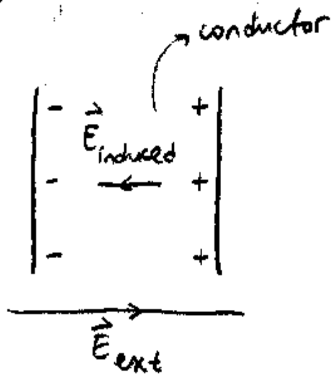
$$+ \int_R^\infty \frac{Q^2}{(4\pi)^2\varepsilon_0^2}\frac{1}{r^4}r^2\sin\theta\, dr\, d\theta\, d\varphi\Big]$$

$$= \frac{\varepsilon_0}{2}\frac{Q^2}{(4\pi)^2\varepsilon_0^2}4\pi\left(\frac{1}{R^{10}}\int_0^R r^8 dr + \int_R^\infty \frac{1}{r^2}dr\right)$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{9}\frac{1}{R^{10}}R^9 + \frac{1}{R}\right)$$

$$= \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{R}\left(\frac{1}{9}+1\right) = \frac{Q^2}{4\pi\varepsilon_0}\frac{1}{R}\frac{5}{9}$$

Conductors



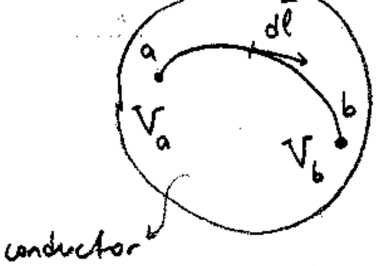
$$\vec{E}_{induced} = -\vec{E}_{external}$$

$\vec{E}_{in} = 0$ (no E-field inside)

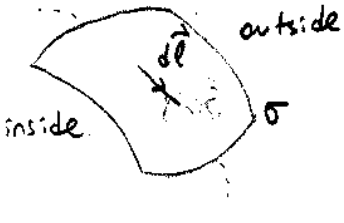
Since $\vec{E}_{in}$ is zero,

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \Rightarrow \rho_f = 0 \text{ inside}, \quad f \text{ for free charge}$$

All free charges are on the surface.



$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{\ell}$$



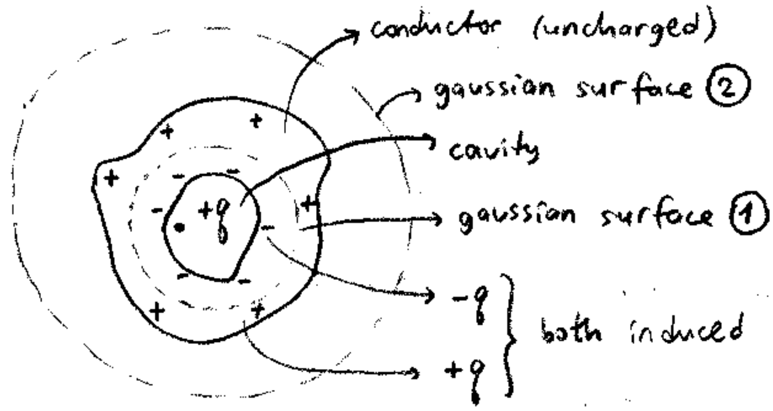
$E_{in}^{\parallel} = 0$ (we know)

$E_{in}^{\parallel} = E_{out}^{\parallel}$ (we have shown)

$\Rightarrow E_{out}^{\parallel} = 0$

Surface of the conductor is an equipotential surface.
$\vec{E}$ is always normal to the surface of conductor (just outside)

ex



①:

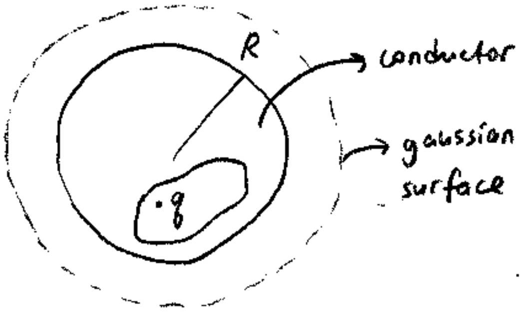
$$\underbrace{\oint \vec{E}\cdot d\vec{a}}_{=0 \text{ (should be)}} = \frac{1}{\varepsilon_0} q_{enc} \Rightarrow q_{enc} = 0 = q + (-q_{ind})$$

q: inside cavity; $-q_{ind}$: induced on cavity wall

②:

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} q_{enc} = \frac{1}{\varepsilon_0} q,$$ q: only charge inside (when $+q_{ind}$ and $-q_{ind}$ cancel out)

ex



$\vec{E}_{out} = ?$

$$\vec{E}_{out} = \frac{1}{4\pi\varepsilon_0} q \frac{\hat{r}}{r^2}$$

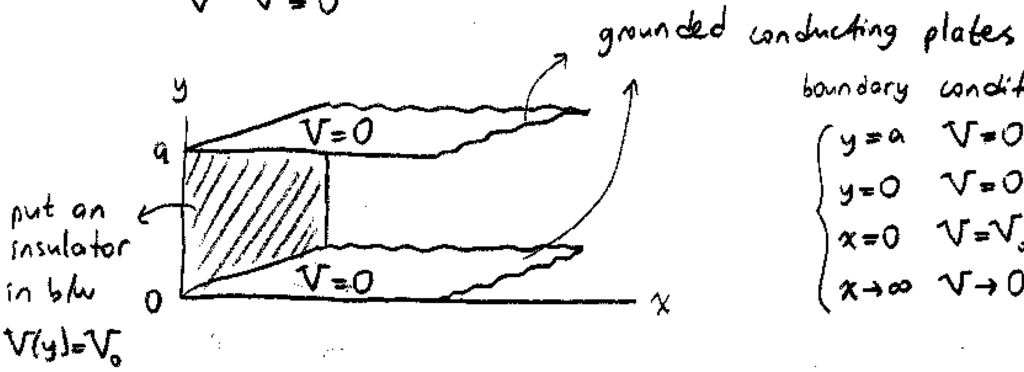
$$\vec{E} = -\vec{\nabla} V$$
$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}^2 V = \frac{1}{\varepsilon_0}\rho$$

Solution is

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\kappa}\rho\, d\tau'$$

Inside conductor

$$\vec{\nabla}^2 V = 0$$

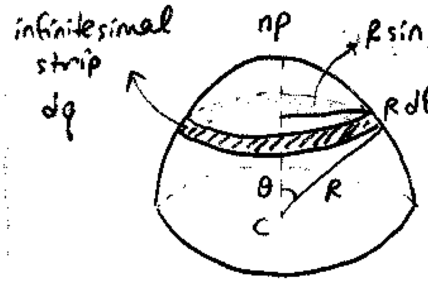


boundary conditions:

$$\begin{cases} y=a & V=0 \\ y=0 & V=0 \\ x=0 & V=V_0 \\ x\to\infty & V\to 0 \end{cases}$$

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0 \rightarrow \text{separation of variables} \Rightarrow \text{unique soln}$$

ex (2-44) Semi-spherical shell, R, σ. Pot. difference b/w north pole and center. Feb 28



$$C \equiv (0,0,0)$$

$$V_{np} - V_c \equiv \Delta V$$

$$V = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma}{\mathscr{r}} da, \quad \sigma da = dq = \sigma 2\pi R \sin\theta R d\theta$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

$$Q = \int dq = \sigma 2\pi R^2 \int_0^{\pi/2} \sin\theta d\theta = 2\pi R^2 \sigma$$

V_{np}:

$$\vec{\mathscr{r}}_{np} = R\hat{z} - R\hat{r}, \quad \mathscr{r}_{np} = \sqrt{R^2 (\hat{z}-\hat{r})\cdot(\hat{z}-\hat{r})}$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\mathscr{r}_{np} = \sqrt{2}\, R\, (1-\cos\theta)^{1/2}$$

$$V_{np} = \frac{1}{4\pi\varepsilon_0} \sigma 2\pi R^2 \int_0^{\pi/2} \frac{\sin\theta d\theta}{R\sqrt{2}\,(1-\cos\theta)^{1/2}}$$

$$= \frac{\sigma R}{2\varepsilon_0} 2(1-\cos\theta)^{1/2} \Big|_0^{\pi/2} = \frac{\sigma R}{\sqrt{2}\,\varepsilon_0}$$

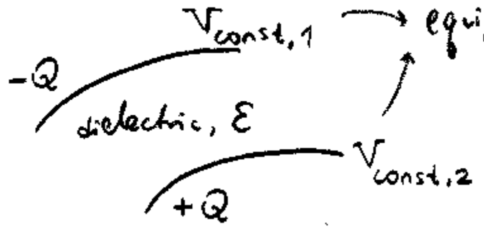
V_c:

$$\vec{\mathscr{r}}_c = \vec{r} - \vec{r}\,' = -R\hat{r}, \quad \mathscr{r}_c = R$$

$$V_c = \frac{1}{4\pi\varepsilon_0} \sigma R^2 2\pi \int_0^{\pi/2} \sin\theta d\theta \frac{1}{R} = \frac{\sigma R}{2\varepsilon_0}$$

$$\Delta V = \frac{\sigma R}{2\varepsilon_0}\left(\sqrt{2} - 1\right)$$

Capacitors



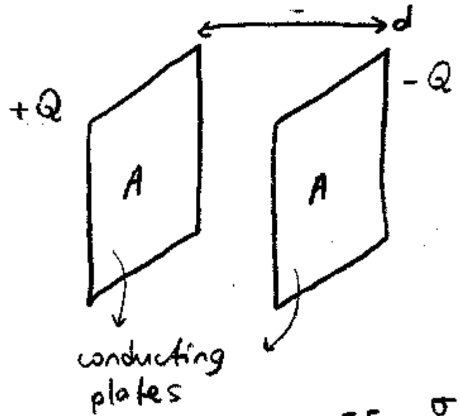
→ equipotential surfaces (conductor property)

$Q \propto V$

$Q = CV$, C: geometrical

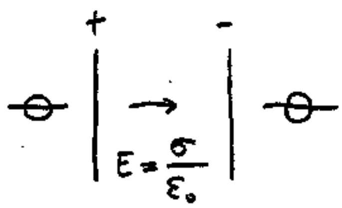
$[C] = \frac{\text{Coulomb}}{\text{volt}} \equiv$ Farad, F

ex Parallel plate capacitor



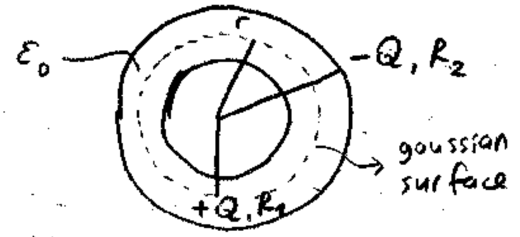
$A \gg d^2$

$V = -\int_{ref}^{r} \vec{E}\cdot d\vec{l}, \quad \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$ per each plate



$$V = \frac{\sigma}{\varepsilon_0} d \Rightarrow C \equiv \frac{Q}{V} = \frac{\sigma A}{\frac{\sigma d}{\varepsilon_0}} = \frac{A\varepsilon_0}{d}$$

Concentric Spherical Metal Shells



$E = 0$ everywhere except $R_1 < r < R_2$

E-field for $R_1 < r < R_2$: Gaussian surface, r

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

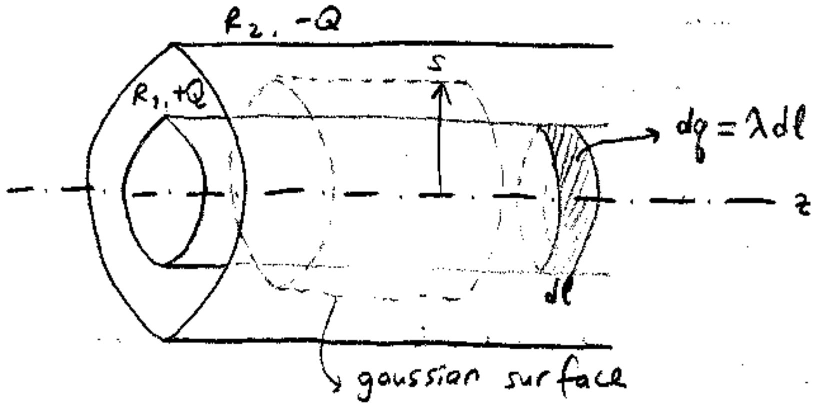
$$E\, 4\pi r^2 = \frac{1}{\varepsilon_0} Q \quad \text{so} \quad E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}$$

$$V = -\int \vec{E} \cdot d\vec{l} = -\int_{R_2}^{R_1} \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} dr$$

$$= \frac{1}{4\pi\varepsilon_0} Q \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{4\pi\varepsilon_0} Q \frac{R_2 - R_1}{R_1 R_2}$$

$$C \equiv \frac{Q}{V} = 4\pi\varepsilon_0 \frac{R_1 R_2}{R_2 - R_1} = \varepsilon \frac{4\pi R_1 R_2}{R_2 - R_1} \begin{array}{l} \rightarrow \sim \text{area} \\ \rightarrow \sim \text{distance} \end{array}$$

<u>ex</u> Coaxial cylindrical shells, very long compared to $(R_2 - R_1)$



$$Q = \lambda L, \quad L \gg (R_2 - R_1)$$

Gauss' law:

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

$$2\pi s L = \frac{1}{\varepsilon_0} \lambda L \quad \text{so} \quad E = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{s}, \qquad \vec{E} = E\hat{s}$$

$$V = -\int_{R_2}^{R_1} \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{s} ds = \frac{\lambda}{2\pi\varepsilon_0} \ln s \Big|_{R_1}^{R_2} = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{R_2}{R_1}$$

$$C \equiv \frac{Q}{V} = \frac{\lambda L}{\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{R_2}{R_1}} = 2\pi\varepsilon_0 L \frac{1}{\ln \frac{R_2}{R_1}}$$

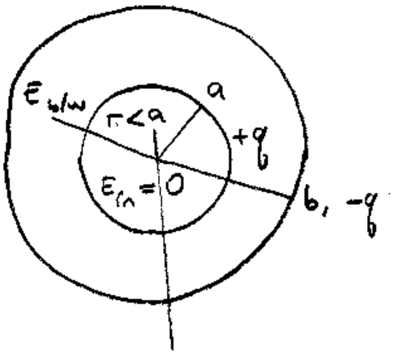
$$\frac{C}{L} = 2\pi\varepsilon_0 \frac{1}{\ln(R_2/R_1)}$$

Energy to charge capacitor from 0 to Q, total charge:

$$dW = \frac{q}{C}\,dq$$

$$W = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{1}{C}Q^2 = \frac{1}{2}CV^2$$

ex (2-34) Spherical shells, $W = ?$



$r > b : E_{out} = 0$

① $a<r<b$: $E_{b/w} = \frac{1}{4\pi r^2}\frac{1}{\varepsilon_0}q$

$$W = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} \vec{E}^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int_{b/w} \frac{1}{(4\pi)^2\varepsilon_0^2} q^2 \frac{1}{r^4} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{2}\frac{1}{\varepsilon_0} q^2 \int_a^b \frac{1}{r^2}dr = \frac{q^2}{2\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

②

$$W = \frac{\varepsilon_0}{2}\left(\int_{\substack{\text{all}\\ \text{space}}} \vec{E}_+^2 d\tau + \int_{\substack{\text{all}\\ \text{space}}} \vec{E}_-^2 d\tau + 2\int_{\substack{\text{all}\\ \text{space}}} \vec{E}_+ \cdot \vec{E}_- d\tau\right)$$

$$\vec{E}_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}, \quad r > a$$

$$= 0 \text{ otherwise}$$

$$\vec{E}_- = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2}\hat{r}, \quad r > b$$

$$= 0 \text{ otherwise}$$

$$W = \frac{\varepsilon_0}{2}\frac{1}{(4\pi)^2\varepsilon_0^2} q^2 \left(\int_a^\infty \frac{1}{r^4} r^2 \sin\theta\, dr\, d\theta\, d\varphi + \int_b^\infty \frac{1}{r^4} r^2 \sin\theta\, dr\, d\theta\, d\varphi \right.$$
$$\left. + 2\int_b^\infty -\frac{1}{r^4} r^2 \sin\theta\, dr\, d\theta\, d\varphi\right)$$

$$= \frac{\varepsilon_0}{2}\frac{1}{(4\pi)^2\varepsilon_0^2} q^2 4\pi \left(\int_a^\infty \frac{1}{r^2}dr + \int_b^\infty \frac{1}{r^2}dr - 2\int_b^\infty \frac{1}{r^2}dr\right)$$

$$= \frac{1}{2}\frac{1}{4\pi\varepsilon_0} q^2 \int_a^b \frac{1}{r^2}dr, \text{ same result}$$

Mar 5

Recit

Problem 1-13

a) $\vec{\nabla} r = ?$ $\quad \hat{r}$

b) $\vec{\nabla}\frac{1}{r} = ?$ $\quad -\frac{1}{r^2}\hat{r}$

c) what is the general formula for $\vec{\nabla} r^n$?

a) $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \vec{\theta}\frac{\partial}{\partial \theta} + \vec{\varphi}\frac{\partial}{\partial \varphi}$

b) $\vec{\nabla}\frac{1}{r} = \hat{r}\frac{\partial}{\partial r}\frac{1}{r} = -\frac{1}{r^2}\hat{r}$

c) In cartesian coor.s :

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$\frac{\partial}{\partial x} r^n = n r^{n-1} \frac{\partial}{\partial x}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\vec{\nabla} r = \hat{x}\,\frac{x}{\sqrt{\dots}} + \hat{y}\,\frac{y}{\sqrt{\dots}} + \hat{z}\,\frac{z}{\sqrt{\dots}}$$

ex $\qquad \vec{\nabla}\cdot\frac{\hat{r}}{r^2} = ? \quad 4\pi\delta^3(\vec{r})$

$$\vec{\nabla}^2 \frac{1}{r} = ? \quad -4\pi\delta^3(\vec{r})$$

$$\int_{\text{sphere}} \vec{\nabla}\cdot\frac{\hat{r}}{r^2}\, d\tau = \oint_{\text{sphere}} \frac{\hat{r}}{r^2}\cdot d\vec{a}, \quad d\vec{a} = \hat{r} r^2 \sin\theta\, d\theta\, d\varphi + \hat{\theta}\dots + \hat{\varphi}\dots$$

$$\oint_{\text{sphere}} \frac{\hat{r}}{r^2}\cdot \hat{r} r^2 \sin\theta\, d\theta\, d\varphi = \oint_{\text{sphere}} \sin\theta\, d\theta\, d\varphi = 4\pi$$

$$\int \vec{\nabla}\cdot\frac{\hat{r}}{r^2}\, d\tau = 4\pi$$

$$\vec{\nabla}^2 \frac{1}{r} = \vec{\nabla}\cdot\vec{\nabla}\frac{1}{r} = \vec{\nabla}\cdot\frac{-\hat{r}}{r^2} = -\vec{\nabla}\cdot\frac{\hat{r}}{r^2}$$

Problem 2-36

Two spherical cavities of radii a, b are hollowed out from the interior of a neutral conducting sphere of radius R. At the center of each cavity, a point charge is placed, q_a and q_b.

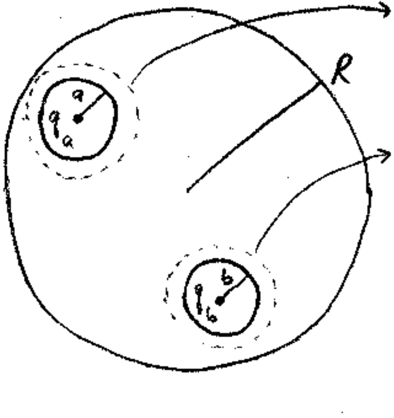
a) Find surface charges, σ_a ; σ_b ; σ_R.

b) Field outside conductor

c) Field within each cavity.

d) Force on q_a and q_b

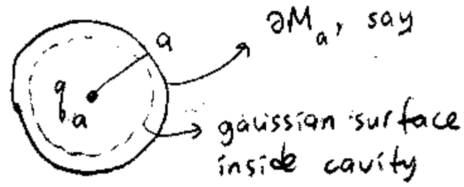
e) Which of these answers would change if a third charge q_c were brought near the conductor?



$\oint \vec{E}\cdot d\vec{a} = 0 \Rightarrow Q_{enc} = q_a + (-q_a)$

$\oint \vec{E}\cdot d\vec{a} = 0 \Rightarrow Q_{enc} = q_b + (-q_b)$

Uniformity of these distributions of induced charges on cavity walls:



$\rightarrow$ spherical symmetry

On ∂M_a,

i) $\quad (\vec{E}_{above} - \vec{E}_{below}) \cdot \hat{n} = \dfrac{\sigma_a}{\varepsilon_0}$, $\quad \hat{n}$: surface normal of ∂M_a

ii) $$\underbrace{E^{\parallel}_{above}}_{0} - E^{\parallel}_{below} = 0$$

$$\Rightarrow E^{\parallel}_{below} = 0 \Rightarrow \sigma_a = \frac{-q_a}{4\pi a^2}$$

Similarly,

$$\sigma_b = \frac{-q_b}{4\pi b^2} \quad \text{and} \quad \sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

Mar 6

CH3 SPECIAL TECHNIQUES

Recall:

$$\vec{\nabla}\frac{1}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\vec{\mathscr{r}})$$

$$\vec{\nabla}^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\vec{\mathscr{r}})$$

$$\vec{E} = -\vec{\nabla}V$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\vec{\nabla}\cdot\vec{\nabla}V = -\frac{\rho(\vec{r})}{\varepsilon_0} \equiv \vec{\nabla}^2 V \qquad \text{Poisson eq}$$

Soln to Poisson:

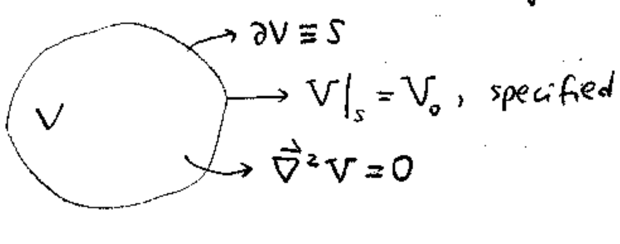
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,')}{\mathscr{r}}d\tau'$$

$$\vec{\nabla}^2 V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\left(\vec{\nabla}^2\frac{1}{\mathscr{r}}\right)\rho(\vec{r}\,')d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int -4\pi\delta^3(\vec{\mathscr{r}})\rho(\vec{r}\,')d\tau'$$

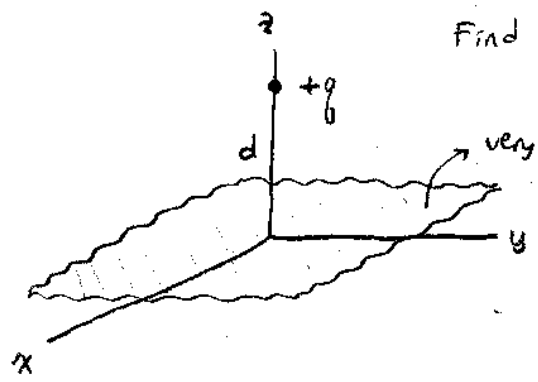
$$= -\frac{\rho(\vec{r})}{\varepsilon_0}$$

Laplace eq: $\vec{\nabla}^2 V = 0$. The soln to this is unique when the potential V is specified over the surface S of a volume $\mathcal{V}$.



$V \to 0$ as $r \to \infty$ is also a boundary.

Image Charges



Find $V(z \geq 0)$.

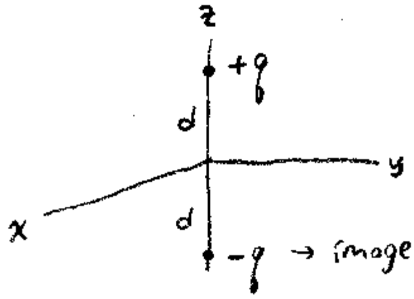
very large grounded conducting plane ($V=0$)

boundaries :

$z=0 \quad V=0$

$z\to\infty \quad V\to 0$

Put another charge $-q$ at $z=-d$ and remove the conductor — a new configuration :



→ image charge : $z=0 \;\; V=0$ ✓

If $V\to 0$ as $z\to\infty$ also holds, then the soln will be unique.

Due to one charge:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

so superpose the two:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{-q}{r_2}\right)$$

This should give 0 at $z=0$:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

In polar coor.s,

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{r^2+(z-d)^2}} - \frac{q}{\sqrt{r^2+(z+d)^2}}\right)$$

As $r\to\infty$ and $z\to\infty$,

$$V\to 0 \quad ✓$$

Induced charge on conductor: From continuity of E-field,

$$\frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0} \quad \text{where} \quad \frac{\partial V}{\partial n} = \vec{\nabla} V\cdot\hat{n}$$

so

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = -\frac{\sigma}{\varepsilon_0}$$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{1}{4\pi\varepsilon_0}\left.\left(\frac{-(z-d)q}{(r^2+(z-d)^2)^{3/2}} + \frac{(z+d)q}{(r^2+(z+d)^2)^{3/2}}\right)\right|_{z=0}$$

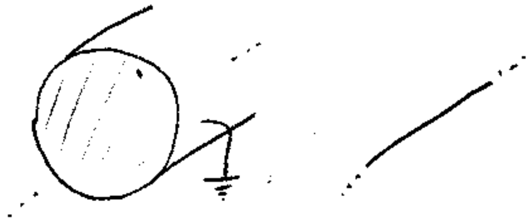
$$= \frac{1}{4\pi\varepsilon_0}\frac{2dq}{(r^2+d^2)^{3/2}}$$

$$\sigma(r) = \frac{-q\,d}{2\pi\,(r^2+d^2)^{3/2}}$$

$$da = r\,dr\,d\theta$$

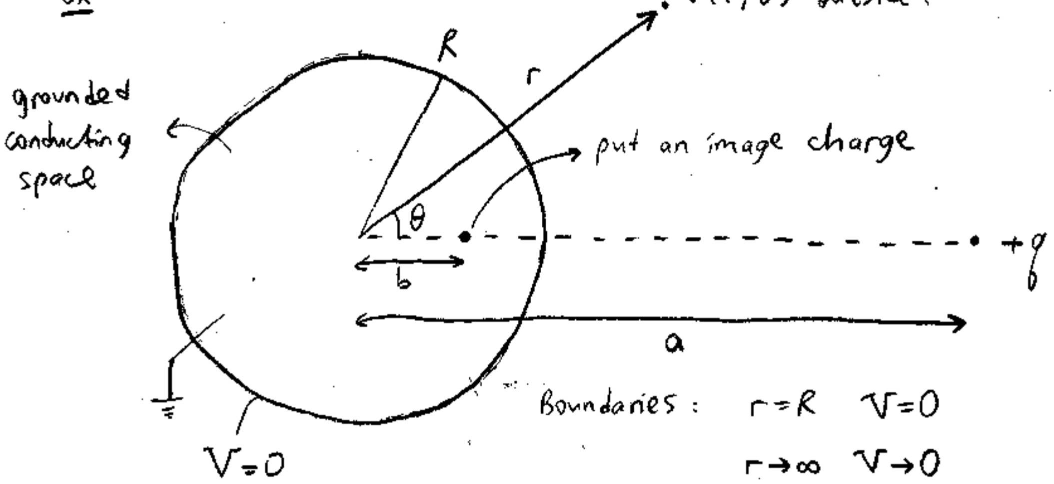
$$q_{ind} = \int \sigma(r)\,r\,dr\,d\theta = -q$$

Study



<u>ex</u>

$\cdot\, V(r,\theta)$ outside?

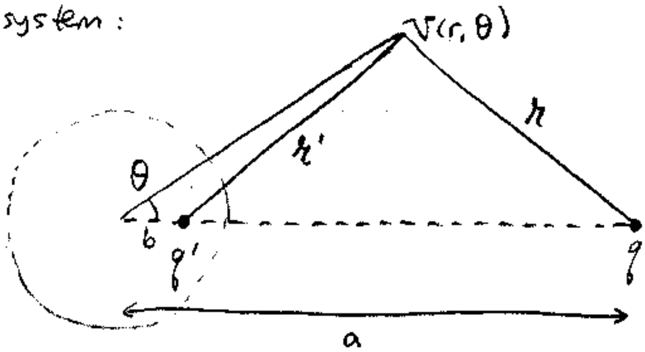


Boundaries : $r=R \quad V=0$

$r\to\infty \quad V\to 0$

$$ab = R^2 \quad \text{or} \quad b = \frac{R^2}{a}$$

Image system :



$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$

$$r = (a^2+r^2-2ar\cos\theta)^{1/2}$$

$$r' = (b^2+r^2-2br\cos\theta)^{1/2}$$

$$= \left(\frac{R^4}{a^2}+r^2-2r\frac{R^2}{a}\cos\theta\right)^{1/2}$$

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(a^2+r^2-2ar\cos\theta)^{1/2}} + \frac{q'}{\left(\frac{R^4}{a^2}+r^2-2r\frac{R^2}{a}\cos\theta\right)^{1/2}}\right)$$

Check boundaries :

$$r = R \quad V = 0$$

$$q' = -\frac{R}{a}q$$

Induced charge :

$$\sigma(R,\theta) = -\varepsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$$

$$= \frac{1}{4\pi\varepsilon_0}\left(q\,\frac{-1}{2}\,\frac{2r-2a\cos\theta}{(a^2+r^2-2ar\cos\theta)^{3/2}} \cdots\right)$$

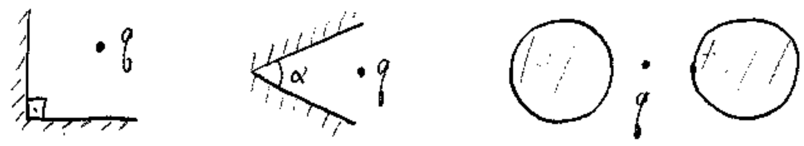
$$\sigma(\theta) = \frac{-q}{4\pi R}(R^2-a^2)\,(R^2+a^2-2Ra\cos\theta)^{-3/2}$$

$$q_{ind} = \int \sigma(\theta)\,da = \int \sigma(\theta)\, r^2 \sin\theta\, d\theta\, d\varphi \Big|_{r=R}$$

$$= \frac{-q}{4\pi R} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} (R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2} R^2 \sin\theta\, d\theta\, d\varphi$$

$$= -\frac{qR}{a}$$

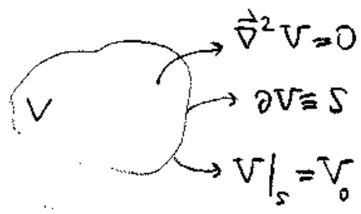
Another problems:



Mar 7

<u>Separation of Variables</u>

Laplace eq for potential V:



$$\vec{\nabla}^2 V = 0$$

$$\vec{\nabla}^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\vec{\nabla}^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}$$

When assuming azimuthal symmetry, $\partial V/\partial\varphi = 0$ and

$$\vec{\nabla}^2 V = \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Here we'll assume

cartesian: $V(x,y) = X(x)Y(y)$

spherical: $V(r,\theta) = R(r)\Theta(\theta)$

so

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{=\text{const},\, k^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{=\text{const},\, -k^2} = 0 \quad\Rightarrow\quad \begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ Y(y) &= C\sin ky + D\cos ky \end{aligned}$$

and

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{=\text{const},\, \ell(\ell+1)} + \underbrace{\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{=\text{const},\, -\ell(\ell+1)} = 0$$

$$R(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}$$

$$\Theta(\theta) = P_\ell(\cos\theta), \quad P_\ell: \text{Legendre polynomials}$$

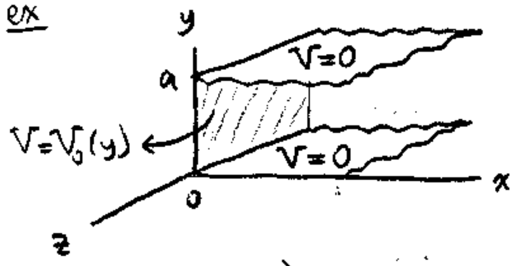
$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

Orthogonality of $P_\ell(x)$:

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \delta_{\ell\ell'}\frac{2}{2\ell'+1}$$

Orthogonality of sin-cos:

$$\int_0^a \sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\, dy = \delta_{nn'}\frac{a}{2}$$

ex 

Potential in b/w?

$$\vec{\nabla}^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = X(x)Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$Y(y) = C\sin ky + D\cos ky$$

Boundaries:

(1) $x \to \infty \quad V \to 0$

(2) $y = 0 \quad V = 0$

(3) $y = a \quad V = 0$

(4) $x = 0 \quad V = V_0(y)$

From (1),

$$A = 0$$

$$V(x,y) = Be^{-kx}(C\sin ky + D\cos ky)$$

From (2),

$$V(x,y) = Be^{-kx}(0 + D) = 0 \Rightarrow D = 0$$

$$V(x,y) = Ce^{-kx}\sin ky$$

From (3),

$$V(x,y) = Ce^{-kx}\sin ka = 0$$

$$\sin ka = 0 \quad \text{so} \quad ka = n\pi \quad \text{or} \quad k = \frac{n\pi}{a}$$

$$V(x,y) = C_n e^{-n\pi x/a}\sin\frac{n\pi y}{a}$$

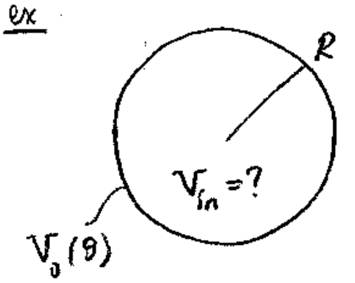
$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin\frac{n\pi y}{a}$$

$$V(0,y) = V_0(y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi y}{a}$$

$$\int_0^a V_0(y)\sin\frac{m\pi y}{a}\,dy = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin\frac{n\pi y}{a}\sin\frac{m\pi y}{a}\,dy}_{\delta_{mn}\frac{a}{2}}$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\frac{n\pi y}{a}\,dy$$

ex



Potential inside shell?

Spherical Laplace: Indep of $\varphi \to \varphi$-symmetry

$$R(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}$$

$$\Theta(\theta) = P_\ell(\cos\theta)$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Boundaries:

$$r = R \qquad V = V_0(\theta)$$
$$r = 0 \qquad V = \text{finite}$$

so

$$B = 0$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

At $r = R$,

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = V_0(\theta)$$

From orthogonality,

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell \underbrace{\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta}_{\delta_{\ell\ell'} \frac{2}{2\ell'+1}} = \int_0^\pi V_0(\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$A_\ell = \frac{1}{R^\ell} \frac{2\ell+1}{2} \int_0^\pi V_0(\theta) P_\ell(\cos\theta)\sin\theta\, d\theta$$

For ex,

$$V_0(\theta) = k\sin^2\frac{\theta}{2}$$

which does not seem like a Legendre polynomial, but

$$V_0(\theta) = k\left(\frac{1-\cos\theta}{2}\right) = \frac{k}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$$

$$\frac{k}{2}\int_0^\pi P_0(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta - \frac{k}{2}\int_0^\pi P_1(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$= \sum_{\ell=0}^{\infty} A_\ell r^\ell \underbrace{\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta}_{\delta_{\ell\ell'} \frac{2}{2\ell+1}}$$

Expanding RHS, only A_0 and A_1 terms will appear.

$$RHS = A_0 R^0 2 + A_1 R \frac{2}{3}$$

$$LHS = \frac{k}{2} 2 - \frac{k}{2}\frac{2}{3}$$

So

$$A_0 R^0 = A_0 = \frac{k}{2}$$

$$A_1 = -\frac{k}{2}$$

Potential:

$$V(r,\theta) = \sum_{\ell=0}^{1} A_\ell r^\ell P_\ell(\cos\theta)$$
$$= A_0 r^0 P_0(\cos\theta) + A_1 r^1 P_1(\cos\theta)$$
$$= \frac{k}{2} - \frac{k}{2R} r\cos\theta, \quad r < R$$

Potential outside: $r > R$

new boundary: $r \to \infty \quad V \to 0$

so

$$A = 0$$
$$\frac{1}{r^{\ell+1}} \neq 0 \quad \text{since } r > R$$
$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Mar 12

(no recit)

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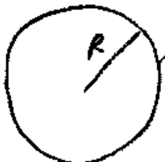
Spherical shell, R



$$V\big|_S = V_0(\theta) = \tilde{k}\sin^2\frac{\theta}{2}$$
$$= \frac{\tilde{k}}{2}(1-\cos\theta) = \frac{\tilde{k}}{2}(1-\cos\theta) = \frac{\tilde{k}}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$$

Orthogonality relation:

$$\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \delta_{\ell\ell'} \frac{2}{2\ell'+1}$$
$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Inside $(r<R)$: $B = 0$

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = \frac{\tilde{k}}{2}\left(1 - \frac{r}{R}\cos\theta\right), \quad r < R$$

Outside $(r>R)$: $A = 0$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta), \quad r > R$$
$$V(R,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = V_0(\theta)$$
$$= \frac{\tilde{k}}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$$
$$\sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$
$$= \frac{\tilde{k}}{2}\left(\int_0^\pi P_0(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta - \int_0^\pi P_1(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta\right)$$

B_0 and B_1 exist only.

$$\sum_{\ell=0}^{1} \frac{B_\ell}{R^{\ell+1}} \delta_{\ell\ell'} \frac{2}{2\ell+1} = \frac{\tilde{k}}{2}\left(\delta_{\ell\ell'} \frac{2}{2\ell+1} - \delta_{\ell\ell'} \frac{2}{2\ell+1}\right)$$

$$\frac{B_0}{R}\delta_{00} + \frac{B_1}{R^2}\delta_{11} = \frac{\tilde{k}}{2}\delta_{00} - \frac{\tilde{k}}{2}\delta_{11}$$

$$B_0 = \frac{R\tilde{k}}{2} \qquad B_1 = -\frac{R^2\tilde{k}}{2}$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{1} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$= \frac{B_0}{r} P_0(\cos\theta) + \frac{B_1}{r^2} P_1(\cos\theta) = \frac{R\tilde{k}}{2r} - \frac{R^2\tilde{k}}{2r^2}\cos\theta$$

Check: $V_{in}(R,\theta) = V_{out}(R,\theta)$

$$\frac{\tilde{k}}{2}\left(1 - \frac{r}{R}\cos\theta\right)\Big|_R \overset{?}{=} \frac{R\tilde{k}}{2r} - \frac{R^2\tilde{k}}{2r^2}\cos\theta\Big|_R$$

$$\frac{\tilde{k}}{2} - \frac{\tilde{k}}{2}\cos\theta = \frac{\tilde{k}}{2} - \frac{\tilde{k}}{2}\cos\theta \quad \checkmark$$

ex Insulating shell, R, covered with surface charge $\sigma_0(\theta)$. $V_{in} = ?$ $V_{out} = ?$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Inside, $B=0$; outside, $A=0$ (due to finiteness)

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta), \quad r<R$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta), \quad r>R$$

$\sigma_0(\theta) \sim$ continuity of potential across surface

$$V_{out}(R,\theta) = V_{in}(R,\theta) \quad \ldots\ldots\ldots\ldots (1)$$

$$\sum_{\ell=0}^{\infty} \underbrace{A_\ell R^\ell}\, P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \underbrace{\frac{B_\ell}{R^{\ell+1}}}\, P_\ell(\cos\theta)$$

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad \text{or} \quad B_\ell = A_\ell R^{2\ell+1}$$

(More formal way: Take integrals of orthogonality)

[Derivation of $\vec{\nabla}^2 V = 0$!]

$$\frac{\partial V_{out}}{\partial r}\Big|_R - \frac{\partial V_{in}}{\partial r}\Big|_R = -\frac{\sigma_0}{\varepsilon_0} \quad \ldots\ldots\ldots\ldots (2)$$

So, (1) and (2) are boundaries.

$$\frac{\partial V_{out}}{\partial r} = \sum_{\ell=0}^{\infty} A_\ell R^{2\ell+1} (-\ell-1) \frac{1}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$\frac{\partial V_{in}}{\partial r} = \sum_{\ell=0}^{\infty} A_\ell \ell r^{\ell-1} P_\ell(\cos\theta)$$

So,

$$\sum_{\ell=0}^{\infty} A_\ell R^{\ell-1}(-\ell-1)P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty} A_\ell \ell R^{\ell-1} P_\ell(\cos\theta) = -\frac{\sigma_0}{\varepsilon_0}$$

$$\sum_{\ell=0}^{\infty} A_\ell[-R^{\ell-1}(\ell+1) - \ell R^{\ell-1}] P_\ell(\cos\theta) = -\frac{\sigma_0}{\varepsilon_0}$$

$$\sum_{\ell=0}^{\infty} A_\ell R^{\ell-1}(2\ell+1)\int_0^{\pi} P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta$$

$$= \int_0^{\pi} \frac{\sigma_0}{\varepsilon_0} P_{\ell'}(\cos\theta)\sin\theta d\theta$$

$$A_\ell R^{\ell-1}(2\ell+1)\frac{2}{2\ell+1} = \frac{1}{\varepsilon_0}\int_0^{\pi} \sigma_0 P_\ell(\cos\theta)\sin\theta d\theta$$

$$A_\ell = \frac{1}{2R^{\ell-1}\varepsilon_0}\int_0^{\pi} \sigma_0 P_\ell(\cos\theta)\sin\theta d\theta$$

For ex,

$$\sigma_0 = \tilde{k}\cos\theta = \tilde{k} P_1(\cos\theta)$$

so

$$A_\ell = \frac{1}{2R^{\ell-1}\varepsilon_0}\tilde{k}\int_0^{\pi} P_1(\cos\theta)P_\ell(\cos\theta)\sin\theta d\theta$$

only A_1 exists:

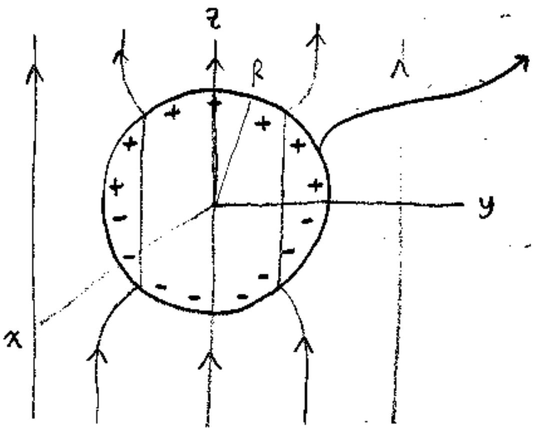
$$A_1 = \frac{\tilde{k}}{2\varepsilon_0}\frac{2}{2+1} = \frac{\tilde{k}}{3\varepsilon_0}$$

$$B_\ell = A_\ell R^{2\ell+1} \Rightarrow B_1 = \frac{\tilde{k}}{3\varepsilon_0}R^{2\ell+1}, \text{ all others are } 0.$$

$$V_{out} = \frac{\tilde{k}}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta, \quad r > R$$

$$V_{in} = \frac{\tilde{k}}{3\varepsilon_0} r\cos\theta, \quad r < R$$

ex Conducting shell in an E-field $\vec{E} = E_0\hat{z}$.



symmetrical and opposite charges will be induced hence equator (or xy-plane) will have zero potential.

Boundaries:

$$r \gg R \quad V \to -E_0 z = -E_0 r\cos\theta$$

$$r = R \quad V = 0$$

$$(\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

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Conducting shell (cont'd)

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$V = -E_0 r\cos\theta \quad \text{for } r \gg R$$

$$V = 0 \quad \text{for } r = R$$

so

$$V(r,\theta)=\sum_\ell \underbrace{\left(A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}}\right)}_{0} P_\ell(\cos\theta) = 0$$

$$A_\ell R^\ell = -\frac{B_\ell}{R^{\ell+1}} \quad \text{or} \quad B_\ell = -A_\ell R^{2\ell+1}$$

When $r \gg R$, then $B_\ell = 0$ and

$$V(r,\theta) = -E_0 r\cos\theta$$

$$\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

Even w/o orthogonal integrations, $\ell=1$ exists only s.t $A_1 = -E_0$, or in the formal way,

$$\sum_{\ell=0}^{\infty} A_\ell r^\ell \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$= -E_0 r \int_0^\pi P_1(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$A_\ell r^\ell \frac{2}{2\ell+1} = -E_0 r\, \delta_{1\ell} \frac{2}{2\ell+1}$$

so

$$\begin{cases} A_1 = -\dfrac{E_0 r\, \delta_{11}}{r^1} = -E_0 \\ A_\ell = 0, \quad \ell \neq 1 \end{cases}$$

$$B_\ell = -A_\ell R^{2\ell+1}$$

$$B_1 = -A_1 R^3 = E_0 R^3$$

Therefore, the potential outside is

$$V_{out}(r>R,\theta) = \left(A_1 r^1 + \frac{B_1}{r^{1+1}}\right) P_1(\cos\theta)$$

$$= \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

$$= \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\cos\theta$$

$$V_{out} = -E_0 r\cos\theta + \frac{E_0 R^3}{r}\cos\theta, \quad r>R$$

Continuity of potential across surface charge:

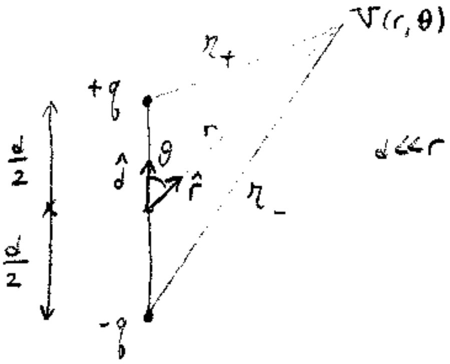
$$V_{out}\big|_R = V_{surface} = 0$$

E-field inside: $E_{in} = 0$

Note: $V_{surface} = 0$ since

$$\left.\begin{array}{l}\text{- conductor is equipotential surface} \\ \text{- at equator, } V=0\end{array}\right\} \therefore V\big|_R = 0$$

Electric Potential of a Dipole



$d \ll r$

$$V(r,\theta) = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$r_+^2 = r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta \cong r^2\left(1 - \frac{d}{r}\cos\theta\right)$$

$$r_-^2 = r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta \cong r^2\left(1 + \frac{d}{r}\cos\theta\right)$$

$$r_+ \cong r\left(1 - \frac{d}{r}\cos\theta\right)^{1/2} \qquad r_- \cong r\left(1 + \frac{d}{r}\cos\theta\right)^{1/2}$$

$$\frac{1}{r_+} \cong \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right) \qquad \frac{1}{r_-} \cong \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$

so

$$V(r,\theta) = kq\frac{d\cos\theta}{r^2}, \qquad V \propto \frac{1}{r^2}$$

Define

$$q\vec{d} \equiv \vec{p}, \text{ electric dipole moment}$$

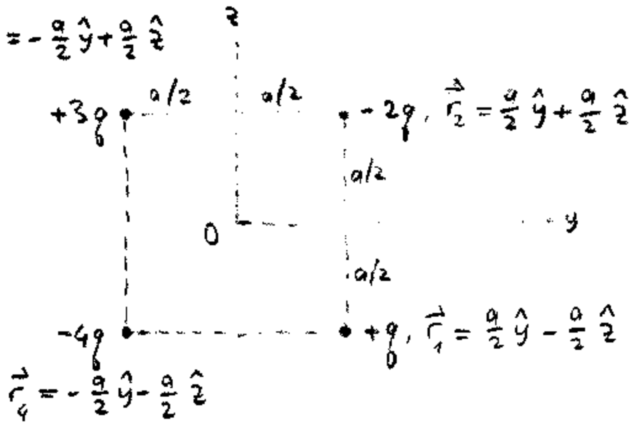
so

$$V(r,\theta) = k\,\frac{\vec{p}\cdot\hat{r}}{r^2}$$

General case:

$$\vec{p} = \vec{r}_i{}' q_i$$

<u>ex (HW problem)</u> ... V far away from O.



$$\vec{p} = \vec{r}_i{}' q_i = \vec{r}_1{}' q - 2\vec{r}_2{}' q + 3\vec{r}_3{}' q - 4\vec{r}_4{}' q$$

$$= q\frac{a}{2}\left(\hat{r}_1' - 2\hat{r}_2' + 3\hat{r}_3' - 4\hat{r}_4'\right)$$

$$= q\frac{a}{2}\left(\hat{y} - \hat{z} - 2\hat{y} - 2\hat{z} - 3\hat{y} + 3\hat{z} + 4\hat{y} + 4\hat{z}\right)$$

$$= q\frac{a}{2}4\hat{z} = 2qa\hat{z} = 2qa(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

$$\vec{p}\cdot\hat{r} = 2qa(\hat{r}\cos\theta - \hat{\theta}\sin\theta)\cdot\hat{r} = 2qa\cos\theta$$

$$V_{dip} = k\,\frac{\vec{p}\cdot\hat{r}}{r^2} = k\,\frac{2qa\cos\theta}{r^2}$$

This is the potential due to dipole. Monopole potential is like

$$V_{mon} = k\,\frac{\Sigma q}{r} = k\,\frac{-2q}{r}$$

$$V \cong V_{mon} + V_{dip}$$

Electric Potential For f[illegible] with Cont Charge Distribution

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\vec{r}\,')d\tau'$$

$$\mathscr{r} = (r^2+r'^2-2rr'\cos\theta)^{1/2} = r\Big(1+\underbrace{\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\theta}_{\equiv\varepsilon,\ \text{small since } r'\ll r}\Big)^{1/2}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1+\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\theta\right)^{-1/2} \equiv \frac{1}{r}(1+\varepsilon)^{-1/2}$$

Binomial expansion :

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta), \quad r\gg r'$$

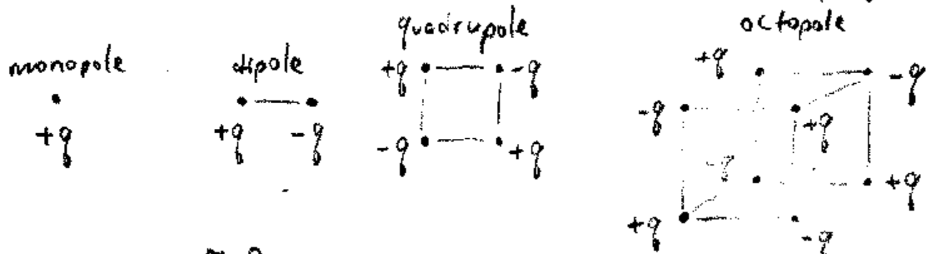
$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{\substack{\text{over}\\ \text{source}}} r'^n P_n(\cos\theta)\rho(\vec{r}\,')d\tau'$$

$n=0 \Rightarrow$ monopole term

$n=1 \Rightarrow$ dipole term

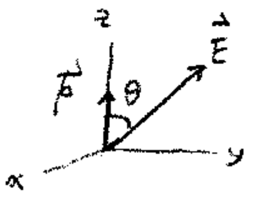
$\vdots$

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{\underbrace{\frac{1}{r}\int\rho(\vec{r}\,')d\tau'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\int r'\cos\theta\,\rho(\vec{r}\,')d\tau'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\int r'^2\frac{3\cos^2\theta-1}{2}\rho(\vec{r}\,')d\tau'}_{\text{quadrupole}} + \right.$$



ex Sphere, $\rho(r,\theta) = \tilde{k}\frac{R}{r^2}(R-2r)\sin\theta$. Find potential far away, up to octopole.

Electric Field of a Dipole



$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

$$\vec{E} = -\vec{\nabla}V$$

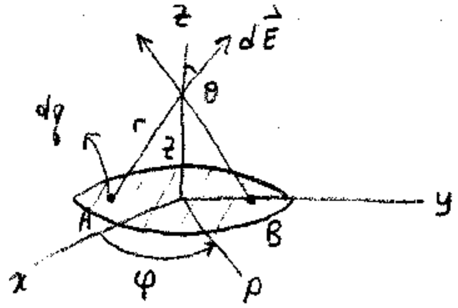
$$E_r = -\frac{\partial V}{\partial r}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}, \quad E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}$$

$$\vec{E}_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(2\cos\theta\,\hat{r}+\sin\hat{\theta})$$

Mar 17

Recit

1. Uniformly charged disk, centered at origin. $V, \vec{E} = ?$ at z.



Cylindrical polar coor.s :

$$\rho \equiv \sqrt{x^2+y^2}$$

$$x \equiv \rho\cos\varphi$$

$$y \equiv \rho\sin\varphi$$

we expect $\vec{E} = \vec{E}(z)\hat{z}$.

Justify: Consider A and B which are also generic points. For all pts A, we can find some point B s.t

$$dE_A \sin\theta + dE_B \sin\theta = 0$$

$$\int_{\substack{\text{over}\\ \text{disk}}} dE \cos\theta \Big|_z = E$$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\rho\, d\rho\, d\varphi}{\rho^2+z^2}$$

$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{z^2+\rho^2}} \Rightarrow -\sin\theta\, d\theta = \frac{-z\rho}{(z^2+\rho^2)^{3/2}}\, d\rho$$

$$E = \int_{\substack{\text{over}\\ \text{disk}}} \frac{\sigma}{4\pi\varepsilon_0}\frac{z\rho\, d\rho\, d\varphi}{(\rho^2+z^2)^{3/2}} = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R \frac{z\rho\, d\rho}{(\rho^2+z^2)^{3/2}}$$

$$= \frac{\sigma}{2\varepsilon_0}\int_{\rho=0}^{R} \sin\theta\, d\theta = \frac{\sigma}{2\varepsilon_0}\left(-\cos\theta\Big|_{\rho=0}^{R}\right)$$

$$E = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right), \quad \vec{E} = E\hat{z}$$

Observe:

$R \to \infty$ $\quad E = \frac{\sigma}{2\varepsilon_0}$, field of ∞-plate

$R \to 0$ $\quad E$: expected to be of a point charge

but

$$\sigma = \frac{Q}{\pi R^2}$$

Taylor expansion of $(z^2+R^2)^{-1/2}$

$$\frac{1}{\sqrt{z^2+R^2}} \cong \frac{1}{z}\left(1 - \frac{R^2}{2z^2}\right)$$

so

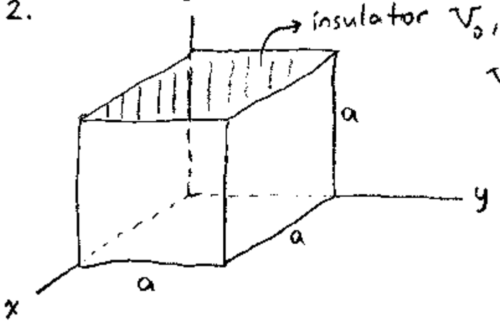
$$E \cong \frac{1}{2\varepsilon_0}\frac{Q}{\pi r^2}\left(1 - 1 + \frac{R^2}{z^2}\right) = \frac{Q}{4\pi\varepsilon_0 z^2}$$

$$V = \int_{\substack{\text{over}\\ \text{disk}}} \frac{dq}{4\pi\varepsilon_0 r} = \frac{1}{4\pi\varepsilon_0}\int_{\substack{\text{over}\\ \text{disk}}} \frac{\sigma\rho\, d\rho\, d\varphi}{\sqrt{\rho^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\sqrt{\rho^2+z^2}\Big|_0^R$$

$$= \frac{\sigma}{2\varepsilon_0}\sqrt{\rho^2+z^2}\Big|_0^R = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2} - z\right)$$

$R \gg z \quad V = \frac{\sigma}{2\varepsilon_0} R$

2.



→ insulator V_0, all other surfaces are grounded conductors

$V_{inside} = ?$

$$\vec{\nabla}^2 V = 0 \qquad V = V(x,y,z)$$

Boundaries:

(1) $z=0 \quad V=0$

(2) $x=0 \quad V=0$

(3) $y=0 \quad V=0$

(4) $x=a \quad V=0$

(5) $y=a \quad V=0$

(6) $z=a \quad V=V_0$

Propose a soln:

$V(x,y,z) = X(x)Y(y)Z(z)$, separable in cartesian coors

Laplace eq:

$$\vec{\nabla}^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)V$$

$$0 = YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2}$$

Divide both sides by V:

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\equiv -k_x^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\equiv -k_y^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\equiv k_z^2} = 0$$

X, Y: sinusoidal since they are periodic

Z: hyperbolic

$$k_z^2 = k_x^2 + k_y^2$$

$$\frac{d^2X}{dx^2} + k_x X = 0, \qquad \frac{d^2Y}{dy^2} + k_y Y = 0, \qquad \frac{d^2Z}{dz^2} - k_z Z = 0$$

$X(x) = A_x \sin k_x x + B_x \cos k_x x$, $B_x = 0$ from 2

$\sin k_x x = 0$ from 4, so $k_x a = m\pi$ or $k_x = m\pi/a$

$X(x) = A_{x,m} \sin\frac{m\pi x}{a}$, sum over m

$$X(x) = \sum_{m=1}^{\infty} A_{x,m} \sin\frac{m\pi x}{a}$$

Similarly for Y:

$$Y(y) = \sum_{n=1}^{\infty} A_{y,n} \sin\frac{n\pi y}{a}$$

For Z:

$$Z(z) = \alpha e^{k_z z} + \beta e^{-k_z z}$$

$z=0 \quad V=0 \Rightarrow \alpha+\beta=0$

$z=a \quad V=V_0$

$$\sum_{n,m}^{\infty} \alpha_{n,m}\left(e^{k_z a} - e^{-k_z a}\right)\sin(\ldots)\sin(\ldots) = V_0$$

$$k_z = k_z^{(n,m)}$$

$$\sum_{m,n=1}^{\infty} \underbrace{A_{x,m} A_{y,n} \alpha_{n,m}}_{\equiv C_{n,m}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \underbrace{\left(e^{\pi\sqrt{n^2+m^2}} - e^{-\pi\sqrt{n^2+m^2}}\right)}_{2\sinh(\pi\sqrt{n^2+m^2})} = V_0$$

Fourier trick:

$$\sum_{m,n=1}^{\infty} C_{n,m} \int_0^a \sin\frac{m\pi x}{a} \sin\frac{m'\pi x}{a}\,dx \int_0^a \sin\frac{n\pi y}{a} \sin\frac{n'\pi y}{a}\,dy \cdot 2\sinh(\pi\sqrt{n^2+m^2})$$

$$= V_0 \int_0^a \sin\frac{n'\pi y}{a}\,dy \int_0^a \sin\frac{m'\pi x}{a}\,dx$$

$$\sum_{m,n}^{\infty} C_{n,m}\, \delta_{mm'} \frac{a}{2}\, \delta_{nn'} \frac{a}{2} \cdot 2\sinh(\pi\sqrt{n^2+m^2})$$

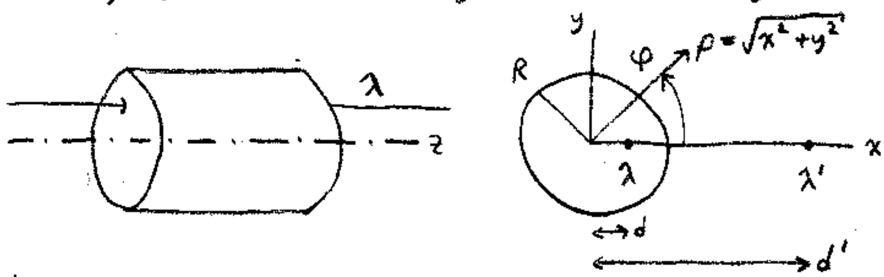
$$= V_0 \left.\frac{\cosh(n'\pi y/a)}{n'\pi/a}\right|_a^0 \left.\frac{\cos(m'\pi x/a)}{m'\pi/a}\right|_a^0$$

Answer:

$$C_{pr}\, 2\sinh(\pi\sqrt{p^2+r^2}) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a dx \int_0^a dy\, \sin\frac{p\pi x}{a} \sin\frac{r\pi y}{a}$$

$$= \begin{cases} 0 & \text{if } p \text{ or } r \text{ is even} \\ \dfrac{16 V_0}{\pi^2 p r} & \text{if both are odd} \end{cases}$$

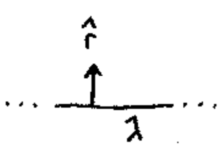
3. Conducting cylinder, unknown if grounded. Line charge λ inside.



Boundaries:

$$\rho = R \quad V = V_0 \quad \text{or, equivalently,} \quad \left.\frac{\partial V}{\partial \varphi}\right|_{\rho=R} = 0$$

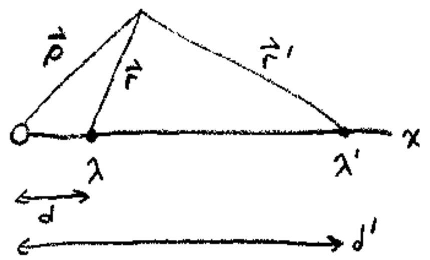
Potential of a line charge:



$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

$$U = -\int_{\rho_0}^{\rho} \vec{E}\cdot d\vec{\ell}, \quad d\vec{\ell} = dr\,\hat{r}$$

$$U = -\int_{r_0}^{r} \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{r''}\,dr'' = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{r_0}{r}$$



$$V = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{r_0}{r} + \frac{\lambda'}{2\pi\varepsilon_0} \ln\frac{r_0}{r}, \quad \left.\frac{\partial V}{\partial \varphi}\right|_{\rho=R} = 0$$

$$\vec{r} = \vec{\rho} - d\hat{x} \Rightarrow r = (\rho^2 + d^2 - 2\rho d\cos\varphi)^{1/2}$$

$$\vec{r}' = \vec{\rho} - d'\hat{x}' \Rightarrow r' = (\rho^2 + d'^2 - 2\rho d'\cos\varphi)^{1/2}$$

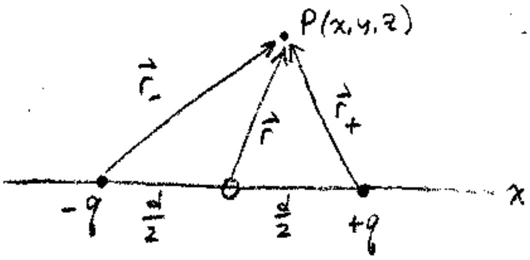
Answer: $\left.\begin{array}{l}\lambda' = -\lambda \\ d' = \dfrac{R^2}{d}\end{array}\right\} \; V = \dfrac{\lambda}{2\pi\varepsilon_0} \ln \dfrac{r'}{r}$

Recit Mar 19

3-33

Show that the electric field of a pure dipole can be written in the coordinate-free form

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{3\vec{p}\cdot\hat{r}\hat{r} - \vec{p}}{r^3}, \quad \vec{p}: \text{dipole moment}$$



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+^2}\hat{r}_+ - \frac{q}{r_-^2}\hat{r}_-\right)$$

Question really asks to find $\vec{E}$ for $r \gg d$.

$$\vec{r}_+' = \vec{r} - \frac{d}{2}\hat{x} \qquad \vec{r}_-' = \vec{r} - \frac{-d}{2}\hat{x} = \vec{r} + \frac{d}{2}\hat{x}$$

$$\frac{\hat{r}_\pm}{r_\pm^2} = \frac{\vec{r}_\pm}{r_\pm^3}$$

$$r_+^3 = \left(r^2 + \frac{d^2}{4} - 2\frac{d}{2}\hat{x}\cdot\vec{r}\right)^{3/2} = \left(r^2 + \frac{d^2}{4} - xd\right)^{3/2}$$

$$r_-^3 = \left(r^2 + \frac{d^2}{4} + xd\right)^{3/2}$$

$$r \gg d \equiv \frac{d}{r} \ll 1$$

Taylor expansion: $\varepsilon \ll 1$

$$(1+\varepsilon)^n = 1 + n\varepsilon + \ldots$$

Expand $r_\pm^3$ up to first order in d/r:

$$r_\pm^{-3} = \left(r^2 + \frac{d^2}{4} \mp xd\right)^{-3/2} = \frac{1}{r^3}\left(1 + \frac{d^2}{4r^2} \mp \frac{xd}{r^2}\right)^{-3/2} \cong$$

$r \gg d \Rightarrow x \gg d$ since $r = \sqrt{x^2+y^2+z^2}$

$$O\left(\frac{x}{r}\right) \sim 1$$

$$\cong \frac{1}{r^3}\left(1 - \frac{3}{2}\varepsilon\right) = \frac{1}{r^3}\left(1 - \frac{3}{2}\left(\frac{d^2}{4r^2} \mp \frac{xd}{r^2}\right)\right) \simeq \frac{1}{r^3}\left(1 \pm \frac{3}{2}\frac{xd}{r^2}\right)$$

$$q\left(\frac{\vec{r}_+}{r_+^3} - \frac{\vec{r}_-}{r_-^3}\right) \simeq q\left[\frac{\vec{r} - \frac{d}{2}\hat{x}}{r^3}\left(1 + \frac{3xd}{2r^2}\right) - \frac{\vec{r} + \frac{d}{2}\hat{x}}{r^3}\left(1 - \frac{3xd}{2r^2}\right)\right]$$

$$= \frac{q}{r^3}\left(\vec{r}\,\frac{3xd}{r^2} - d\hat{x}\right), \quad \vec{p} \equiv qd\hat{x} \quad -q \bullet \xrightarrow{d\hat{x}} \bullet +q$$

$$= \frac{q}{r^3}\left(\frac{\vec{r}}{r}\,\frac{3xd}{r} - d\hat{x}\right) = \frac{q}{r^3}\left(\hat{r}\,(3d\hat{x})\cdot\frac{\vec{r}}{r} - d\hat{x}\right)$$

$$= \frac{1}{r^3}\left(3\vec{p}\cdot\hat{r}\hat{r} - \vec{p}\right)$$

$$\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3\vec{p}\cdot\hat{r}\hat{r} - \vec{p}\right)$$

3-40 A thin insulating rod running from $z=-a$ to $+a$ carries the indicated line charges. In each case find the leading term in the multipole expansion of the potential.

a) $\lambda = k\cos\frac{\pi z}{2a}$, $k=$ const

General formula of multipole expansion of potential:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\underbrace{\int_{\substack{\text{over}\\ \text{source}}} d\tau'\, r'^n P_n(\cos\theta)\rho(\vec{r}')}_{\equiv I_0}$$

$\vec{r}$: point of interest

$\vec{r}'$: source point

$$\rho(\vec{r}') = \delta(x')\delta(y')\lambda(z')H(a-z')H(a+z')$$

$$I_0 = \int \underbrace{d\tau'}_{dx'dy'dz'} \cdot 1\cdot 1\cdot \delta(x')\delta(y')H(a-z')H(a+z')\,k\cos\frac{\pi z'}{2a}$$

$$= k\int_{-a}^{a} dz'\cos\frac{\pi z}{2a} = k\frac{2a}{\pi}\sin\frac{\pi z'}{2a}\Big|_{-a}^{a} = \frac{4ak}{\pi}$$

is the leading term. Otherwise, check $n=1,2,\dots$

Mar 20

$$\mathscr{r}^{-1} = |\vec{r}-\vec{r}'|^{-1} = \left[(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')\right]^{-1/2}, \quad r' \ll r$$

$$\frac{1}{\mathscr{r}} = \sum_{n=0}^{\infty}\frac{r'^n}{r^{n+1}}P_n(\cos\theta')$$

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}')}{\mathscr{r}}d\tau' = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{\substack{\text{over}\\ \text{source}}} r'^n P_n(\cos\theta')\rho(\vec{r}')d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\Big\{\frac{1}{r}\int\rho(\vec{r}')d\tau' \rightarrow \text{monopole term}$$

$$+\frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r}')d\tau' \rightarrow \text{dipole term}$$

$$+\frac{1}{r^3}\int r'^2\frac{3\cos^2\theta-1}{2}\rho(\vec{r}')d\tau' \rightarrow \text{quadrupole term}$$

$$+\dots\Big\}$$

Specific case:

$$V_{dip} = \frac{k}{r^2}\cos\theta\int r'\rho(\vec{r}')d\tau' \equiv \frac{k}{r^2}\cos\theta\cdot|\vec{p}|, \quad k\equiv\frac{1}{4\pi\varepsilon_0}$$

For point charges

$$\vec{p} = \vec{r}_i'\,q_i$$

$$V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}\cdot\vec{p}$$

ex Find leading term in potential. (3-26)

$$\rho(r,\theta) = \tilde{k}\,\frac{R}{r^2}(R-2r)\sin\theta$$

$$V_{mon} = \frac{1}{4\pi\varepsilon_0}\tilde{k}\,\frac{1}{r}\int_V \frac{R}{r'^2}(R-2r')\sin\theta'\,r'^2\sin\theta'\,dr'\,d\theta'\,d\varphi'$$

$$= \frac{\tilde{k}}{4\pi\varepsilon_0}\,\frac{R}{r}\underbrace{\int_0^R (R-2r')\,dr'}_{0}\int_0^\pi \sin^2\theta'\,d\theta' \cdot 2\pi$$

$$= 0$$

$$V_{dip} = \frac{\tilde{k}}{4\pi\varepsilon_0}\,\frac{1}{r^2}\int_V r'\cos\theta'\,\frac{R}{r'^2}(R-2r')\sin\theta'\,r'^2\sin\theta'\,dr'\,d\theta'\,d\varphi'$$

$$= \frac{\tilde{k}}{4\pi\varepsilon_0}\,\frac{1}{r^2}\,2\pi\int_0^R r' R(R-2r')\,dr'\int_0^\pi \cos\theta'\sin^2\theta'\,d\theta'$$

$$= 0$$

$$V_{quad} = \frac{\tilde{k}}{4\pi\varepsilon_0}\,\frac{1}{r^3}\int_V r'^2\,\frac{3\cos^2\theta'-1}{2}\,\frac{R}{r'^2}(R-2r')\sin\theta'\,r'^2\sin\theta'\,dr'\,d\theta'\,d\varphi'$$

$$= \frac{\tilde{k}}{4\pi\varepsilon_0}\,\frac{1}{r^3}\,R\,2\pi\int_0^R r'^2(R-2r')\,dr'\int_0^\pi (3\cos^2\theta'-1)\sin^2\theta'\,d\theta'\,\frac{1}{2}$$

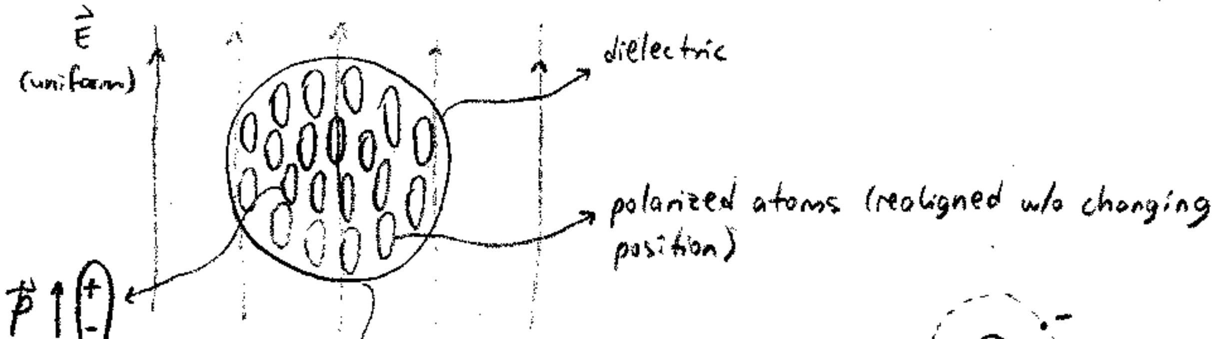
$$= \frac{\tilde{k}}{4\pi\varepsilon_0}\,\pi^2\,\frac{R^5}{48r^3}$$

$\sigma_0 = \tilde{k}\cos\theta$ (shell)

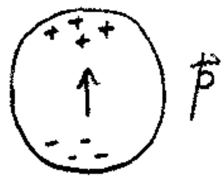
$\vec{p}$: dipole moment

Dipole per unit volume:

$\vec{P} = \vec{p}/V$, polarization of the object (dielectric)



seems like a dipole from far away



polarization : dipole moment per unit volume

$\vec{P} = \vec{p}/V$

$\vec{P} = \alpha\vec{E}$, α : atomic polarizibility, linear polarization

For some materials,

$$\vec{P} = \alpha_{\parallel}\vec{E}_{\parallel} + \alpha_{\perp}\vec{E}_{\perp}$$

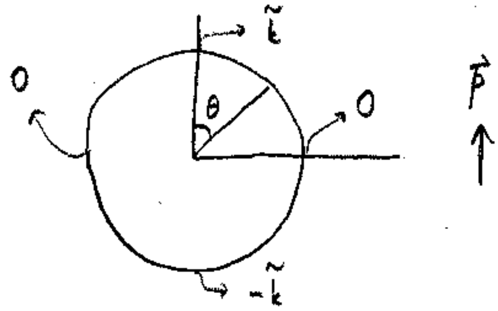
Thus, in general,

$$P_i = \alpha_{ij} E_j, \quad j=1,2,3 \text{ for each } i=1,2,3 \text{ (cartesian)}$$

Therefore we may need 9 α_{ij}'s totally.

Back to $\sigma_0 = \tilde{k}\cos\theta$

$$V = \begin{cases} \frac{\tilde{k}}{3\varepsilon_0} r\cos\theta, & r>R \\ \frac{\tilde{k}}{3\varepsilon_0}\frac{R^2}{r^2}\cos\theta, & r<R \end{cases}$$



behaves like dipole from far away

$$V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

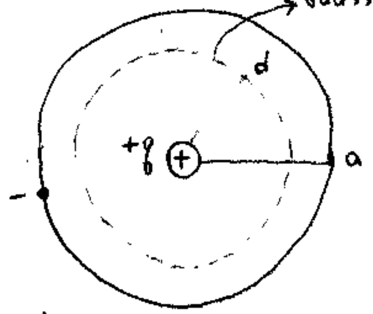
(If asked to find V far away, you may use

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2} \;)$$

From potential above,

$$p = \frac{\tilde{k}}{3\varepsilon_0}R^3 \quad \text{thus} \quad P = \frac{p}{V}$$

Atomic polarizibility:



$\vec{p} = \alpha\vec{E}$

$V = \frac{4\pi a^3}{3}$

$\rho = q \Big/ \frac{4\pi a^3}{3}$, very rough approximation

E-field inside atom:

$$\int \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} q_{enc}$$

$$q_{enc} = q\,\frac{4\pi d^3}{3}\Big/\frac{4\pi a^3}{3} = q\,\frac{d^3}{a^3}$$

$$E\, 4\pi d^2 = q\,\frac{d^3}{a^3}\frac{1}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0} q\frac{d}{a^3}, \quad qd \equiv p$$

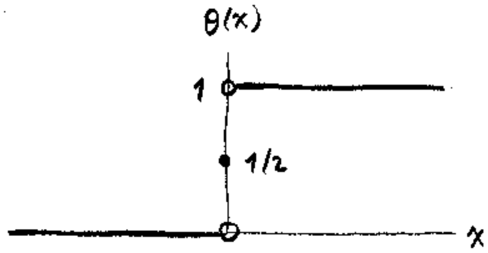
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\vec{p} \quad \text{so } \alpha = 4\pi\varepsilon_0 a^3, \text{ approximately}$$

<u>Mar 21</u>

(class missed)

Step fn:

$$\theta(x) \equiv \begin{cases} 1, & x>0 \\ 1/2, & x=0 \\ 0, & x<0 \end{cases}$$ → up to convention; $x=0$ can be added to any of the other intervals



e.g.

$$\int_{-\infty}^{\infty} f(x)\,\theta(x-a) = \int_a^{\infty} f(x)\,dx$$

Prop.s:

$$\theta(-x) = \begin{cases} 1, & x<0 \\ 0, & x>0 \end{cases}$$

$$\frac{d}{dx}\theta(x) \equiv \delta(x) \quad : \quad \int_{x_1}^{x_2} f(x)\,\delta(x-a)\,dx \equiv \begin{cases} f(a) & \text{if } x_1<a<x_2 \\ 0 & \text{otherwise} \end{cases}$$

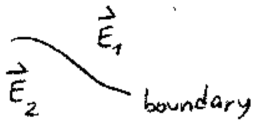
$\theta(x)$ is a piecewise cont fn.

ex Consider a charge distribution.

$$\rho(\vec r) = \begin{cases} kr, & 0 \le r < R \\ 0, & r \geq R \end{cases}$$

$$\tilde\rho(\vec r) \equiv \rho(\vec r)\,\theta(r-R)$$

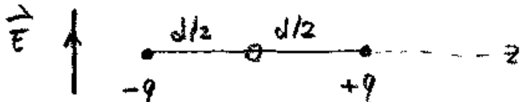
ex



$\vec E$-field may have jump at boundary.

Problem 4-7 Show that energy of an ideal dipole $\vec p$ in an electric field $\vec E$ is $U = -\vec p \cdot \vec E$.

Pure dipole in a uniform external field, $\vec E$:



$$\vec F_{total} = q\vec E + (-q)\vec E = 0$$

What if $\vec E$ is not uniform:

$$\vec F_{on\ q} = q\vec E(x_q, y_q, z_q)$$

$$\vec F_{on\ -q} = -q\vec E(x_{-q}, y_{-q}, z_{-q})$$

$$\vec F_{on\ dipole} = (\vec p \cdot \vec\nabla)\vec E$$

$$U(\vec r_2) - U(\vec r_1) = -\int_{\vec r_1}^{\vec r_2} \vec F \cdot d\vec\ell$$

$$\vec F = -\vec\nabla U$$

$$\int_C \vec F \cdot d\vec\ell = 0 \quad \text{for} \quad \vec F \perp d\vec\ell$$

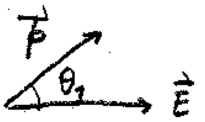
Torque:

$$\vec{N} = \vec{p} \times \vec{E} \quad \text{acting on } \vec{p}.$$

We can make use of $\vec{N}$ to define U.

$$\cos\theta \equiv \frac{\vec{p}\cdot\vec{E}}{|\vec{p}||\vec{E}|}$$

Assume that $\vec{p}$ is initially aligned s.t



Consider an external agent which applies a torque $-\vec{N}$ on $\vec{p}$ to align it s.t

$$W_{ext} = \int_{\theta_1}^{\theta_2} -N d\theta = -\int_{\theta_1}^{\theta_2} |\vec{p}||\vec{E}|\sin\theta d\theta$$

$$= |\vec{p}||\vec{E}|\cos\theta\Big|_{\theta_1}^{\theta_2}$$

$$\equiv -[U(\theta_2) - U(\theta_1)]$$

$$\vec{p}\cdot\vec{E}\Big|_{\theta_2} - \vec{p}\cdot\vec{E}\Big|_{\theta_1} = -U(\theta_2) + U(\theta_1)$$

Therefore,

$$U(\theta) = -\vec{p}\cdot\vec{E} + U_0$$

up to a const.

Notice that $\vec{p}\cdot\vec{E} = 0$ if $\theta = \pi/2$. Then assign $U(\pi/2) = 0 \Rightarrow U_0 = 0$.

Note: We did not proceed with $\int \vec{F}\cdot d\vec{l}$ since this integral gives zero for at least one curve. So, go on with "orientation" of the dipole.

Problem 4-10 Sphere, R, $\vec{P}(\vec{r}) = k\vec{r}$. Find bound charges and field inside sphere.

Mar 27

Linear Dielectrics

For a linear dielectric, the polarization is proportional to E-field.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

χ_e: electric susceptibility

ε_0: permittivity of free space

Recall:

$$\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho_f + \rho_b, \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho_f - \vec{\nabla}\cdot\vec{P}$$

$$\vec{\nabla}\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_f, \quad \varepsilon_0\vec{E} + \vec{P} \equiv \vec{D}$$

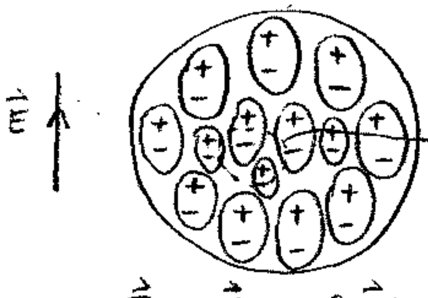
$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

Express $\vec{P}$ in terms of $\vec{D}$ for a linear dielectric, then

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\frac{\varepsilon_0\chi_e}{\varepsilon}\vec{D} = -\frac{\chi_e}{1+\chi_e}\rho_f$$

so, for a linear dielectric

$$\rho_f = 0 \Rightarrow \rho_b = 0$$



inside dielectric, not on the surface, +'s and −'s cancel out, leaving $\rho_b = 0$

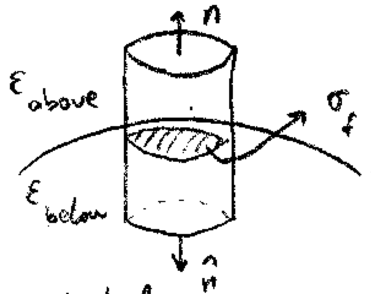
$$\vec{\nabla}\cdot(\varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E}) = \rho_f$$

$$\vec{\nabla}\cdot\varepsilon_0(1+\chi_e)\vec{E} = \rho_f$$

$$\vec{D} = \varepsilon_0(1+\chi_e)\vec{E}, \quad \varepsilon_0(1+\chi_e) \equiv \varepsilon, \text{ permittivity of material}$$

relative permittivity, $\varepsilon_r \equiv \dfrac{\varepsilon}{\varepsilon_0} = 1+\chi_e$

Continuity of displacement vector, $\vec{D}$:



$$\oint \vec{D}\cdot d\vec{a} = \int \sigma_f\, da$$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

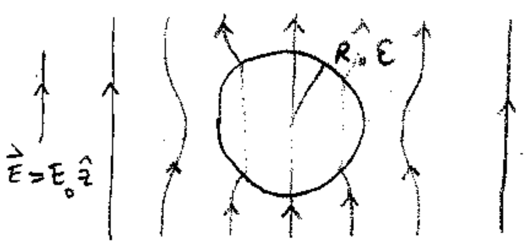
$$\varepsilon_{above} E^{\perp}_{above} - \varepsilon_{below} \vec{E}_{below} = \sigma_f$$

Consider potentials:

$$V_{above} = V_{below}$$

$$\varepsilon_{above}\frac{\partial V_{above}}{\partial n} - \varepsilon_{below}\frac{\partial V_{below}}{\partial n} = -\sigma_f$$

ex 4.7



Find potential inside and outside.
Find σ_b on the surface.

Here,

$\rho_b = 0$ is expected since $\rho_f = 0$

$\sigma_b \neq 0$ is expected due to distorted $\vec{E}_{ext}$

$$\vec{\nabla}^2 V = 0$$

Soln in spherical coor.s:

$$V_{in}(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

$$V_{out}(r,\theta) = ?$$

$$\vec{E} = E_0\hat{z} \Rightarrow V(r,\theta) = -E_0 r\cos\theta, \quad r \gg R$$

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

Boundaries:

(1) $V_{in}(R,\theta) = V_{out}(R,\theta)$

(2) $\varepsilon_{above} \frac{\partial V_{above}}{\partial n} - \varepsilon_{below} \frac{\partial V_{below}}{\partial n} = -\sigma_f = 0$

$$\varepsilon \frac{\partial V_{in}}{\partial r} = \varepsilon_0 \frac{\partial V_{out}}{\partial r} \quad @ \; r = R$$

(3) $V = -E_0 r\cos\theta, \quad r \gg R$

(1):

$$\sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = -E_0 R\cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{R^{n+1}} P_n(\cos\theta)$$

$$\sum_{n=0}^{\infty} A_n R^n \int_0^{\pi} P_n P_{n'} \sin\theta d\theta = -E_0 R \int_0^{\pi} P_1 P_{n'} \sin\theta d\theta + \sum_{n=0}^{\infty} \frac{B_n}{R^{n+1}} \int_0^{\pi} P_n P_{n'} \sin\theta d\theta$$

$n = 1$: $\quad A_1 R = -E_0 R + \frac{B_1}{R^2}$ .... (a)

$n \neq 1$: $\quad A_n R^n = B_n \frac{1}{R^{n+1}}$ .... (b)

(2):

$$\varepsilon_r \sum_{n=0}^{\infty} A_n n R^{n-1} P_n(\cos\theta) = -E_0 P_1(\cos\theta) + \sum_{n=0}^{\infty} B_n[-(n+1)] \frac{1}{R^{n+2}} P_n(\cos\theta), \quad \varepsilon_r \equiv \frac{\varepsilon}{\varepsilon_0}$$

$$\varepsilon_r \sum_{n=0}^{\infty} A_n n R^{n-1} \underbrace{\int_0^{\pi} P_n P_{n'} \sin\theta d\theta}_{\delta_{nn'}(\cancel{..})}$$

$$= -E_0 \underbrace{\int_0^{\pi} P_1 P_{n'} \sin\theta d\theta}_{\delta_{1n'}(\cancel{.})} - \sum_{n=0}^{\infty} B_n (n+1) \frac{1}{R^{n+2}} \underbrace{\int_0^{\pi} P_n P_{n'} \sin\theta d\theta}_{\delta_{nn'}(\cancel{.})}$$

$n = 1$: $\quad \varepsilon_r A_1 = -E_0 - B_1 \, 2 \cdot \frac{1}{R^3}$ .... (c)

$n \neq 1$: $\quad \varepsilon_r A_n n R^{n-1} = -B_n (n+1) \frac{1}{R^{n+1}}$ ... (d)

From (a)-(d):

$$A_n = B_n = 0, \quad n \neq 1$$

$$A_1 = -\frac{3}{\varepsilon_r + 2} E_0$$

$$B_1 = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_0$$

$$V_{in}(r,\theta) = A_1 r P_1(\cos\theta) = -\frac{3}{\varepsilon_r + 2} E_0 \underbrace{r\cos\theta}_{z}$$

E-field inside:

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\frac{\partial V}{\partial z}\hat{z} = \frac{3}{\varepsilon_r + 2} E_0 z \hat{z}$$

Outside:

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_0 \frac{1}{r^2}\cos\theta$$

Bound charges: $\sigma_b = ?$

$$\sigma_b = \vec{P}\cdot\hat{n}\Big|_s \,,\quad \vec{P} \text{ from } V_{out} = V_{ext} + \underbrace{V_{material}}_{\frac{\varepsilon_r-1}{\varepsilon_r+2}R^3 \underbrace{\frac{\cos\theta}{r^2}}_{\sim\text{potential of dipole}} E_0}$$

Recall:

$$V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

$$\vec{P} = \frac{\vec{p}}{V} \quad \text{or} \quad \vec{p} = \vec{P}V$$

By comparison,

$$\frac{\varepsilon_r-1}{\varepsilon_r+2}R^3E_0 = \frac{1}{4\pi\varepsilon_0}p$$

$$p = 4\pi\varepsilon_0\frac{\varepsilon_r-1}{\varepsilon_r+2}R^3E_0$$

$$P = \frac{p}{V} = \frac{1}{\frac{4\pi R^3}{3}}4\pi\varepsilon_0\frac{\varepsilon_r-1}{\varepsilon_r+2}R^3E_0 = 3\varepsilon_0\frac{\varepsilon_r-1}{\varepsilon_r+2}E_0$$

$$\vec{P} = P\hat{z} = P(\hat{r}\cos\theta - \hat{\theta}\sin\theta) \Rightarrow \sigma = \vec{P}\cdot\hat{n}\Big|_s = \vec{P}\cdot\hat{r}\Big|_R = 3\varepsilon_0\frac{\varepsilon_r-1}{\varepsilon_r+2}E_0\cos\theta$$

(MT1 @ 1740, energy of dielectric excluded)

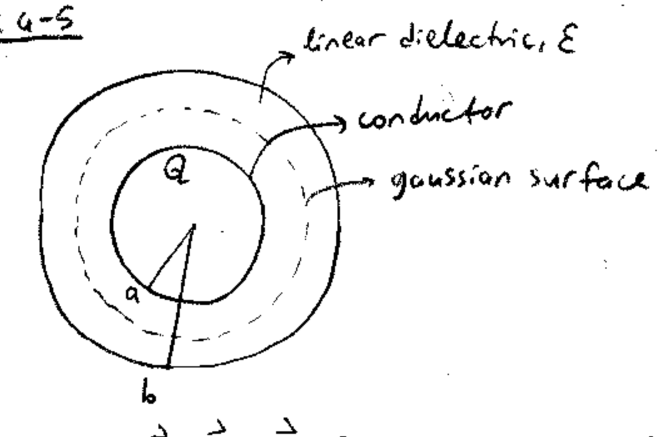
Mar 28

Linear Dielectrics

$$\vec{P} = \varepsilon_0\chi_e\vec{E}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E} = \varepsilon_0(1+\chi_e)\vec{E} \equiv \varepsilon\vec{E}$$

ex 4-5



Find $\vec{E}$ everywhere.
Potential at center
Bound charges.

$\vec{E} = \vec{P} = \vec{D} = 0$, $r < a$, since, inside, there is no polarization

$\oint_S \vec{D}\cdot d\vec{a} = Q_f$, revised Gauss' law

$$D\,4\pi r^2 = Q \qquad \vec{D} = \frac{Q}{4\pi r^2}\hat{r} \quad \text{so} \quad \vec{E} = \frac{1}{\varepsilon}\vec{D}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2}\hat{r},\ a<r<b$$

$$\oint\vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q$$

$E\,4\pi r^2 = \frac{1}{\varepsilon_0}Q \qquad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$, $r>b$, no polarization outside

$$V = -\int_\infty^0 \vec{E}\cdot d\vec{l} = -\int_\infty^b \vec{E}\cdot d\vec{r} - \int_b^a \vec{E}\cdot d\vec{r} - \int_a^0 \vec{E}\cdot d\vec{r}$$

$$= -\int_\infty^b \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,dr - \int_b^a \frac{1}{4\pi\varepsilon}\frac{Q}{r^2}\,dr - \int_a^0 \cancel{0\,dr}^{\,0}$$

$$= -\frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0}\int_\infty^b \frac{1}{r^2}\,dr + \frac{1}{\varepsilon}\int_b^a \frac{1}{r^2}\,dr\right)$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\,\frac{-1}{b} + \frac{1}{\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

$$= \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon}\left(\frac{1}{b} - \frac{1}{a}\right)\right]$$

Bound charges: σ_b, ρ_b.

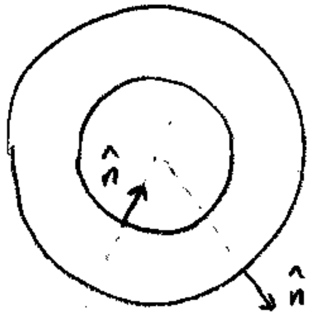
$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\varepsilon_0\chi_e\vec{E} = -\vec{\nabla}\cdot\frac{Q}{4\pi\varepsilon}\frac{1}{r^2}\hat{r}\,\varepsilon_0\chi_e$$

$$= -\frac{Q}{4\pi\varepsilon}\varepsilon_0\chi_e\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right)$$

$$= 0$$

For σ_b: Normal directions are always outwards.

$$\sigma_b = \vec{P}\cdot\hat{n}\Big|_s = \frac{Q}{4\pi\varepsilon}\varepsilon_0\chi_e\frac{\hat{r}}{r^2}\cdot\hat{n}\Big|_s$$



inner: $\hat{n} = -\hat{r}$

outer: $\hat{n} = \hat{r}$

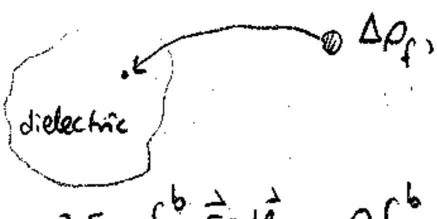
$$\sigma_{b,in} = \frac{-Q}{4\pi\varepsilon}\varepsilon_0\chi_e\frac{1}{r^2}\Big|_{r=a} = -\frac{Q}{4\pi\varepsilon}\varepsilon_0\chi_e\frac{1}{a^2}$$

$$\sigma_{b,out} = \frac{Q}{4\pi\varepsilon}\varepsilon_0\chi_e\frac{1}{r^2}\Big|_{r=b} = \frac{Q}{4\pi\varepsilon}\varepsilon_0\chi_e\frac{1}{b^2}$$

Energy

$$W = \frac{1}{2}\int_{\substack{\text{over}\\\text{source}}} \rho_f V\,d\tau = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\\text{space}}} |\vec{E}|^2\,d\tau \qquad \text{w/o dielectrics}$$

With dielectrics, there comes polarization.



$\Delta\rho_f$, incremental free charge introduced to material (Δ acts as derivative)

$$W = \int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l} = Q\left[V(b) - V(a)\right]$$

$$\Delta W = \int \Delta\rho_f V\,d\tau, \quad \Delta\rho_f = \vec{\nabla}\cdot\Delta\vec{D}, \quad \vec{\nabla}\cdot(\Delta\vec{D}V) = \vec{\nabla}\cdot(\Delta\vec{D})V + \Delta\vec{D}\cdot\vec{\nabla}V$$

$$= \int_V \vec{\nabla}\cdot(\Delta\vec{D}V)\,d\tau - \int_V \Delta\vec{D}\cdot\vec{\nabla}V\,d\tau$$

$$= \underbrace{\oint_S \Delta\vec{D}\, V \cdot d\vec{a}}_{\to 0 \text{ as } S\to\infty} + \int_V \Delta\vec{D}\cdot\vec{E}\, d\tau$$

Now consider a linear dielectric material: $\vec{D} = \varepsilon\vec{E}$

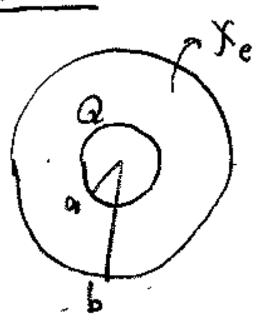
$$\Delta\vec{D}\cdot\vec{E} = \varepsilon\,\Delta\vec{E}\cdot\vec{E} = \frac{1}{2}\varepsilon\,\Delta(\vec{E}\cdot\vec{E}) = \frac{1}{2}\varepsilon\left[(\Delta\vec{E}\cdot\vec{E}) + (\vec{E}\cdot\Delta\vec{E})\right]$$

$$= \frac{1}{2}\Delta(\vec{D}\cdot\vec{E})$$

$$\Delta W = \int_V \frac{1}{2}\,\Delta(\vec{D}\cdot\vec{E})\,d\tau = \Delta\left(\frac{1}{2}\int_V \vec{D}\cdot\vec{E}\,d\tau\right)$$

$$W = \frac{1}{2}\int_{\substack{\text{all}\\\text{space}}} \vec{D}\cdot\vec{E}\,d\tau$$

<u>Problem 4.8</u>



Total energy of the system?

$$W = \frac{1}{2}\int_{\substack{\text{all}\\\text{space}}} \vec{D}\cdot\vec{E}\,d\tau$$

$$\oint \vec{D}\cdot d\vec{a} = Q_f$$

Outside:

$$D\,4\pi r^2 = Q \qquad \vec{D} = \frac{Q}{4\pi}\frac{\hat{r}}{r^2},\ r>b$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2},\ r>b$$

B/w:

$$D\,4\pi r^2 = Q \qquad \vec{D} = \frac{Q}{4\pi}\frac{\hat{r}}{r^2},\ b>r>a$$

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0(1+\chi_e)\vec{E}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon}\frac{\hat{r}}{r^2},\ a<r<b$$

Inside:

$$\vec{D} = 0$$

$$\vec{E} = 0$$

$$W = \frac{1}{2}\left(\cancelto{0}{\int_0^a} + \int_a^b + \int_b^\infty\right)$$

$$= \frac{1}{2}\left(\int_{r=a}^b \frac{Q}{4\pi}\frac{1}{r^2}\frac{Q}{4\pi\varepsilon}\frac{1}{r^2}r^2\,dr\,d\Omega + \int_{r=b}^\infty \frac{Q}{4\pi}\frac{1}{r^2}\frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}r^2\,dr\,d\Omega\right)$$

$$= \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left(\frac{1}{\varepsilon}\int_a^b\frac{1}{r^2}dr + \frac{1}{\varepsilon_0}\int_b^\infty\frac{1}{r^2}dr\right)$$

$$= \frac{Q^2}{8\pi}\left[-\frac{1}{\varepsilon}\left(\frac{1}{b}-\frac{1}{a}\right) - \frac{1}{\varepsilon_0}\left(0-\frac{1}{b}\right)\right] = \frac{Q^2}{8\pi}\left[\frac{1}{\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{\varepsilon_0 b}\right],\ \varepsilon \equiv \varepsilon_0(1+\chi_e)$$

Study 4-27 (for energy), 4-2, 5, 6, 7, 10, 18, 28 and 3-33 and ex 3-9 (for MT1).

Past exam question: $\vec{P} = kr^3\hat{r}$, spherical shell, σ_b, ρ_b, $\vec{E}$ everywhere? ($kr^3\hat{r} \Rightarrow$

$\rho_b \neq 0 \Rightarrow$ not linear dielectric)

Apr 2

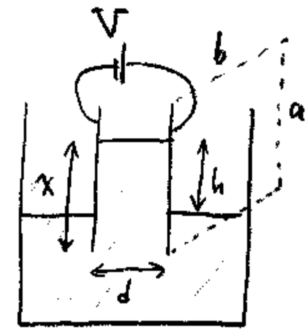
Recit

ex A capacitor consisting of two plane parallel plates separated by a distance d is immersed vertically in a dielectric fluid of dielectric const K and density ρ. Calculate the height to reach the fluid raises b/w plates

a) When the capacitor is connected to a battery that maintains a const voltage V across the plates.

b) When the capacitor carries a charge, but is not connected to a battery.

a)



$F_{gravity} = F_{electrostatic}$

↳ force on dielectric

$|F_e dx| = |\frac{1}{2} C V^2|$

$$\frac{dW}{dx} = F_e = \frac{V^2}{2}\frac{dC}{dx}$$

$$C = \frac{b}{4\pi d}\left[(K-1)x + a\right]$$

$$\frac{dC}{dx} = \frac{b(K-1)}{4\pi d} \Rightarrow F_e = \frac{V^2 b(K-1)}{8\pi d}$$

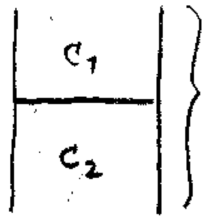
$$E = \frac{\sigma}{\varepsilon_0} = \frac{V}{d}$$

$$\sigma = \frac{Q}{ab} \Rightarrow \frac{Q}{ab\varepsilon_0} = \frac{V}{d},$$ take $[\varepsilon_0] = 1$ in this unit system

$$C = \frac{Q}{V} \qquad \frac{Q}{V} = \frac{a_1 b_1 \varepsilon_0}{d} = C_1$$ (w/o dielectric in b/w)

$$C_1 = \frac{a_1 b_1}{d} \Rightarrow C_2 = \frac{K a_2 b_2}{d},$$ with dielectric

$a_1 = a - x \qquad a_2 = x$



total capacitance

$C_1 + C_2$

Recall

const $V \Rightarrow$ parallel connection

$C = \Sigma C$

const $Q \Rightarrow$ series connection

$\frac{1}{C} = \Sigma \frac{1}{C}$

With proper units,

$$C_1 = \frac{(a-x)b}{4\pi d}, \qquad C_2 = \frac{Kxb}{4\pi d}$$

$$\left.\begin{array}{l} F_e = \dfrac{V^2 b(K-1)}{8\pi d} \\ F_g = abh\rho g \end{array}\right\}$$ find h.

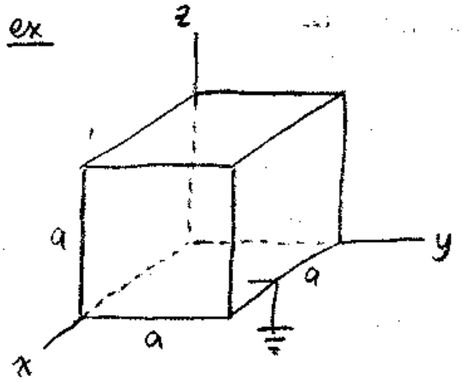
b) $$|F_e| = \frac{dW}{dx} = \left|\frac{d}{dx}\frac{1}{2}\frac{Q^2}{C}\right| = \left|-\frac{Q^2}{2C^2}\frac{dC}{dx}\right|$$

With same C_1 and C_2,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$|F_e| = \frac{2\pi d (K-1) Q^2}{b[(K-1)x+a]} \Rightarrow h = \frac{2\pi Q^2 (K-1)}{\rho g b^2 [(K-1)x+a]^2}$$

ex



$V_{inside} = ?$

$\vec{\nabla}^2 V = 0, \quad V(x,y,z) = X(x)Y(y)Z(z)$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = X''YZ + XY''Z + XYZ'' = 0$$

$$\underbrace{\frac{X''}{X}}_{-k_x^2} + \underbrace{\frac{Y''}{Y}}_{-k_y^2} + \underbrace{\frac{Z''}{Z}}_{k_x^2+k_y^2} = 0$$

$$X(x) = A\sin k_x x + B\cos k_x x$$

$$Y(y) = C\sin k_y y + D\cos k_y y$$

$$Z(z) = Ee^{\sqrt{k_x^2+k_y^2}\,z} + Fe^{-\sqrt{k_x^2+k_y^2}\,z}$$

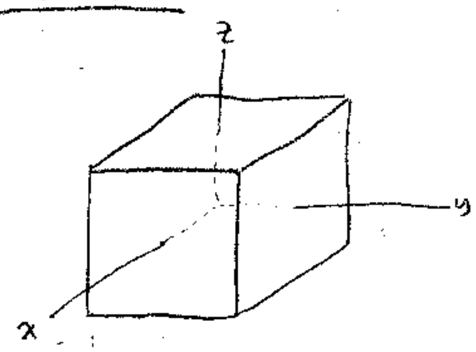
$$V(x,y,z) = \sum C \sin k_x x \sin k_y y \sinh\sqrt{k_x^2+k_y^2}\,z$$

But z is not a special direction to choose sinh for it; x and y can have similar forms. In this case, take linear combination of the three solns.

Griffiths 3-22

Apr 3

Problem 3-41



A cube made of a dielectric material with sides each of length a has a frozen-in polarization $\vec{P} = k\vec{r}$. Find σ_b, ρ_b and Q_T.

$$\vec{P} = k\vec{r} = k(x\hat{x} + y\hat{y} + z\hat{z})$$

$$\sigma_b = \vec{P}\cdot\hat{n}$$

Surfaces in

$\hat{x}$ direction: $\sigma_b = kx\big|_{x=a/2} = ka/2$

$\hat{y}$ direction: $\sigma_b = ky\big|_{y=a/2} = ka/2$

$\hat{z}$ direction: $\sigma_b = kz\big|_{z=a/2} = ka/2$

$-\hat{x}$ direction: $\sigma_b = -kx\big|_{x=-a/2} = ka/2$

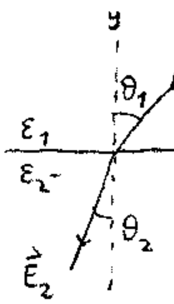
$-\hat{y}$ direction: $\sigma_b = -ky\big|_{y=-a/2} = ka/2$

$-\hat{z}$ direction: $\sigma_b = -kz\big|_{z=-a/2} = ka/2$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -k\cdot 3 = -3k$$

$$Q_T = \sum \int \sigma_b \, da + \int \rho_b \, d\tau = 6 \frac{ka}{2} a^2 + (-3k) a^3 = 0$$

Problem 4-33 (Recall $\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}$ for linear dielectrics)



no surface charge within boundary

Show that

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_2}{\varepsilon_1}$$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f = 0$$

$$D^{\perp}_{above} = D^{\perp}_{below}$$

$$\varepsilon_1 E^{\perp}_{above} = \varepsilon_2 E^{\perp}_{below}$$

$$\varepsilon_1 E_{1y} = \varepsilon_2 E_{2y} \quad \text{------ (1)}$$

Parallel components are cont:

$$E_{1x} = E_{2x} \qquad (2)$$

$$\tan\theta_2 = \frac{E_{2x}}{E_{2y}} \qquad \tan\theta_1 = \frac{E_{1x}}{E_{1y}}$$

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{E_{2x}/E_{2y}}{E_{1x}/E_{1y}} = \frac{E_{1x}/(\varepsilon_1/\varepsilon_2) E_{1y}}{E_{1x}/E_{1y}} = \frac{\varepsilon_2}{\varepsilon_1}$$

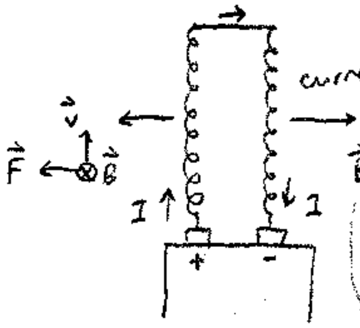
For small θ's:

$$\frac{\theta_2}{\theta_1} \simeq \frac{\varepsilon_2}{\varepsilon_1}$$

Observe:

$$\varepsilon_2 > \varepsilon_1 \Rightarrow \theta_2 > \theta_1 \sim \text{ divergent lens}$$

CH5 MAGNETOSTATICS

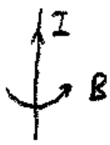


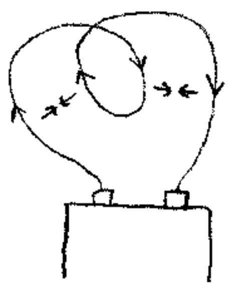
currents repel each other

$\vec{B} \otimes \longrightarrow \vec{F} \to$ Lorentz force

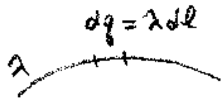
$\vec{v} \to$ velocity of current on the right

due to current on the left



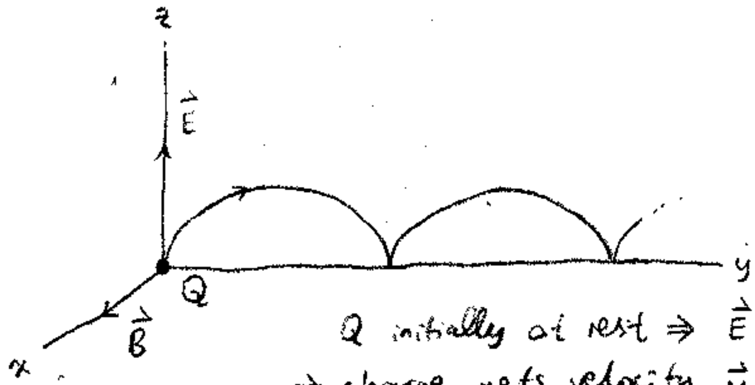


Now they attract each other



$I = \frac{dq}{dt} = \lambda \frac{dl}{dt} = \lambda v$ in Amper, $A \equiv \frac{\text{coulomb, C}}{\text{second, s}}$

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ Lorentz Force



Q initially at rest $\Rightarrow$ $\vec{E}$ affects Q first
$\Rightarrow$ charge gets velocity $\vec{v}\uparrow$ (in z-direction)
$\Rightarrow$ force in y direction $\Rightarrow$ motion in y direction

$x(0) = y(0) = z(0) = 0$

$\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$

$\vec{B} = B\hat{x} \qquad \vec{E} = E\hat{z}$

$\vec{v}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t)) = (0, \dot{y}, \dot{z})$

$\vec{F} = m\vec{a}$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = (0, B\dot{z}, -B\dot{y}) = B\dot{z}\hat{y} - B\dot{y}\hat{z}$$

$\vec{F} = Q(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}) = m(\ddot{y}\hat{y} + \ddot{z}\hat{z})$

$w \equiv \frac{QB}{m}$

$y: \; QB\dot{z} = m\ddot{y} \quad$ so $\quad QB\ddot{z} = m\dddot{y}$

$z: \; QE - QB\dot{y} = m\ddot{z}$

$\ddot{z} = \frac{QE}{m} - \frac{QB}{m}\dot{y}$

$QB\left(\frac{QE}{m} - \frac{QB}{m}\dot{y}\right) = m\dddot{y}, \quad Q = \frac{mw}{B}$

$w^2\frac{E}{B} - w\dot{y} = \dddot{y} \quad$ or $\quad \dddot{y} + w^2\dot{y} = \frac{w^2 E}{B}$

$r^3 + w^2 r = 0$

$r(r^2 + w^2) = 0$

$r = \{0, \pm iw\}$

$y(t) = C_1 \cos wt + C_2 \sin wt + C_3 + \frac{E}{B}t$

$z(t) = C_4 \cos wt + C_5 \sin wt + C_6$

Using initial conditions,

$C_4 = C_1, \quad C_5 = -C_2$

Then the soln:

$$y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t)$$

$$z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$$

$$R \equiv \frac{E}{\omega B}$$

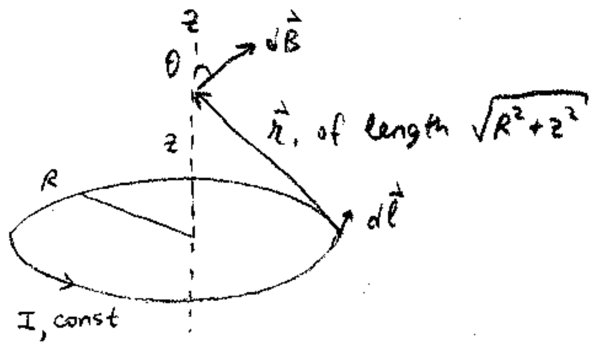
$(y - R\omega t)^2 + (z - R)^2 = R^2$ $\Rightarrow$ center of circle is at $(y, z) = (R\omega t, R)$
$\Rightarrow$ cycloid, moving with speed $R\omega$

$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\vec{F} = \int \vec{v} \times \vec{B}\, \lambda\, dl, \qquad \lambda\vec{v} = \vec{I}$$

$$\vec{F} = \vec{I} \times \int \vec{B}\, dl$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dl \qquad \text{Biot - Savart Law}$$



$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, $\vec{r} = \vec{z}$, $\vec{r}\,' = R\hat{r}$ in cylindrical coor.s

$$\vec{\mathscr{r}} = \vec{z} - R\hat{r}$$

$$\vec{B} = \frac{\mu_0}{4\pi} I \int d\vec{l} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}, \quad d\vec{l} \perp \hat{\mathscr{r}}$$

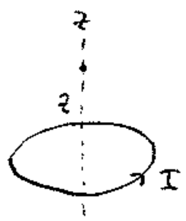
$\vec{B} = \vec{B}(z) = B(z)\hat{z}$ due to symmetry

$$\cos\theta = \frac{R}{\sqrt{R^2+z^2}}$$

$$|\vec{B}| = \frac{\mu_0 I}{4\pi}\frac{2\pi R^2}{(R^2+z^2)^{3/2}}$$

Apr 4

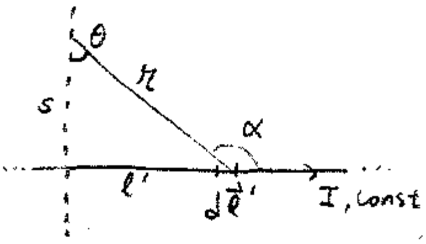
The magnetic field $\vec{B}$ of a current of a line segment:



$\vec{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{z}$, field of a current loop

$\vec{B}$ at the center, $\vec{B}(z=0) = \frac{\mu_0 I}{2R}\hat{z}$

(Recall Biot - Savart Law: $\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$, I const.)



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{r}}{r^2}, \quad r^2 = l'^2 + s^2$$

$$|d\vec{l}' \times \hat{r}| = dl' \sin\alpha$$

$$\sin\left(\theta + \frac{\pi}{2}\right) = \sin\alpha$$

$$\cos\theta = \sin\alpha$$

$$\tan\theta = l'/s$$

$$s\tan\theta = l'$$

$$s\sec^2\theta\, d\theta = dl'$$

$$|d\vec{l}' \times \hat{r}| = dl' \cos\theta$$

$$r^2 = l'^2 + s^2 = s^2\tan^2\theta + s^2 = s^2\sec^2\theta$$

$$|d\vec{l}' \times \hat{r}| = dl'\cos\theta = s\sec^2\theta\cos\theta\, d\theta$$

$$|\vec{B}| = \frac{\mu_0 I}{4\pi}\int \frac{s\sec^2\theta\cos\theta\, d\theta}{s^2\sec^2\theta} = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

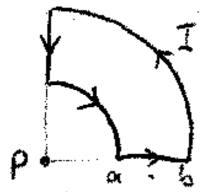
For ∞-long line: $\theta_1 = -\pi/2$, $\theta_2 = \pi/2$, thus

$$|\vec{B}| = \frac{\mu_0 I}{4\pi s}\left(\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right) = \frac{\mu_0 I}{4\pi s}\cdot 2$$

$$|\vec{B}| = \frac{\mu_0 I}{2\pi s}$$

<u>Problem 5-9</u> $\vec{B}$-field at P.

a)

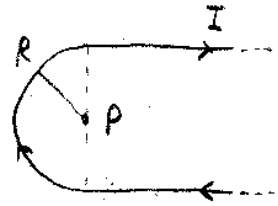


Linear portions contribute nothing.

$$B = \frac{\mu_0 I}{2a}\frac{1}{4}\otimes + \frac{\mu_0 I}{2b}\frac{1}{4}\odot$$

$$B = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)\otimes$$

b)



From line segments, we have $\otimes$. From the circular segment, also $\otimes$.

$$B = \underbrace{\frac{\mu_0 I}{2R}\frac{1}{2}\otimes}_{\text{half circle}} + \underbrace{\frac{\mu_0 I}{2\pi R}\frac{1}{2}\otimes + \frac{\mu_0 I}{2\pi R}\frac{1}{2}\otimes}_{\text{semi-}\infty}$$

Lorentz force:

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

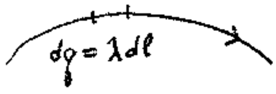
Let $\vec{E} = 0$.

$$F = ma = \frac{mv^2}{R}$$

$$QvB = m\frac{v^2}{R}, \quad v = \omega R$$

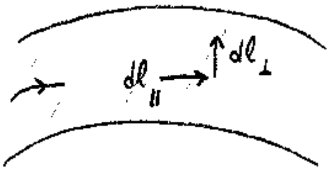
$$QB = m\omega$$

$$\omega = \frac{QB}{m}, \text{ cyclotrone frequency}$$



$$I = \frac{dq}{dt} = \lambda v$$

For surface currents:



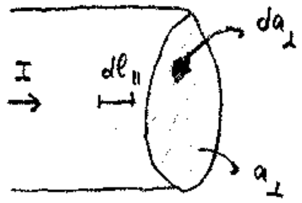
$$dq = \sigma da = \sigma\, dl_{\parallel}\, dl_{\perp}$$

$$\frac{dq}{dt} = \sigma v\, dl_{\perp} = dI$$

$$\vec{K} \equiv \frac{d\vec{I}}{dl_{\perp}}, \text{ current density}$$

$$\vec{K} = \sigma\vec{v}\,\frac{dl_{\perp}}{dl_{\perp}} = \sigma\vec{v}, \text{ surface current density}$$

Vol current density:



$$d\tau = da_{\perp}\, dl_{\parallel}$$

$$dq = \rho\, d\tau = \rho\, da_{\perp}\, dl_{\parallel}$$

$$\frac{dq}{dt} = \rho\, da_{\perp}\, v$$

$$\vec{J} \equiv \frac{d\vec{I}}{da_{\perp}} = \rho\vec{v}, \text{ vol current density}$$

$\vec{B}$ in terms of these densities:

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Apr 9

(No problems solved in recit)

Apr 10

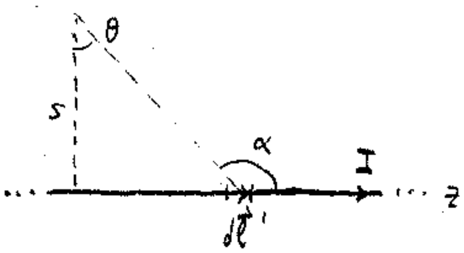
(MT2: at class hour, includes chapters [5, ...])

Magnetic field $\vec{B}$:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\,' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \text{Biot - Savart Law}$$

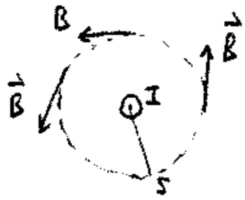
$$I = \int \vec{J} \cdot d\vec{a}$$

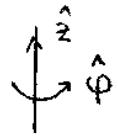
$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau', \quad \vec{J} = \vec{J}(\vec{r}\,')$$



$$\vec{B} = \frac{\mu_0 I}{2\pi}\,\frac{\hat{\varphi}}{s}$$

front view





Consider $\oint \vec{B}\cdot d\vec{l}$.

$$d\vec{l} = ds\,\hat{s} + s\,d\varphi\,\hat{\varphi} + dz\,\hat{z}$$

$$\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s}\hat{\varphi}\cdot(ds\,\hat{s} + s\,d\varphi\,\hat{\varphi} + dz\,\hat{z})$$

$$= \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\varphi = \mu_0 I$$

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \qquad \text{Ampère's Law}}$$

(Gaussian surface vs. Amperian loop)

So far, we have considered static magnetic field $\vec{B}$.

$$\vec{\nabla}\cdot\vec{B} = ? \qquad \vec{\nabla}\times\vec{B} = ?$$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\vec{\nabla}\times\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{J}\,\underbrace{\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}_{4\pi\delta^3(\vec{\mathscr{r}})} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\overset{\to 0}{\cancel{(\vec{J}\cdot\vec{\nabla})}}$$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\,4\pi\delta^3(\vec{r}-\vec{r}\,')\,d\tau' = \mu_0\vec{J}(\vec{r})$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\vec{\nabla}\cdot\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\underbrace{\cancel{\vec{\nabla}\times\vec{J}}}_{0 \text{ since } \vec{J}=\vec{J}(\vec{r}\,')} - \vec{J}\cdot\overset{\to 0}{\cancel{\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}} = 0$$

$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ both in static and dynamic case

$\vec{\nabla}\cdot\vec{B}$ means that there is a fn $\vec{A}$: $\vec{B} = \vec{\nabla}\times\vec{A}$ since $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$$

$$-\vec{\nabla}^2\vec{A} + \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{A}}_{0}) = \mu_0\vec{J}$$

One can let $\vec{\nabla}\cdot\vec{A} = 0$, which is called a coulomb gauge. A gauge means that

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \;:\; \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A}$$

(or $\vec{B} = \vec{\nabla}\times\vec{A}$ satisfies $\vec{\nabla}\cdot\vec{B} = 0 \;\wedge\; \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$)

$-\vec{\nabla}^2\vec{A} = \mu_0\vec{J}$ (~poisson eq: $\vec{\nabla}^2 V = -\rho/\varepsilon_0$ with soln $V = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho\,d\tau'}{\mathscr{r}}$)

$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\,d\tau'}{\mathscr{r}}$, called magnetic vector potential

Now we can find $\vec{A}'$: $\vec{\nabla}\cdot\vec{A}' = 0$.

$$\vec{\nabla}\cdot\vec{A}' = 0 = \vec{\nabla}\cdot\vec{A}_0 + \vec{\nabla}^2\lambda, \quad \vec{\nabla}\cdot\vec{A}_0 \neq 0$$

$$\vec{\nabla}\cdot\vec{A}_0 = -\vec{\nabla}^2\lambda$$

$$\lambda = \int \frac{\vec{\nabla}\cdot\vec{A}_0\, d\tau'}{r}$$

Potential far away from source:

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint\frac{d\vec{l}'}{r}, \quad \text{using} \quad \int\vec{J}\cdot d\vec{a} = I \quad \text{and} \quad da\,dl = d\tau$$

$$\frac{1}{r} = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}r'^n P_n(\cos\theta)$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint r'^n P_n(\cos\theta)\,d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi}\left\{\frac{1}{r}\underbrace{\oint d\vec{l}'}_{0} + \frac{1}{r^2}\oint r'\cos\theta\, d\vec{l}' + \frac{1}{r^3}\int r'^2\,\frac{3\cos^2\theta - 1}{2}\,d\vec{l}' + \dots\right\}$$

monopole term is 0 (no single magnetic charge, or monopole)

$$\vec{\nabla}\cdot\vec{B} = 0$$

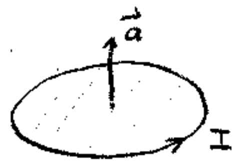
Dipole term:

$$\int r'\cos\theta\, d\vec{l}' = \int \vec{r}'\cdot\hat{r}\, d\vec{l} = -\int \hat{r}\times d\vec{a}'$$

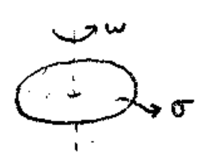
$$\vec{A}_{dip} = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int d\vec{a}\times\hat{r} \sim V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{where } \vec{p} \equiv \int \vec{r}'\rho(\vec{r}')\,d\tau$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \quad \text{where } \vec{m} \equiv I\int d\vec{a}, \text{ magnetic dipole moment}$$

(Note: I goes under integral when not const).

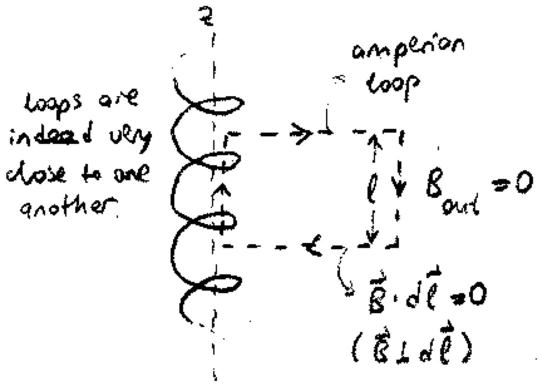


idea



rotation creates nonuniform current

<u>ex</u> (Solonoid) Magnetic field?



since we can take amperian loop very large so $B \to 0$ (or, consecutive loops cancel out each other in terms of B-field)

Ampère law:

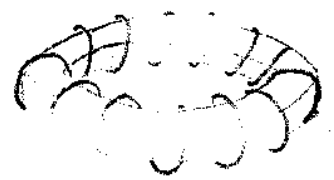
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

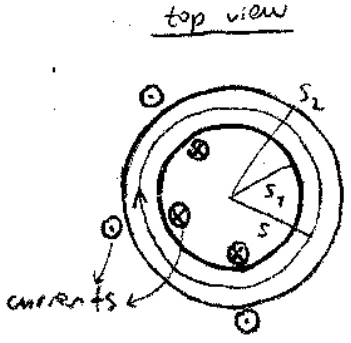
$$\int \vec{B}_{in}\cdot d\vec{l} = \mu_0 (nl) I$$

$$Bl = \mu_0 n l I$$

$$\vec{B} = \mu_0 n I\,\hat{z}$$

ex (Cylindrical toroid)





$$\int \vec{B}\cdot d\vec{l} = \mu_0 n I$$

$$B 2\pi s = \mu_0 n I$$

$$\vec{B} = \frac{\mu_0 n I}{2\pi}\frac{\hat{\varphi}}{s}, \quad s_1 < s < s_2$$

For $s > s_2$ (outside), take amperian loop $s > s_2$. Then opposite currents cancel out, so $B_{out} = 0$. Similarly for $s < s_1$ (inside), $B_{in} = 0$ since $I_{enc} = 0$.

Magnetic Flux

$$\Phi_B \equiv \int_S \vec{B}\cdot d\vec{a} = \int_S \vec{\nabla}\times\vec{A}\cdot d\vec{a} = \oint_{\partial S} \vec{A}\cdot d\vec{l}$$

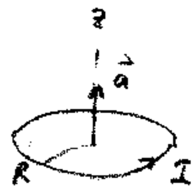
Apr 11

Recall

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau', \qquad \vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}}{\mathscr{r}}\,d\tau'$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}, \quad \vec{m} \equiv \int dI(\xi)\,\vec{a}(\xi), \quad \vec{a}: \text{area}$$



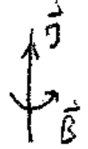
$\vec{m} = I\pi R^2\hat{z}$

Similarly to $V, \vec{E}, \rho$ - triangle:

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\,d\tau'}{\mathscr{r}} \qquad \vec{B} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\vec{A} \xrightarrow{\ } \vec{J}, \quad -\nabla^2\vec{A} = \mu_0\vec{J}, \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}, \quad \vec{J} \to \vec{B}$$

$$\vec{A} \xrightarrow{\vec{\nabla}\times\vec{A} = \vec{B}} \vec{B}$$



(problems 5-51,

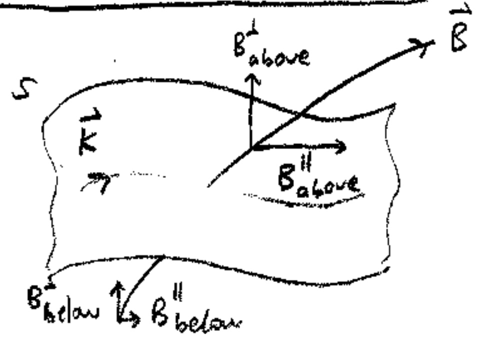
we can find magnitude of $\vec{A}$:

$$\int_S \vec{B}\cdot d\vec{a} = \int_S \vec{\nabla}\times\vec{A}\cdot d\vec{a} = \oint_{\partial S}\vec{A}\cdot d\vec{l}$$

$$\downarrow$$

$$\Phi_B = \oint_S \vec{B}\cdot d\vec{a} = 0 \quad \text{since} \quad \vec{\nabla}\cdot\vec{B} = 0$$

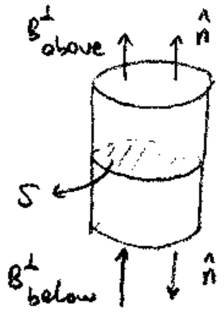
Continuity of Magnetic Field



For E-field, we have considered $\oint\vec{E}\cdot d\vec{l} = 0$ and $\oint\vec{E}\cdot d\vec{a} = \sigma/\varepsilon_0$ (Gauss law)

$(\vec{K} = \sigma\vec{v}, \quad \vec{J} = \rho\vec{v})$

Take a pillbox on the surface:

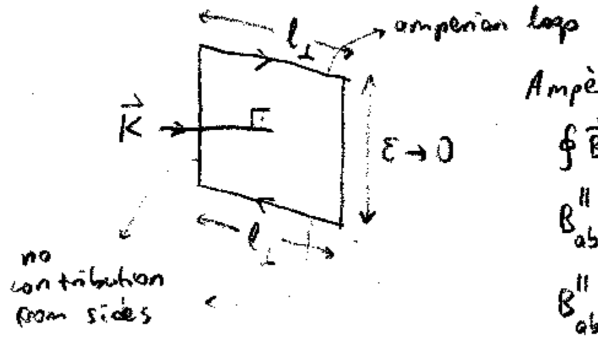


$$\oint \vec{B}\cdot d\vec{a} = 0$$

$$B^{\perp}_{above}A - B^{\perp}_{below}A = 0$$

$$B^{\perp}_{above} = B^{\perp}_{below}$$

Take a loop perpendicular to surface and to $\vec{K}$:



Ampère's law:

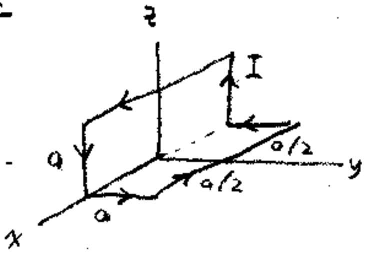
$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 \int K\, d\ell_{\perp}$$

$$B^{\parallel}_{above}\,\ell_{\perp} - B^{\parallel}_{below}\,\ell_{\perp} = \mu_0 K \ell_{\perp}$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$

$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$	$B^{\perp}_{above} = B^{\perp}_{below}$
$E^{\parallel}_{above} = E^{\parallel}_{below}$	$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$

ex



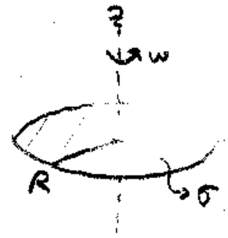
$$\vec{m} = ?$$

$$\vec{m} = \vec{m}_{xz\text{-plane}} + \vec{m}_{xy\text{-plane}}$$

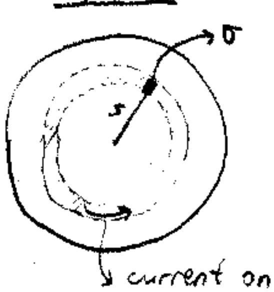
$$= Ia^2\hat{y} + Ia^2\hat{z}$$

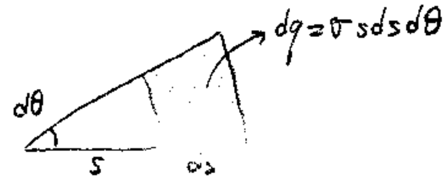
$$= Ia^2(\hat{y}+\hat{z})$$

ex Disk, R, σ; $m = ?$



top view





$$\Delta I = \frac{dq}{dt} = \frac{d}{dt}\left(\sigma s\,ds\,d\theta\right)$$

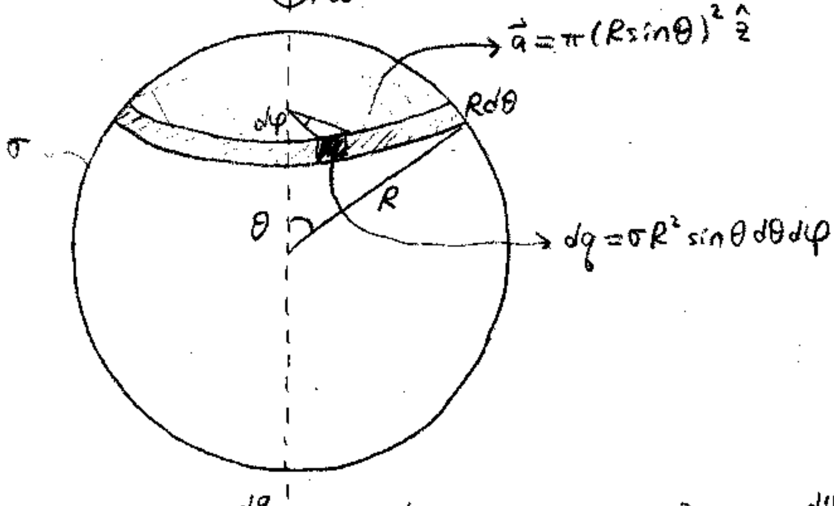
$$= \sigma s\,ds\,\frac{d\theta}{dt} = \sigma w s\,ds$$

$$\vec{m} = \int dI(s)\vec{a}(s) = \int_0^R \sigma w s\,ds\,\underbrace{\pi s^2\hat{z}}_{\vec{a}} = \sigma w \pi \int_0^R s^3 ds\ \hat{z}$$

$$\vec{m} = \frac{\sigma w \pi R^4}{4}\hat{z}$$

Problem 5-36

$\vec{m} = ?$



$$\Delta I = \frac{dq}{dt} = \sigma R^2 \frac{d}{dt}(\sin\theta d\theta d\varphi) = \sigma R^2 \sin\theta d\theta \frac{d\varphi}{dt} = \sigma R^2 \omega \sin\theta \, d\theta$$

$$\vec{a}(\theta) = \pi R^2 \sin^2\theta \, \hat{z}$$

$$\vec{m} = \int dI \, \vec{a} = \sigma R^4 \omega \pi \int_0^\pi \sin^3\theta d\theta \, \hat{z} = \frac{4}{3} \sigma R^4 \pi \, \hat{z}$$

$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r} d\tau'$ or, in this case, $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}}{r} da'$, exact potential

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{1}{r^2} \vec{m} \times \hat{r} = \frac{\mu_0}{4\pi} \frac{1}{r^2} \frac{4\pi R^4}{3} \sigma \omega \hat{z} \times \hat{r}, \quad \hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\hat{r} \times \hat{r} = 0, \quad -\hat{\theta} \times \hat{r} = \hat{\varphi}$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^2} \frac{4\pi R^4}{3} \sigma \omega \sin\theta \, \hat{\varphi}$$

$$= \frac{\mu_0 R^4 \sigma \omega}{3} \frac{\sin\theta}{r^2} \hat{\varphi}$$

$$V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2}$$

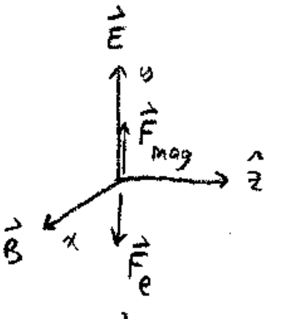
$$A_{dip} \sim \frac{\sin\theta}{r^2} \qquad V_{dip} \sim \frac{\cos\theta}{r^2}$$

Apr 16

Problem 5-3

a) Pass the beam thru uniform $\vec{E}$ and $\vec{B}$ fields ($\vec{E} \perp \vec{B} \perp$ beam). Adjust $\vec{E}$ until zero deflection. What is the speed of particles in terms of $\vec{E}$ and $\vec{B}$?
b) Turn off $\vec{E}$. Measure radius of curvature. What is q/m in E, B, and R?

a)



no deflection $\Rightarrow F_e = F_{mag}$

$$qE = qvB$$

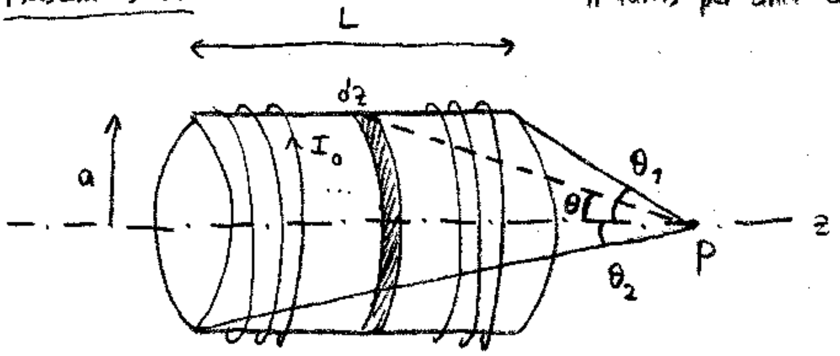
$$v = \frac{E}{B}$$

b)

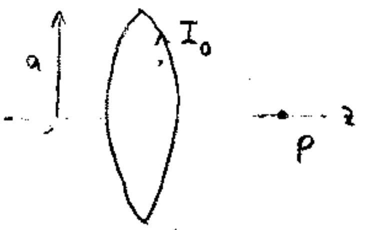
$$\frac{mv^2}{r} = qvB$$

$$\frac{q}{m} = \frac{v}{RB} = \frac{E}{RB^2}$$

Problem 5-11 $\qquad$ n turns per unit length. B-field at P?



$dI = I_0 n dz$



$$\vec{B}(P) = \frac{\mu_0 I_0}{2} \frac{a^2}{(a^2+z^2)^{3/2}}$$

$$dB = \frac{\mu_0}{2} dI \frac{a^2}{(a^2+z^2)^{3/2}} = \frac{\mu_0}{2} n I_0 dz \frac{a^2}{(a^2+z^2)^{3/2}}$$

$$\frac{a}{\sqrt{a^2+z^2}} = \sin\theta$$

$$\frac{-za}{(a^2+z^2)^{3/2}} dz = \cos\theta \, d\theta$$

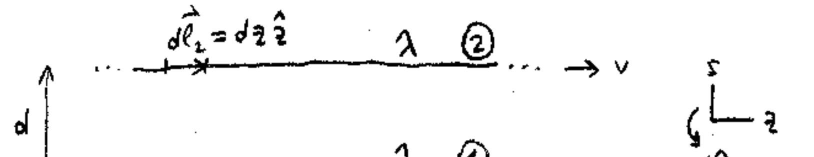
$$\frac{-a}{a^2+z^2} dz = -\frac{\sin^2\theta}{a} dz = d\theta \qquad / \times \frac{-a}{\sqrt{a^2+z^2}}$$

$$\frac{a^2}{(a^2+z^2)^{3/2}} dz = -\frac{1}{a}\sin\theta \, d\theta$$

$$\int_{①}^{②} dB = \int_{①}^{②} \frac{\mu_0 I_0 n \, dz}{2} \frac{a^2}{(a^2+z^2)^{3/2}} = \frac{\mu_0 I_0}{2} \int_{-\theta_2}^{\theta_1} (-\sin\theta) d\theta$$

$$B = \frac{\mu_0 I_0 n}{2} (\cos\theta_1 - \cos\theta_2), \quad \vec{B} = -|B| \hat{z}$$

Problem 5-12 $\quad \lambda$: charge per unit length



currents in same direction $\Rightarrow$ attraction

B of a long wire:

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi s} \hat{\varphi}$$

Force on a current I_2 in external field $\vec{B}_1$:

$$\vec{F}_{21} = \int I_2 d\vec{l}_2 \times \vec{B}_1 = \int_{-\infty}^{\infty} I_2 dz \, \hat{z} \times B_1 \hat{\varphi}, \qquad \hat{z} \times \hat{\varphi} = -\hat{s}$$

$$= -I_2 \frac{\mu_0 I_1}{2\pi s} \int_{-\infty}^{\infty} dz \, \hat{s} = \infty$$

Force per unit length:

$$|f| = \frac{\mu_0 I_1 I_2}{2\pi s}\Bigg|_{s=d} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

I_1, I_2 in terms of λ and v:

$$I_1 = \lambda v \qquad I_2 = \lambda v$$

Electrostatic force per unit length: $\vec{E}$ of a long wire

$$\vec{E}_1 = \frac{\lambda_1}{2\pi\varepsilon_0 s}\,\hat{s}$$

Force per unit length on λ_2 by λ_1:

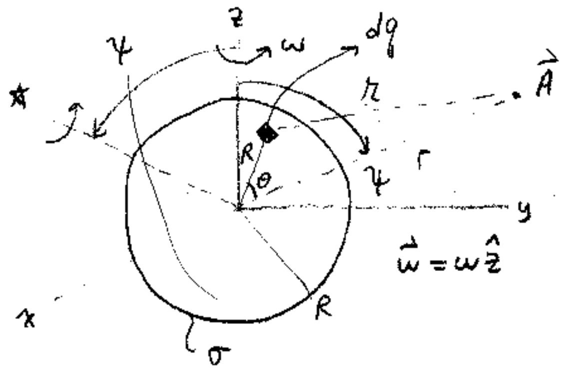
$$f_{21} = \frac{\lambda_1\lambda_2}{2\pi\varepsilon_0 s}\Bigg|_{s=d} = \frac{\lambda_1\lambda_2}{2\pi\varepsilon_0 d}$$

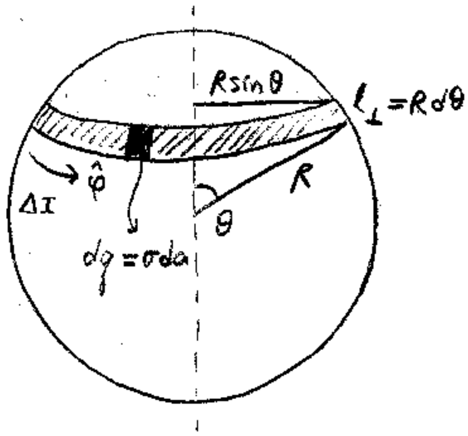
$$\frac{\mu_0\lambda_1\lambda_2 v^2}{2\pi d} = \frac{\lambda_1\lambda_2}{2\pi\varepsilon_0 d}$$

$$v^2 = \frac{1}{\mu_0\varepsilon_0} \equiv c^2 \rightarrow$$ to create electrostatic force balancing magnetic force

Apr 17

Ex 5-11 Shell, R, σ. Rotates with ω. Find $\vec{A}$ and $\vec{B}$.





Infinitesimal current on the strip:

$$\Delta I = \frac{dq}{dt}$$

$$|\vec{K}| = \frac{\Delta I}{\Delta s_\perp}$$

(★) In order to use spherical coors, we shall look for $\vec{A}$ on the z-axis, thus shift ψ on the xz-plane so $\vec{\omega}$ is around this new axis.

$$dq = \sigma R\sin\theta\,d\varphi\cdot R\,d\theta = \sigma R^2\sin\theta\,d\theta\,d\varphi$$

$$\Delta I = \frac{dq}{dt} = \frac{d}{dt}\left(\sigma R^2\sin\theta\,d\theta\,d\varphi\right) = \sigma R^2\sin\theta\,d\theta\,\frac{d\varphi}{dt} \equiv \sigma R^2\omega\sin\theta\,d\theta$$

$$K = \frac{\Delta I}{\Delta s_\perp} = \frac{\sigma R^2\omega\sin\theta\,d\theta}{R\,d\theta} = \sigma R\omega\sin\theta$$

$$\vec{K} = \sigma\underbrace{\vec{\omega}\times\vec{r}}_{\vec{v}},$$ coor. system-indep.

$\vec{\omega}$ after shifted:

$$\vec{\omega} = \omega\cos\psi\,\hat{z} + \omega\sin\psi\,\hat{x}$$

$\vec{r}$ is on the surface now:

$$\vec{r} = \vec{R} = R\sin\theta'\cos\varphi'\hat{x} + R\sin\theta'\sin\varphi'\hat{y} + R\cos\theta'\hat{z}$$

$$\vec{K} = \sigma\vec{w}\times\vec{R}$$

$$\vec{w}\times\vec{R} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ w\sin\psi & 0 & w\cos\psi \\ R\sin\theta'\cos\varphi' & R\sin\theta'\sin\varphi' & R\cos\theta' \end{vmatrix}$$

$$= Rw\left[\hat{x}(-\cos\psi\sin\theta'\sin\varphi') - \hat{y}(\sin\psi\cos\theta' - \cos\psi\sin\theta'\cos\varphi') + \hat{z}(\sin\psi\sin\theta'\sin\varphi')\right]$$

Put $\vec{w}\times\vec{R}$ into integral; there are terms

$$\int_0^{2\pi}\sin\varphi'\,d\varphi' = \int_0^{2\pi}\cos\varphi'\,d\varphi' = 0, \text{ where } d\varphi' \text{ comes from } da'$$

The only expression which survives in $\vec{w}\times\vec{R}$ is $-Rw\sin\psi\cos\theta'$:

$$\vec{A} = \frac{-\mu_0}{4\pi}\int_S Rw\sin\psi\cos\theta'\, R^2\sin\theta'\,d\theta'\,d\varphi'\,\frac{1}{\sqrt{R^2+r^2-2rR\cos\theta'}}$$

$$= -\frac{\mu_0 R^3 w\sin\psi}{4\pi}(2\pi)\int_0^{\pi}\frac{\sin\theta'\cos\theta'\,d\theta'}{\sqrt{R^2+r^2-2rR\cos\theta'}}, \quad u\equiv\cos\theta, \quad du = -\sin\theta\, d\theta$$

$$\int_{+1}^{-1}\frac{u\,du}{\sqrt{R^2+r^2-2rRu}} = -\frac{1}{3R^2r^2}\left[(R^2+r^2+2Rr)|R-r| - (R^2+r^2-2Rr)|R-r|\right]$$

$$= \begin{cases}\dfrac{2r}{3R^2}, & r<R \\ \dfrac{2R}{3r^2}, & r>R\end{cases}$$

$$\vec{A} = \begin{cases}\dfrac{\mu_0\sigma R}{3}\vec{w}\times\vec{r}, & r\le R \\ \dfrac{\mu_0\sigma R^4}{3}\dfrac{\vec{w}\times\vec{r}}{r^3}, & r>R \quad \sim \dfrac{\sin\theta}{r^2} \Rightarrow \text{dipole term (kind of)}\end{cases}$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}, \quad \vec{m}\equiv\int dI(\theta)\vec{a}(\theta), \quad \vec{a} = \pi(R\sin\theta)^2\hat{z}$$

$$\vec{A}(\vec{r}) = \begin{cases}\dfrac{\mu_0\sigma Rw}{3}r\sin\theta\hat{\varphi}, & r<R \\ \dfrac{\mu_0\sigma R^4 w}{3}\dfrac{\sin\theta}{r^2}\hat{\varphi}, & r>R\end{cases}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & 0 & r\sin\theta A_\varphi\end{vmatrix}$$

$$\vec{B} = \frac{2}{3}\mu_0\sigma Rw(\cos\theta\hat{r} - \sin\theta\hat{\theta}) = \frac{2}{3}\mu_0\sigma R\vec{w}$$

Apr 18

Recall:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}'}{\mathscr{r}}$$

$$\frac{1}{\mathscr{r}} = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}r'^n P_n(\cos\theta)$$

so.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint r'^n P_n(\cos\theta')\,d\vec{l}'$$

Expand for $n=0$ and $n=1$: For $n=0$, $\vec{A}_{mon}=0$. For $n=1$,

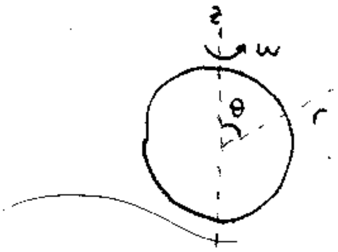
$$\vec{A}_{dip} = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\theta\, dl' = \frac{\mu_0 I}{4\pi r^2}\,\hat{r}\cdot\underbrace{\oint \vec{r}'\,dl'}_{-\hat{r}\times d\vec{a}'}$$

so

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} I\int d\vec{a}\times\hat{r} = \frac{\mu_0}{4\pi r^2}\vec{m}\times\hat{r}, \quad \vec{m}\equiv\int dI(\xi)\,\vec{a}(\xi)$$

$$\vec{A}_{dip}\sim\frac{\sin\theta}{r^2}$$

Shell, R, σ.



$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0\sigma R}{3}\,\vec{w}\times\vec{r}, & r<R \\ \dfrac{\mu_0\sigma R^4}{3}\,\dfrac{\vec{w}\times\vec{r}}{r^3}, & r>R \end{cases}$$

$$\vec{w}=w\hat{z}, \quad \hat{z}=\hat{r}\cos\theta-\hat{\theta}\sin\theta$$

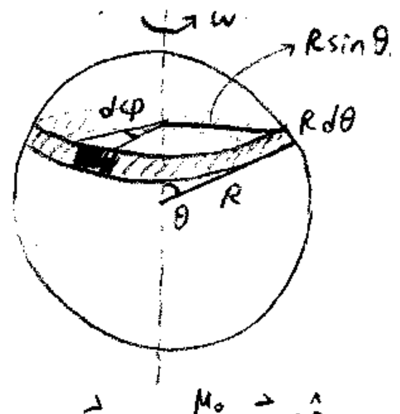
$$\vec{w}\times\vec{r} = w\hat{z}\times r\hat{r} = wr\sin\theta\,\hat{\varphi}$$

In spherical coor.s,

$$\vec{A}(r,\theta) = \begin{cases} \dfrac{\mu_0\sigma R w}{3}\, r\sin\theta\,\hat{\varphi}, & r<R \\ \dfrac{\mu_0\sigma R^4 w}{3}\,\dfrac{\sin\theta}{r^2}\,\hat{\varphi}, & r>R, \quad \sim\dfrac{\sin\theta}{r^2}, \text{ dipole term} \end{cases}$$

Find $\vec{m}$:

$$\vec{m} = \int dI(\xi)\,\vec{a}(\xi)$$



$$\Delta I = \sigma(R\sin\theta)\frac{d\varphi}{dt}(R\,d\theta) = \sigma R^2 w\sin\theta\, d\theta$$

$$\vec{a} = \pi(R\sin\theta)^2\,\hat{z} = \pi R^2\sin^2\theta\,\hat{z}$$

$$\vec{m} = \int dI(\theta)\,\vec{a}(\theta) = \int\sigma R^2 w\sin\theta\, d\theta\ \pi R^2\sin^2\theta\,\hat{z}$$

$$= \sigma R^4 w\pi\underbrace{\int_0^{\pi}\sin^3\theta\, d\theta}_{4/3}\,\hat{z} = \frac{4}{3}\pi R^4\sigma w\,\hat{z}$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

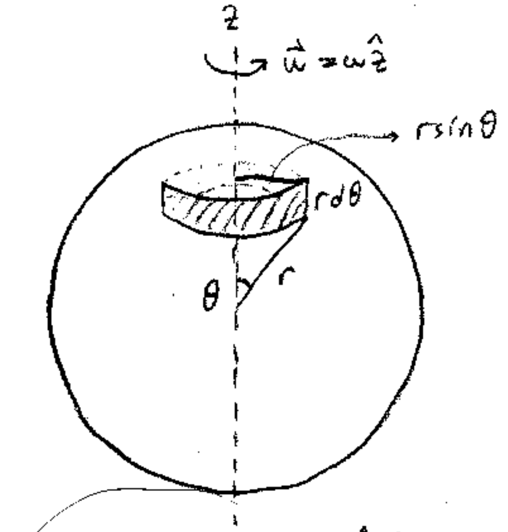
$$= \frac{\mu_0}{4\pi}\frac{1}{r^2}\frac{4\pi R^4}{3}\sigma w\,\hat{z}\times\hat{r}, \quad \hat{z}=\hat{r}\cos\theta-\hat{\theta}\sin\theta, \quad \hat{z}\times\hat{r}=\sin\theta\,\hat{\varphi}$$

$$= \frac{\mu_0\sigma R^4 w}{3}\sin\theta\,\hat{\varphi}\,\frac{1}{r^2} = \frac{\mu_0\sigma R^4 w}{3}\frac{\sin\theta}{r^2}\hat{\varphi} \overset{\text{exact}}{=} \vec{A}(\vec{r}), \ r>R$$

so there is no need to calculate for $n=2,\dots$

$$\vec{A}_{dip}(r,\theta) \overset{\text{exact}}{=} \vec{A}(\vec{r})$$

ex Solid sphere, R, ρ, ω. Find $\vec{m}$.



top view on infinitesimal element

area, $\vec{a} = \vec{a}(r,\theta) = \pi(r\sin\theta)^2\,\hat{z} = \pi r^2\sin\theta\,\hat{z}$

$\Delta I = \rho r\sin\theta \frac{d\varphi}{dt}\, r\,d\theta\,dr = \rho\omega r^2\sin\theta\,dr\,d\theta$

$$\vec{m} = \int dI(r,\theta)\,\vec{a}(r,\theta)$$
$$= \int \rho\omega r^2\sin\theta\,dr\,d\theta \cdot \pi r^2\sin^2\theta\,\hat{z}$$
$$= \rho\omega\pi \int_{\theta=0}^{\pi}\sin^3\theta\,d\theta \int_{r=0}^{R} r^4\,dr\,\hat{z}$$
$$= \rho\omega\pi\,\frac{4}{3}\,\frac{R^5}{5}\,\hat{z} = \frac{4}{15}\pi R^5\rho\omega\hat{z} = \frac{1}{5}QR^2\,\vec{\omega}$$

Potential far away:

$$\vec{A} \cong \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\,\frac{1}{r^2}\,\frac{1}{5}QR^2\underbrace{\vec{\omega}\times\hat{r}}_{\omega\sin\theta\hat{\varphi}} = \frac{\mu_0}{4\pi}\,\frac{QR^2\omega}{5}\,\frac{\sin\theta}{r^2}\,\hat{\varphi}$$

Magnetic moment per unit volume:

magnetic moment density, $\vec{M} \equiv \frac{\vec{m}}{V}$

so

$$\vec{m} = \int \vec{M}(\vec{r}\,')\,d\tau'$$

Then

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{\nabla}\frac{1}{\mathscr{r}}$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{M}\times\vec{\nabla}\frac{1}{\mathscr{r}}\,d\tau'$$

$$\vec{\nabla}\times\left(\frac{1}{\mathscr{r}}\vec{M}\right) = \frac{1}{\mathscr{r}}\vec{\nabla}\times\vec{M} - \vec{M}\times\vec{\nabla}\frac{1}{\mathscr{r}}$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\left\{-\int \vec{\nabla}\times\frac{\vec{M}}{\mathscr{r}}\,d\tau' + \int \frac{\vec{\nabla}\times\vec{M}}{\mathscr{r}}\,d\tau'\right\}$$

$$= \frac{\mu_0}{4\pi}\left\{\underbrace{\oint \frac{\vec{M}\times d\vec{a}}{\mathscr{r}}}_{\text{Eq 1.606}} + \int \frac{\vec{\nabla}\times\vec{M}}{\mathscr{r}}\,d\tau'\right\}, \quad \vec{M}\times d\vec{a}' = \vec{M}\times\hat{n}\,da'$$

For surface charges,

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}\,da'$$

For volume charges,

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{\imath}\, d\tau'$$

From comparison,

$$\boxed{\vec{M}\times\hat{n} = \vec{K}_b \qquad \vec{\nabla}\times\vec{M} = \vec{J}_b}$$

$\vec{K}_b, \vec{J}_b$: bound current densities (occurs when material is placed in an external B-field)

Paramagnets: $\vec{M}$ is antiparallel to $\vec{B}$.

Diamagnets: $\vec{M}$ is parallel to $\vec{B}$.

ex Uniformly magnetized sphere. Take $\vec{M} = M\hat{z}$. Find $\vec{B}$.

No free current in the system. (There appear small dipoles b/w e^- and their nucleus, but they are all bound.)

$$\vec{K}_b = \vec{M}\times\hat{n}\Big|_S = \vec{M}\times\hat{r}\Big|_{r=R} = M\hat{z}\times\hat{r}\Big|_{r=R} = M\sin\theta\,\hat{\varphi}$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = 0 \quad \text{since } \vec{M} = \text{const.}$$

$$\vec{K} = \sigma\omega R\sin\theta\,\hat{\varphi} \quad \text{for a shell (Ex 5-11)}$$

By comparison,

$$M = \sigma\omega R \quad \text{or} \quad \vec{M} = \sigma\vec{\omega}R$$

From Ex 5-11,

$$\vec{B} = \frac{2}{3}\mu_0\sigma R\vec{\omega}$$

so, here,

$$\vec{B} = \frac{2}{3}\mu_0\vec{M}$$

For volume charges,

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}} d\tau'$$

From comparison,

$$\boxed{\vec{M} \times \hat{n} = \vec{K}_b \qquad \vec{\nabla} \times \vec{M} = \vec{J}_b}$$

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$$\vec{K} = \sigma\omega R\sin\theta\,\hat{\varphi} \quad \text{for a shell (Ex 5-11)}$$

By comparison,

$$M = \sigma\omega R \quad \text{or} \quad \vec{M} = \sigma\vec{\omega}R$$

From Ex 5-11,

$$\vec{B} = \frac{2}{3}\mu_0\sigma R\vec{\omega}$$

so, here,

$$\vec{B} = \frac{2}{3}\mu_0\vec{M}$$

Apr 24

(no class)

Apr 25

(no class)

Apr 30

Recitation

Section 6.3.1

Pure electricity:

$$\vec{E} \rightarrow \text{free charges}$$

Material medium $\vec{E} \Rightarrow \vec{P} \Rightarrow$ bound charges

Recall Gauss' law:

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc} \quad \text{or} \quad \oint \vec{D} \cdot d\vec{a} = Q_{f,enc} \quad \text{in dielectric medium}$$

Similarly for magnetism:

$$\vec{B} \rightarrow \text{free current}$$

Material medium $\Rightarrow \vec{M}$, magnetization $\Rightarrow$ bound charges

Ampère's law in magnetized materials:

Pure electricity:

$$Q = Q_f + Q_b \quad \equiv \quad \rho = \rho_f + \rho_b$$

Q_f: can be controlled in experiments (flow of charge)

Q_b: cannot be directly controlled in experiments

$$V = -\int \vec{E} \cdot d\vec{l}$$ can be controlled as well.

Magnetism:

$$\vec{J} = \vec{J}_f + \vec{J}_b$$

"Free" stuff can be controlled: $\rho_f, \vec{J}_f$. Bound-ones cannot.

$$\left.\begin{aligned} \vec{J} &= \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} \\ \vec{J}_b &= \vec{\nabla} \times \vec{M} \end{aligned}\right\} \quad \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \vec{J}_f + \vec{\nabla} \times \vec{M}$$

so

$$\boxed{\vec{J}_f = \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right)}$$

Note that $\vec{J} = \vec{J}_f + \vec{J}_b$ comes from experiment.

$$\boxed{\vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}}$$

Ampère's law in terms of free current:

$$\left.\begin{aligned} \vec{J}_f &= \vec{\nabla} \times \vec{H} \\ I_{f,enc} &= \oint \vec{H} \cdot d\vec{l} \end{aligned}\right\} \text{ Ampère's law in a magnetized material}$$

$\therefore$ We indeed can control $\vec{E}$ and $\vec{H}$, so

$$V = -\int \vec{E} \cdot d\vec{l}$$

$$I_{f,enc} = \oint \vec{H} \cdot d\vec{l}$$

$\vec{H}$ is called, in some textbooks, "magnetic field" often.

<u>Ferromagnetism</u>

$\vec{E}, \vec{D}, \vec{B}, \vec{H}$: They are related but not always linearly.

$\vec{D} = \varepsilon \vec{E}$ linear case (or $D_i = \varepsilon_{ij} E_j$)

$\vec{H} = \vec{H}(\vec{B}) = ?$

Linear case:

$$\vec{B} \propto \vec{H}$$

(Ohm's law: $\vec{J} = \sigma \vec{E}$ linear resistance, σ: conductivity tensor)

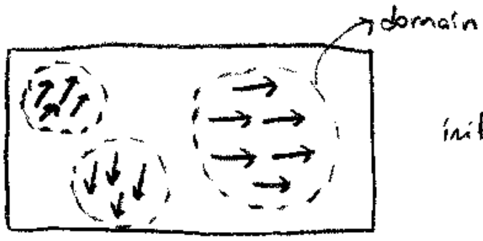
Iron:

under magnetic field, linearly magnetized.

Even when magnetic field is removed, magnetization remains.

So, ferromagnetics:

- dipoles align with $\vec{B}_{ext}$.
- magnetization is "frozen-in."
  magnetic history is relevant.

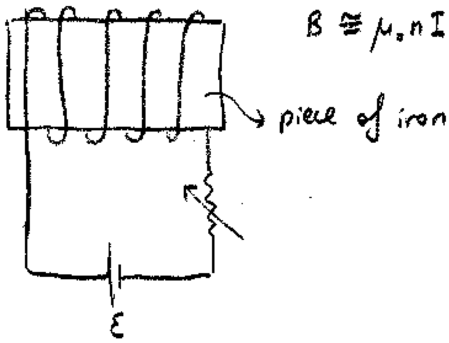


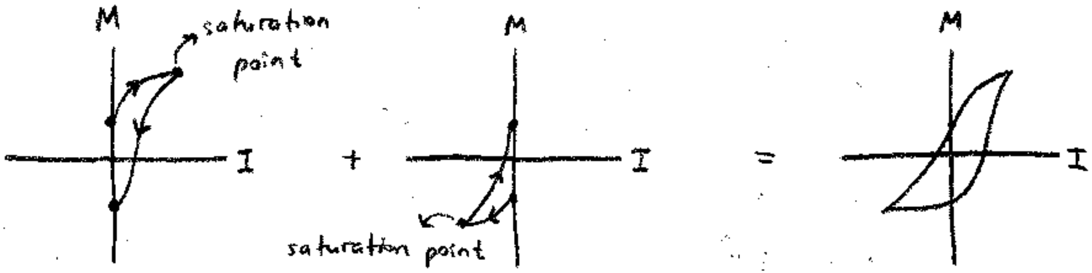
initially

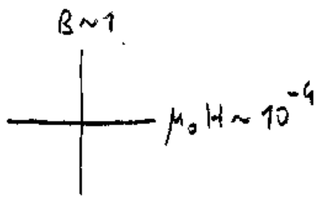
Apply an external B-field to the right ($\rightarrow$), then domains of the same magnetization expand.

Hysteresis:

$B \cong \mu_0 n I$







$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

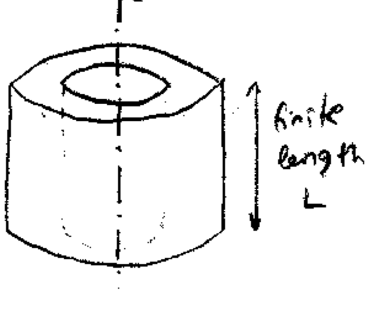
May 2

Bound currents due to magnetization:

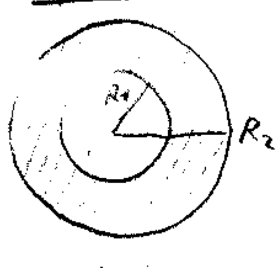
$$\vec{K}_b = \vec{M}\times\hat{n}\Big|_S$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M}$$

ex Magnetized $\vec{M} = \frac{\tilde{k}}{s^2}\hat{\varphi}$. (HW: $\vec{M} = \tilde{k}\,\hat{z}/s^2$) Inner radius R_1, outer radius R_2, shell.



top view



(observe that there is no free currents but only bound currents)

a) What is the total current?

b) What is the $\vec{B}$ field for $R_1 < s < R_2$

a) $$\vec{K}_b = \vec{M} \times \hat{n} \Big|_S$$

Inner surface:

$$\vec{K}_1 = \frac{\tilde{k}}{s^2}\,\hat{\varphi} \times -\hat{s}\Big|_S = \frac{\tilde{k}}{s^2}\,\hat{z}\Big|_{s=R_1} = \frac{\tilde{k}}{R_1^2}\,\hat{z}$$

Outer surface:

$$\vec{K}_2 = \frac{\tilde{k}}{s^2}\,\hat{\varphi} \times \hat{s}\Big|_S = \frac{-\tilde{k}}{s^2}\,\hat{z}\Big|_{s=R_2} = -\frac{\tilde{k}}{R_2^2}\,\hat{z}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$= \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\varphi} & \hat{z} \\ \partial_s & \partial_\varphi & \partial_z \\ 0 & s\frac{\tilde{k}}{s^2} & 0 \end{vmatrix} = \frac{1}{s}\left(\hat{s}\cdot 0 - s\hat{\varphi}\cdot 0 + \hat{z}\left(-\frac{\tilde{k}}{s^2}\right)\right) = -\frac{\tilde{k}}{s^3}\,\hat{z}, \quad R_1 < s < R_2$$

$$I_{total} = \underbrace{I_f}_{0} + I_b = I_{sur} + I_{vol} = \int \vec{K}_1 \cdot d\vec{s}_1 + \int \vec{K}_2 \cdot d\vec{s}_2 + \int \vec{J}_b \cdot d\vec{a}$$

$$= \int \frac{\tilde{k}}{R_1^2}\,\hat{z}\cdot d\vec{s}_1 + \int -\frac{\tilde{k}}{R_2^2}\,\hat{z}\cdot d\vec{s}_2 + \int -\frac{\tilde{k}}{s^3}\,\hat{z}\cdot \underbrace{d\vec{a}}_{s\,ds\,d\varphi\,\hat{z}}$$

($d\vec{s}$'s are $l_\perp$'s since $\vec{K}$'s are const — otherwise ds will be $dl_\perp$ — so, they come out of integral as $2\pi R_1$ and $2\pi R_2$, resp.)

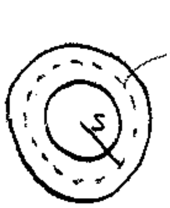
$$= \frac{\tilde{k}}{R_1^2}\,2\pi R_1 - \frac{\tilde{k}}{R_2^2}\,2\pi R_2 - \tilde{k}\int_{R_1}^{R_2} \frac{1}{s^3}\,s\,ds \int_0^{2\pi} d\varphi$$

$$= \frac{2\pi\tilde{k}}{R_1} - \frac{2\pi\tilde{k}}{R_2} + \tilde{k}\left(\frac{s^{-1}}{+1}\Big|_{R_1}^{R_2}\,2\pi\right) = 2\pi\tilde{k}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_2} - \frac{1}{R_1}\right)$$

$$= 0$$

b) Ampère's law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$



$$I_{enc} = \begin{matrix}\text{bound current} \\ \text{on inner surface}\end{matrix} + \begin{matrix}\text{vol bound current} \\ \text{from } R_1 \text{ to } s\end{matrix}$$

$$B\,2\pi s = \mu_0\left(\int \frac{\tilde{k}}{R_1^2}\,\hat{z}\cdot d\vec{s}_1 + \int -\frac{\tilde{k}}{\tilde{s}^3}\,\hat{z}\cdot d\vec{a}\right)$$

$$= \mu_0\left(\frac{\tilde{k}}{R_1^2}\,2\pi R_1 - \tilde{k}\int_0^{2\pi} d\varphi \int_{R_1}^{s} \tilde{s}\,d\tilde{s}\,\frac{1}{\tilde{s}^3}\right)$$

$$= \mu_0\left(\frac{2\pi\tilde{k}}{R_1} + \tilde{k}\,2\pi\left(+\frac{1}{\tilde{s}}\Big|_{R_1}^{s}\right)\right) = \mu_0\left(\frac{2\pi\tilde{k}}{R_1} + 2\pi\tilde{k}\left(\frac{1}{s} - \frac{1}{R_1}\right)\right)$$

$$= 2\pi\tilde{k}\mu_0\left(\frac{1}{R_1} + \frac{1}{s} - \frac{1}{R_1}\right) = \frac{2\pi\tilde{k}\mu_0}{s}$$

$$B = \frac{2\pi \tilde{k} \mu_0}{s} \frac{1}{2\pi s} = \frac{\tilde{k}\mu_0}{s^2}, \quad \vec{B} = B\hat{\varphi}$$

$$\left(\begin{array}{ll} I = \int \vec{K}_b \cdot d\vec{s}_\perp & \vec{K}_b = K_b \hat{z} \Rightarrow d\vec{s}_\perp = R\, d\varphi\, \hat{z} \\ I = \int \vec{J}_b \cdot d\vec{a}_\perp & \vec{J}_b = J_b \hat{z} \Rightarrow d\vec{a}_\perp = s\, ds\, d\varphi\, \hat{z} \end{array} \right)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}, \quad \vec{J} = \vec{J}_f + \vec{J}_b, \quad \vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$\vec{J}_f + \vec{\nabla} \times \vec{M} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}$$

$$\begin{cases} \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}_f \\ \vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}, \text{ "auxiliary field" (or "magnetic field" in some other contents,} \end{cases}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \quad \sim \quad \vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\oint \vec{H} \cdot d\vec{l} = I_f \quad \overset{?}{\Leftarrow} \quad \oint \vec{D} \cdot d\vec{a} = Q_f$$

Ampère's law in terms of $\vec{H}$-field:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \qquad \oint \vec{H} \cdot d\vec{l} = I_f$$

For some materials,

$$\vec{M} = \chi_m \vec{H}, \qquad \chi_m : \text{magnetic susceptibility}$$

For linear media,

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (\vec{H} + \chi_m \vec{H}) = \mu_0 (1 + \chi_m) \vec{H} \equiv \mu \vec{H}$$

$\mu \equiv \mu_0 (1 + \chi_m)$, magnetic permeability of material

Continuity of $\vec{H}$:

$$H^\perp_{above} - H^\perp_{below} = -(M^\perp_{above} - M^\perp_{below})$$

$$\vec{H}^\parallel_{above} - \vec{H}^\parallel_{below} = \vec{K}_f \times \hat{n}$$

$$B^\perp_{above} - B^\perp_{below} = 0$$

$$\vec{B}^\parallel_{above} - \vec{B}^\parallel_{below} = \mu_0 (\vec{K} \times \hat{n})$$

$$\left(\begin{array}{l} E^\perp_{above} - E^\perp_{below} = \frac{\sigma}{\varepsilon_0} \\ E^\parallel_{above} - E^\parallel_{below} = 0 \\ D^\perp_{above} - D^\perp_{below} = \sigma_f \\ D^\parallel_{above} - D^\parallel_{below} = P^\parallel_{above} - P^\parallel_{below} \end{array} \right)$$

Expand the integration region to some very large space:

$$\varphi_m = \frac{1}{4\pi}\Big(\underbrace{-\oint_{\partial V} \frac{\vec{M}}{\mathscr{r}} \cdot d\vec{a}}_{\to 0 \text{ as } V \to \infty} + \int_V \vec{M} \cdot \big(\vec{\nabla}' \frac{1}{\mathscr{r}}\big) d\tau' \Big)$$

$$= \frac{1}{4\pi} \int \vec{M} \cdot \vec{\nabla}' \frac{1}{\mathscr{r}} d\tau'$$

Now, $\vec{\nabla}' \to \vec{\nabla}$, which will bring a minus sign:

$$\varphi_m = -\frac{1}{4\pi} \int \vec{M} \cdot \vec{\nabla} \frac{1}{\mathscr{r}} d\tau'$$

$$= -\frac{1}{4\pi} \vec{\nabla} \cdot \int \frac{\vec{M}}{\mathscr{r}} d\tau'$$

For away from the source:

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \simeq \vec{r}$$

so

$$\varphi_m \simeq -\frac{1}{4\pi} \vec{\nabla} \frac{1}{r} \cdot \int \vec{M} d\tau, \quad \vec{\nabla} \frac{1}{r} = -\frac{\hat{r}}{r^2}, \quad \int \vec{M} d\tau = \vec{m}$$

$$\boxed{\varphi_m = \frac{1}{4\pi} \frac{\vec{m} \cdot \hat{r}}{r^2}}$$

Similar arguments hold for polarization. Long story short: Do not mix things up!

$$\oint \vec{D} \cdot d\vec{a} = Q_{f,enc}$$

$$-\vec{\nabla} \cdot \vec{P} = \rho_b$$

$$\vec{P} \cdot \hat{n}\big|_S = \sigma_b$$

$$Q_f = 0 \Rightarrow \vec{D} = 0$$

$$\vec{D} = \varepsilon_0 \vec{E} - \vec{P} = 0$$

$$\vec{E} = \frac{1}{\varepsilon_0} \vec{P}$$

$$\vec{P} = 0 \text{ outside} \Rightarrow \vec{E} = 0 \text{ outside}$$

This is all wrong!

Electromotive Force, EMF

Not a force.

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} \quad \text{where } \vec{f}\text{: force per unit charge}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{f} = \vec{E} + \vec{v} \times \vec{B}$$

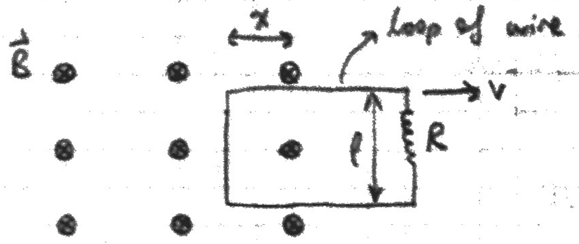
Usually,

$$\mathcal{E} \begin{cases} \oint \vec{f}_m \cdot d\vec{l} \\ \oint \vec{f}_e \cdot d\vec{l} \end{cases}$$

$\mathcal{E} = -\frac{d\Phi_m}{dt}$, Φ_m (magnetic flux) not to be confused with φ_m (scalar potential): $\vec{H} = -\vec{\nabla}\varphi_m$

$$\Phi_m \equiv \int \vec{B}\cdot d\vec{a} \quad \text{or} \quad d\Phi_m = \vec{B}\cdot d\vec{a}$$

$$\mathcal{E} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$



⇒ flux decreases

⇒ to compensate this, $\vec{B}$ ⊙ will be created

⇒ some current ↺ will be created on the loop

(Lenz's law)

$$\mathcal{E} = B\ell\frac{dx}{dt} = B\ell(-v) = -B\ell v$$

For f_e:

$$\oint \vec{f}_e\cdot d\vec{\ell} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$\boxed{\oint_{\partial S} \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}}$$ Faraday's law in integral form

$$\int \vec{\nabla}\times\vec{E}\cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a},$$ assuming area remains const

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's law in diff form

Customarily, anything related to a source is written on RHS, so

$$\boxed{\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$

$$\left(\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx = \int_a^b \frac{\partial f}{\partial \theta}\,dx + f(b(\theta),\theta)\,b'(\theta) - f(a(\theta),\theta)\right)$$

(second hour missed)

May 8

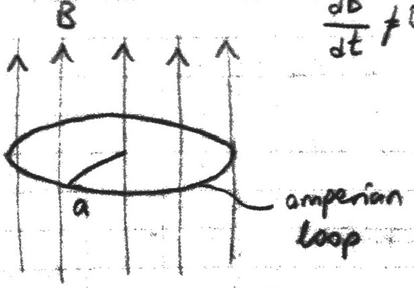
Faraday's Law:

$$\oint_{\partial S} \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \equiv -\frac{d\Phi_B}{dt}$$

$$\mathcal{E} = \int \vec{f}_e\cdot d\vec{\ell}, \quad \vec{f} = \underbrace{\vec{E}}_{\vec{f}_e} + \underbrace{\vec{v}\times\vec{B}}_{\vec{f}_m}$$

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

ex



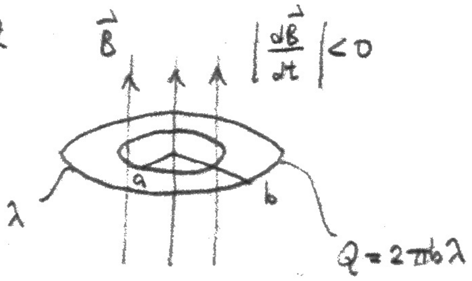
$$\frac{d\vec{B}}{dt} \neq 0$$

$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$E\,2\pi a = -\frac{dB}{dt}\pi a^2$$

$$E = -\frac{1}{2}\frac{dB}{dt}a, \quad \vec{E} = E\hat{\varphi} \quad \left(\frac{dB}{dt} \lessgtr 0 \text{ unknown}\right)$$

ex 7.7



Given:

$\vec{B} = B_0\hat{z}$ @ $t=0$

$\vec{B} = 0$ @ t

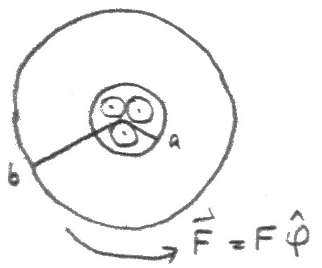
$$\vec{F} = Q\vec{E}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$E\,2\pi b = -\frac{dB}{dt}\pi a^2 \quad \text{(no mag. field further a)}$$

$$E = -\frac{1}{2}\frac{a^2}{b}\frac{dB}{dt}, \qquad \vec{E} = E\hat{\varphi}$$

$$\vec{F} = Q\vec{E} = 2\pi b\lambda\left(-\frac{1}{2}\right)\frac{a^2}{b}\frac{dB}{dt}\hat{\varphi} = -\pi\lambda a^2\frac{dB}{dt} \Rightarrow \text{some torque}$$



$\vec{F} = F\hat{\varphi}$

$$\vec{N} = \vec{r}\times\vec{F} = s\hat{s}\times F\hat{\varphi} = s\hat{s}\times(-\pi\lambda a^2)\frac{dB}{dt}\hat{\varphi}\Big|_{s=b} = -b\pi\lambda a^2\frac{dB}{dt}\hat{z}$$

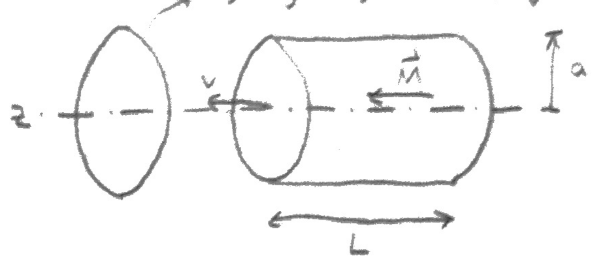
$$\vec{L} = \int_{t_0}^{t_f}\vec{N}\,dt$$

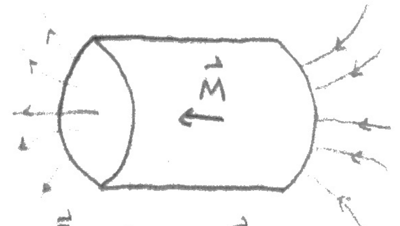
$$\vec{L} = \int_0^{t} -\pi\lambda a^2 b\frac{dB}{dt}\,dt\,\hat{z} = \pi\lambda a^2 bB\Big|_t^0\hat{z} = \pi\lambda a^2 b(B_0 - 0)\hat{z}$$

$$\vec{L} = \pi\lambda a^2 b B_0\hat{z}$$

ex

slightly larger than object





$\vec{J}_b = 0$ since $\vec{M} = \text{const}$: $\nabla\times\vec{M} = 0$

$\vec{K}_b = \vec{M}\times\hat{n} = M\hat{z}\times\hat{s} = M\hat{\varphi}$

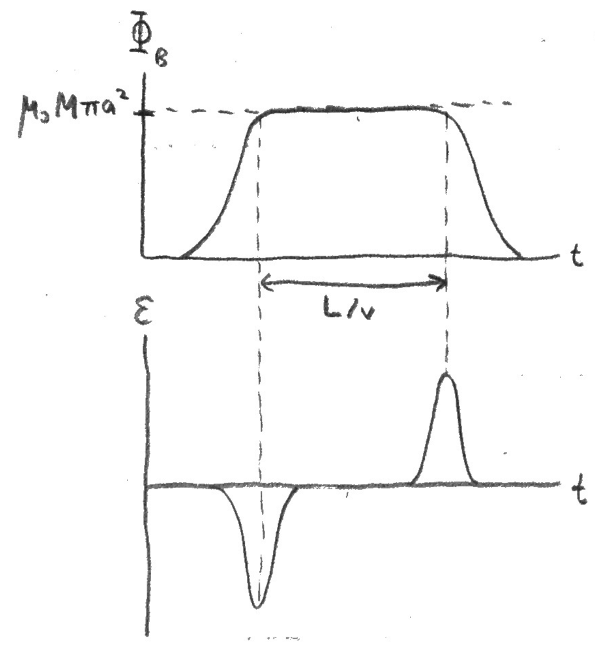
$$\oint \vec{H}\cdot d\vec{l} = I_{f,enc} = 0$$

$$\vec{H} = \frac{1}{\mu_0}(\vec{B} - \vec{M}) = 0$$

$\vec{B} = \mu_0\vec{M}$, just on the edges of the object

$\Phi_B = BA = B\pi a^2 = \mu_0 M\pi a^2$, max flux

$\vec{K}_b = M\hat{\varphi} \Rightarrow$ looks like a solenoid



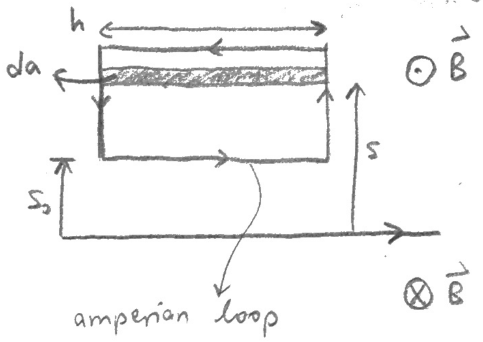
ex (7.9)

$$\xrightarrow{\quad I(t) \quad} \qquad \frac{dI}{dt} \neq 0$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$B 2\pi s = \mu_0 I \qquad \text{or} \qquad B = \frac{\mu_0 I}{2\pi s}$$

$$\frac{dB}{dt} = \frac{\mu_0}{2\pi} \frac{dI}{dt} \frac{1}{s}$$



Vertical segments do no contribution.

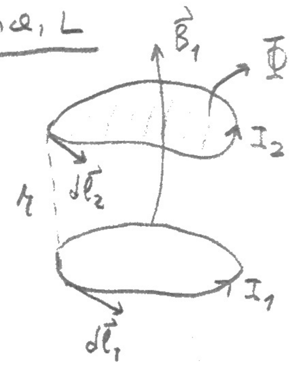
$$E(s_0)h - E(s)h = -\frac{\mu_0}{2\pi} \frac{dI}{dt} h \int_{s_0}^{s} \frac{1}{\tilde{s}} d\tilde{s}$$

$$E(s_0) - E(s) = \frac{\mu_0}{2\pi} \frac{dI}{dt} \ln(s_0/s)$$

$$E(s_0) - E(s) = \frac{\mu_0}{2\pi} \frac{dI}{dt} (\ln s_0 - \ln s)$$

$$E(s) = \frac{\mu_0}{2\pi} \frac{dI}{dt} \ln s + K, \qquad K \equiv E(s_0) - \frac{\mu_0}{2\pi} \frac{dI}{dt} \ln s_0$$

Mutual Inductance, M_{12}, and Self-Inductance, L



$$\vec{B}_1 = \frac{\mu_0 I_1}{4\pi} \int \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\Phi = \int \vec{B}_1 \cdot d\vec{a}_2$$

$$\vec{B}_1 = \vec{\nabla} \times \vec{A}_1$$

$$\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \int \frac{d\vec{l}_1}{\mathscr{r}}$$

$$\Phi = \frac{\mu_0 I_1}{4\pi} \oint \oint \vec{\nabla} \times \frac{d\vec{l}_1}{\mathscr{r}} \cdot d\vec{a}_2 = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}$$

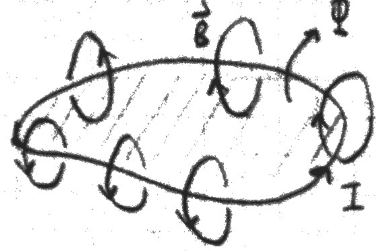
$$\boxed{\Phi = I_1 M_{12} \text{ where } M_{12} \equiv \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}}$$

~ functions like capacitance, (includes permeability and geometry)

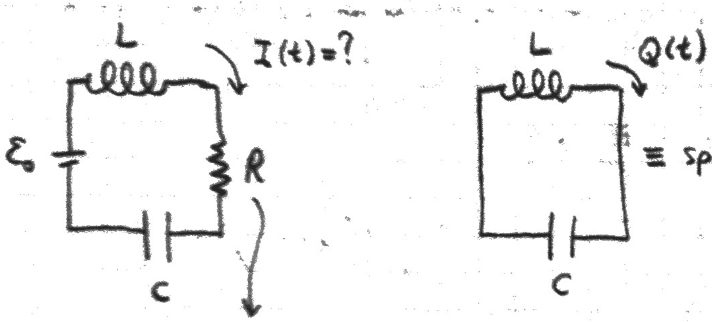
$$M_{12} = M_{21}$$

Similarly,

$$\Phi = I L$$



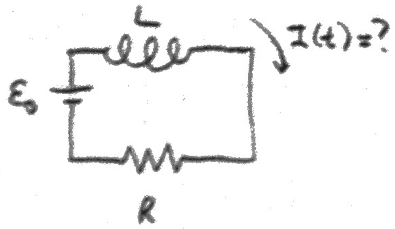
$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$



$\equiv$ spring-mass system

causes dissipation in spring-mass system

ex



$\mathcal{E} = IR$ (eq to be solved)

$\mathcal{E}_0 - L\frac{dI}{dt} = IR$

$L\frac{dI}{dt} + IR = \mathcal{E}_0$ or $\dot{I} + \frac{R}{L} I = \frac{\mathcal{E}_0}{L}$

$I(t) = \frac{\mathcal{E}_0}{R}\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$

(Problem 7.25)

May 9

(class missed)

May 15

(MT2 : 5-6-7)

CH8 CONSERVATION LAWS

Conservation of charge:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

Energy:

$$W_e = \frac{\varepsilon_0}{2}\int \vec{E}^2 d\tau, \quad W_m = \frac{1}{2\mu_0}\int \vec{B}^2 d\tau$$

$$W = \frac{1}{2}\int \left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) d\tau$$

Energy per unit time per unit area:

$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$, Poynting vector

$u_{em} = \frac{1}{2}(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2)$, electromagnetic energy density

$$\frac{d}{dt}\int_V (u_{mech} + u_{em})\, d\tau = -\oint_{\partial V} \vec{S}\cdot d\vec{a}$$

$$= -\int_V \vec{\nabla}\cdot\vec{S}\, d\tau$$

$$\boxed{\frac{\partial}{\partial t}(u_{mech} + u_{em}) + \vec{\nabla}\cdot\vec{S} = 0}$$

Linear momentum density:

$$\frac{\partial}{\partial t}(\wp_{em} + \wp_{mech}) = \vec{\nabla}\cdot\overleftrightarrow{T}$$

$$T_{ij} = \varepsilon_0 (E_i E_j - \frac{1}{2}\delta_{ij}\vec{E}^2) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}\delta_{ij}\vec{B}^2)$$

ex In cartesian coors,

$$T_{xy} = \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y$$

$$T_{zz} = \varepsilon_0 [E_z^2 - \frac{1}{2}(E_x^2 + E_y^2 + E_z^2)] + \frac{1}{\mu_0}[B_z^2 - \frac{1}{2}(B_x^2 + B_y^2 + B_z^2)]$$

$$\vec{\wp}_{em} = \mu_0 \varepsilon_0 \vec{S}$$

Angular momentum density:

$$\vec{\ell}_{em} = \vec{r}\times\vec{\wp}_{em}$$

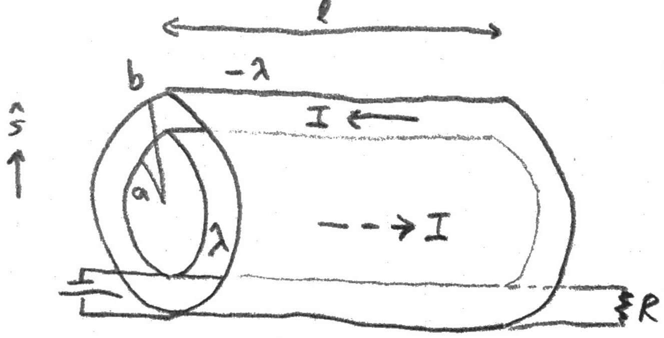
$$\vec{L}_{em} = \int \vec{r}\times\vec{\wp}_{em}\, d\tau = \int \vec{\ell}_{em}\, d\tau$$

$\vec{S}$: energy per unit time per unit area

$\frac{dW}{dt} = P$, power

Ex 8.3 Find power transported thru the line.

$$P = \int_S \vec{S}\cdot d\vec{a}, \quad \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$



$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q$$

$$E 2\pi s \ell = \frac{1}{\varepsilon_0}\lambda \ell$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s}, \quad a < s < b$$

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I$$

$$B 2\pi s = \mu_0 I$$

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{\hat{\varphi}}{s}, \quad a < s < b$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s}\times\frac{\mu_0 I}{2\pi}\frac{\hat{\varphi}}{s} = \frac{\lambda I}{4\pi^2\varepsilon_0}\frac{\hat{z}}{s^2}$$

$$P = \int \vec{S}\cdot d\vec{a} = \iint \frac{\lambda I}{4\pi^2\varepsilon_0}\frac{1}{s^2}\, s\,ds\,d\varphi = \frac{\lambda I}{4\pi^2\varepsilon_0}2\pi \ln(b/a)$$

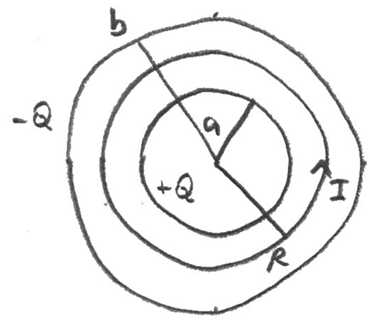
$$P = \frac{\lambda I}{2\pi\varepsilon_0}\ln(b/a)$$

$$P = IV$$

$$V = \frac{\lambda}{2\pi\varepsilon_0}\ln(b/a)$$

ex (Nevermind the figure; here is how it is: ∞-long solenoid, radius R, carrying I. Inside, finite cylindrical shell, radius a, charge $-Q$; outside, cylindrical shell, radius b, charge $+Q$ — also finite).

top view



$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \quad \text{if } I \neq \text{const}$$

$$\vec{F} = Q\vec{E}$$

$$\vec{E}\to\vec{F}\to\vec{N}=\vec{r}\times\vec{F}\to\vec{L}_{mech}=\int\vec{N}\,dt$$

$$\vec{E},\vec{B}\to\vec{p}_{em}=\varepsilon_0\vec{E}\times\vec{B}\to\vec{L}_{em}=\int\vec{r}\times\vec{p}_{em}\,d\tau$$

I: from I_0 to 0 $\Rightarrow$ induced $\vec{E}$ (Faraday's law)

$\vec{p}_{em}$: steady state (em-static linear momentum density)

Static case:

$$E\,2\pi s l = \frac{Q}{\varepsilon_0}$$

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0 l}\frac{\hat{s}}{s}, \quad a<s<b$$

$$Bl = \mu_0 n l I$$

$$\vec{B} = \mu_0 n I\hat{z}, \quad s<R$$

$$\vec{S} = \frac{1}{\mu_0}\frac{Q}{2\pi\varepsilon_0 l}\frac{\hat{s}}{s}\times\mu_0 n I\hat{z} = \frac{QnI}{2\pi\varepsilon_0 l}\frac{-\hat{\varphi}}{s}, \quad a<s<R$$

$$\vec{L}_{em} = \int\vec{r}\times\vec{p}_{em}\,d\tau = \int_V s\hat{s}\times\frac{\mu_0 QnI}{2\pi l}\frac{-\hat{\varphi}}{s}\,s\,ds\,d\varphi\,dz$$

$$= \int\frac{\mu_0 QnI}{2\pi l}(-\hat{z})\,s\,ds\,d\varphi\,dz$$

$$= -\hat{z}\,\frac{R^2-a^2}{2}\,2\pi l\,\frac{\mu_0 QnI}{2\pi l} = \frac{a^2-R^2}{2}\mu_0 QnI\hat{z}$$

$\vec{L}_{em} = \frac{a^2-R^2}{2}\mu_0 QnI\hat{z}$, stored in the field.

Removing one of the fields, we get mechanical angular momentum: Drop $I_0 = \text{const}$ to 0. Change in $\vec{B}$ will induce $\vec{E}$.

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \mu_0 n I\, da$$

$$E\, 2\pi s = -\mu_0 n \frac{dI}{dt} \underbrace{\int da}_{\pi R^2 \text{ or } \pi s^2}$$

$\Rightarrow$ choose Amperian loop accordingly

πR^2: for induced field, $R<s<b$ (outside solenoid)

πs^2: for induced field $s<R$ (inside solenoid)

$$\vec{E}_{ind,in} = \frac{-1}{2\pi s}\mu_0 n \frac{dI}{dt}\pi s^2 \hat{\varphi} = -\frac{1}{2}\mu_0 n \frac{dI}{dt} s\hat{\varphi}$$

$$\vec{E}_{ind,out} = \frac{-1}{2\pi s}\mu_0 n \frac{dI}{dt}\pi R^2 \hat{\varphi} = -\frac{1}{2}\mu_0 n \frac{dI}{dt} R^2 \frac{\hat{\varphi}}{s^2}$$

Inner cylinder will feel:

$$\vec{F} = +Q\vec{E}_{ind,in} \quad @ \quad s=a$$

$$\vec{F} = Q\frac{-1}{2}\mu_0 n \frac{dI}{dt} a\hat{\varphi}$$

$$\vec{N} = \vec{r}\times\vec{F} = a\hat{s}\times Q\frac{-1}{2}\mu_0 n\frac{dI}{dt}a\hat{\varphi} = -\frac{1}{2}\mu_0 n a^2 Q\frac{dI}{dt}\hat{z}$$

$$\vec{L}_{mech} = \int \vec{N}\,dt = \frac{1}{2}\mu_0 n a^2 Q I_0 \hat{z}$$

Outer cylinder will feel:

$$\vec{F} = -Q\vec{E}_{ind,out} \quad @ \quad s=b$$

$$\vec{F} = -Q\frac{-1}{2}\mu_0 n\frac{dI}{dt}R^2\frac{\hat{\varphi}}{b}$$

$$\vec{N} = \vec{r}\times\vec{F} = b\hat{s}\times\frac{1}{2}\mu_0 n Q R^2\frac{dI}{dt}\frac{\hat{\varphi}}{b} = \frac{1}{2}\mu_0 n Q R^2\frac{dI}{dt}\hat{z}$$

$$\vec{L}_{mech} = \int\vec{N}\,dt = -\frac{1}{2}\mu_0 n Q R^2 I_0\hat{z}$$

(Check out Ex 7.8)

Total angular momentum:

$$\vec{L}_{mech,inner} + \vec{L}_{mech,outer} = \frac{1}{2}\mu_0 n Q a^2 I_0\hat{z} - \frac{1}{2}\mu_0 n Q R^2 I_0\hat{z}$$

$$= \frac{1}{2}\mu_0 n Q I_0 (a^2 - R^2)\hat{z}$$

$= \vec{L}_{em}$, angular momentum stored in the fields

$\vec{L}_{mech} \rightarrow \vec{L}_{em}$ (exactly transferred)

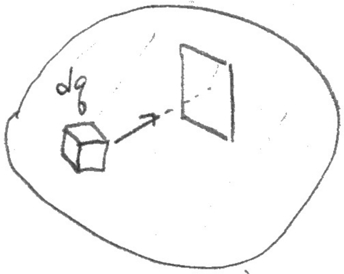
Maxwell Eq's in Matter

May 16

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M}$$

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp$$



$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}, \text{ polarization current density}$$

$$\vec{\nabla} \cdot \vec{J}_p = \vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{P} = - \frac{\partial \rho_b}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J}_p + \frac{\partial \rho_b}{\partial t} = 0 \quad \text{Continuity eq}$$

$\vec{J}_p$ satisfies continuity eq.

$$\rho = \rho_f + \rho_b$$

Experimentally controllable quantities:

- potential difference $\Rightarrow$ electrostatic field $\Rightarrow$ free charge (though $\vec{\nabla} \cdot \vec{D} = \rho_f$)
- free current

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{P}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} (\rho_f - \vec{\nabla} \cdot \vec{P})$$

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$$

$\vec{\nabla} \cdot \vec{D} = \rho_f$ valid for non-static case as well

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \underbrace{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{Maxwell correction to Ampère's law}}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \quad \text{where } \vec{H} \equiv \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$\vec{\nabla} \cdot \vec{B} = 0$ always

$$\begin{cases} \vec{\nabla} \cdot \vec{D} = \rho_f \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \end{cases}$$

We need "constitutive relations" that relate $\vec{E}$ and $\vec{B}$ to $\vec{D}$ and $\vec{H}$. These relations depend on the nature of the material.

For linear media:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad \text{and} \qquad \vec{M} = \chi_m \vec{H}$$

so

$$\boxed{\vec{D} = \varepsilon \vec{E} \quad \text{and} \quad \vec{H} = \frac{1}{\mu}\vec{B}}$$ constitutive relations for linear media

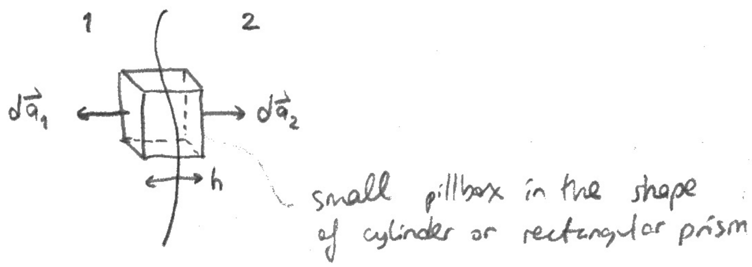
where

$$\varepsilon \equiv \varepsilon_0 (1+\chi_e) \quad \text{and} \quad \mu \equiv \mu_0 (1+\chi_m)$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\equiv \vec{J}_D \text{, displacement current}}$$

<u>Boundary Conditions</u>

1 $$\vec{\nabla}\cdot\vec{D} = \rho_f \quad \Rightarrow \quad \oint_S \vec{D}\cdot d\vec{a} = Q_{f\,enc}$$



$$\oint \vec{D}\cdot d\vec{a} = \int \vec{D}_1\cdot d\vec{a}_1 + \int \vec{D}_2\cdot d\vec{a}_2 + \int \vec{D}\cdot d\vec{a}\Big|_{sides}$$

$$\lim_{h\to 0} \int \vec{D}\cdot d\vec{a}\Big|_{sides} = 0$$

$$d\vec{a}_1 = -d\vec{a}_2$$

Then

$$\vec{D}_1\cdot d\vec{a}_1 - \vec{D}_2\cdot d\vec{a}_2 = \sigma_f\, da_1 , \qquad d\vec{a}_1 = \hat{n}\, da_1$$

$$(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = \sigma_f$$

or

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f}$$

2 $$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \oint_S \vec{B}\cdot d\vec{a} = 0$$

Consider this integral over the same pillbox in 1.

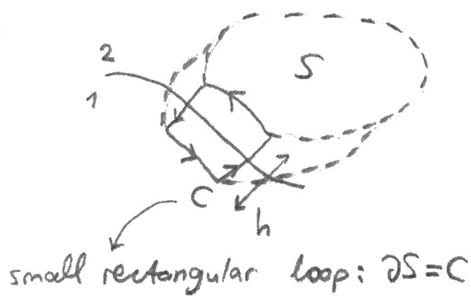
$$\lim_{h\to 0} \int \vec{B}\cdot d\vec{a}\Big|_{sides} = 0$$

$$\Rightarrow (\vec{B}_1 - \vec{B}_2)\cdot\hat{n} = 0$$

or

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0}$$

3



$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$$

$$\oint_C \vec{E}\cdot d\vec{l} = \int \vec{E}_1\cdot d\vec{l}_1 + \int \vec{E}_2\cdot d\vec{l}_2 + \int \vec{E}\cdot d\vec{l}\Big|_{sides}$$

$$\lim_{h\to 0} \int \vec{E}\cdot d\vec{l}\Big|_{sides} = 0$$

$$\lim_{h\to 0} \int_S \vec{B}\cdot d\vec{a} = \oint_S \vec{B}\cdot d\vec{a} = 0$$

$$\vec{dl}_2 = -\vec{dl}_1$$

Therefore,

$$(\vec{E}_2 - \vec{E}_1)\cdot \vec{dl}_1 = 0$$

or

$$\boxed{\vec{E}_1^{\,\parallel} - \vec{E}_2^{\,\parallel} = 0}$$

(4) $$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \Rightarrow \oint_C \vec{H}\cdot\vec{dl} = \underbrace{\int_S \vec{J}\cdot\vec{da}}_{I_{f,enc}} + \frac{d}{dt}\int_S \vec{D}\cdot\vec{da}$$

S, C: same as in (3)

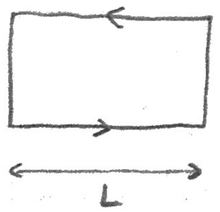
$$\lim_{h\to 0} \int \vec{H}\cdot\vec{dl}\,\Big|_{sides} = 0$$

$$\lim_{h\to 0} \int \vec{D}\cdot\vec{da} = 0 \quad \text{since} \quad \lim_{h\to 0} da = 0$$

$$\lim_{h\to 0} \int (\vec{H}_1 - \vec{H}_2)\cdot \vec{dl}_1 = I_{f,enc}$$

Since no volume is involved now, $I_{f,enc}$ is related to $\vec{K}$, surface current density.

We have a rectangular loop, and assume that $\vec{H}$ is approximately const along $\vec{dl}_1$ and $\vec{dl}_2$.



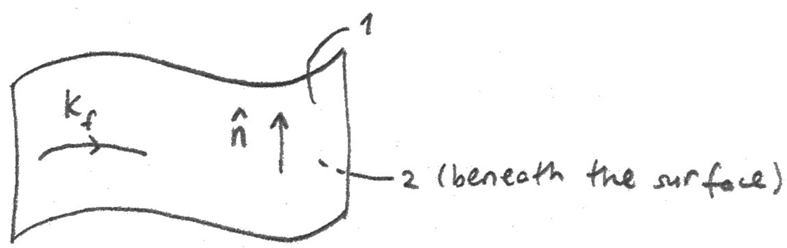
$$\int (\vec{H}_1 - \vec{H}_2)\cdot\vec{dl}_1 = (\vec{H}_1 - \vec{H}_2)\cdot\vec{L}$$

$$I_{f,enc} = \vec{K}_f\cdot(\hat{n}\times\vec{L}) = (\vec{K}_f\times\hat{n})\cdot\vec{L}$$

$\hat{n}$: unit normal directed from 2 to 1

$$(\vec{H}_1 - \vec{H}_2)\cdot\vec{L} = (\vec{K}_f\times\hat{n})\cdot\vec{L}$$

$$\boxed{\vec{H}_1^{\,\parallel} - \vec{H}_2^{\,\parallel} = \vec{K}_f\times\hat{n}}$$



These boundary conditions are valid for any kind of medium.

In linear media:

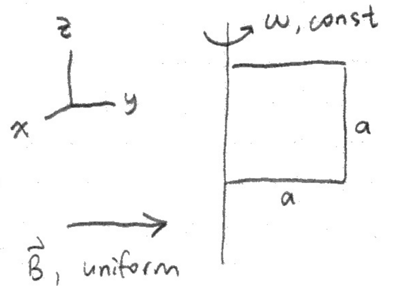
$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

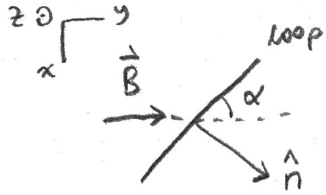
$$\vec{E}_1^{\,\parallel} - \vec{E}_2^{\,\parallel} = 0$$

$$\frac{1}{\mu_1}\vec{B}_1^{\,\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\,\parallel} = \vec{K}_f\times\hat{n}$$

Problem 7-10 $\quad \mathcal{E} = ?$

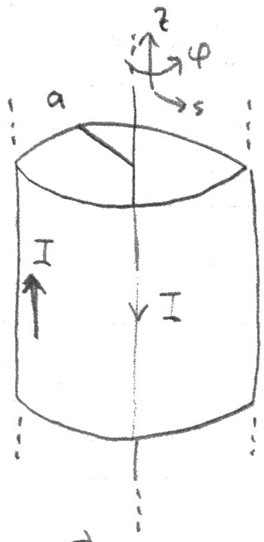


$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \quad \text{over area enclosed by loop}$$



$$\int \vec{B}\cdot d\vec{a} = BA\sin\alpha$$
$$= Ba^2 \sin\alpha$$
$$\frac{d}{dt}\Phi = Ba^2\cos\alpha \frac{d\alpha}{dt} = B\omega a^2 \cos\omega t$$

7-16



$I = I_0 \cos\omega t$

a) In what direction does the induced $\vec{E}$ point?

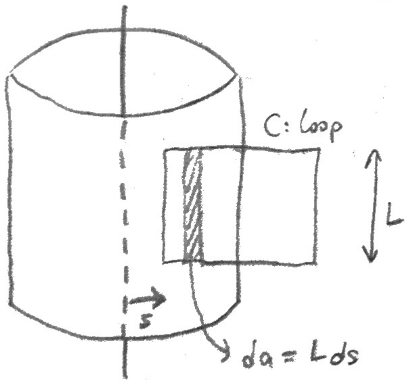
b) Assuming $\vec{E}_{ind} \to 0$ as $s \to a$, find $\vec{E}_{ind}(s,t) = ?$

a) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{E} = E(s,t)\hat{z}$

(Quasistatic approx.: Ignore $\vec{B}_{ind}$ by $\vec{E}_{ind}$ — try the exact soln using Taylor series:

$$\vec{\nabla}\times\vec{B}_{ind} = \vec{J}_f + \mu_0\varepsilon_0 \frac{\partial \vec{E}_{ind}}{\partial t}$$)

b) Here, $\vec{E} = E\hat{z}$ inside and $\vec{E} = \vec{B} = 0$ outside.



$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$E(s,t)\cancel{L} = -\frac{d}{dt}\int B(s,t)\cancel{L}\,ds$$

$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$ in quasistatic approx.
(as if I = const)

$$E(s,t) = -\frac{d}{dt}\int \frac{\mu_0 I}{2\pi s}\,ds$$
$$= -\frac{d}{dt}\frac{\mu_0 I}{2\pi}\ln(a/s)$$
$$= \frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(s/a)$$
$$\vec{E}_{ind} = \frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(s/a)\,\hat{z}$$

Study: 7-20, 33, 51.

May 21

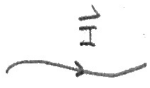
Review of Ch 5-6-7 (Recit)

Ch5 Magnetostatics

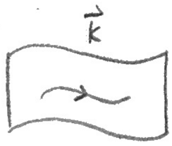
static charges → electrostatics

steady currents → magnetostatics

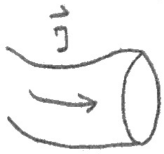
*) Biot-Savart Law



1-dimensional : $\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$



2-dimensional : $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} da'$

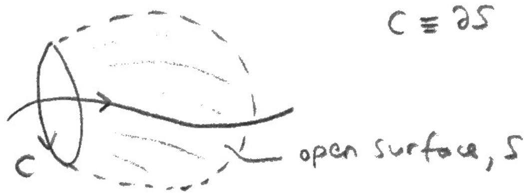


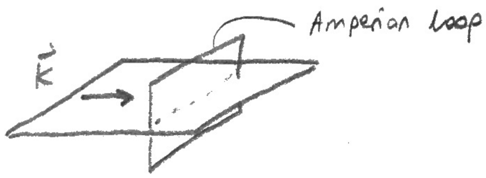
3-dimensional : $\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau'$

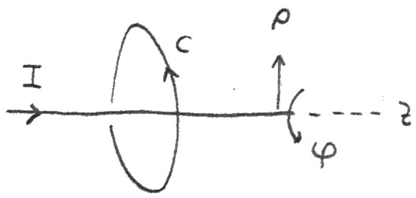
*) Ampère Law

$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ or $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (w/o Maxwell correction)

$C \equiv \partial S$







Assume $\vec{B} = B\hat{\varphi}$ from geometry:

$B\, 2\pi\rho = \mu_0 I$

$B = \frac{\mu_0 I}{2\pi} \frac{1}{\rho} = B(\rho)$

$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2\vec{r} \cdot \vec{r}'} = \sqrt{r^2 + r'^2 - 2rr'\cos\alpha}$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

$\vec{r} \cdot \vec{r}' = xx' + yy' + zz'$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$\vec{r} \cdot \vec{r}' = rr' (\sin\theta\sin\theta'\cos\varphi\cos\varphi' + \sin\theta\sin\theta'\sin\varphi\sin\varphi' + \cos\theta\cos\theta')$

$= rr' (\sin\theta\sin\theta'\cos(\varphi - \varphi') + \cos\theta\cos\theta')$

*) Vector Potential

$\vec{B} = \vec{\nabla} \times \vec{A}$, definition of $\vec{A}$

$\vec{\nabla} \times \vec{\nabla} \psi = 0$

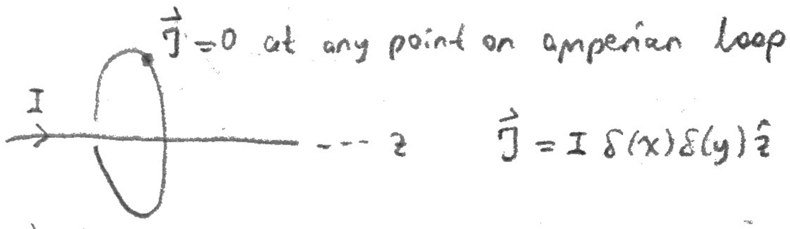
$\vec{A} \to \vec{A} + \vec{\nabla}\psi$, gauge freedom

Gauge choice: $\vec{\nabla} \cdot \vec{A} = 0$ (Coulomb gauge)

In magnetostatics,

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}, \quad \vec{B} = \vec{B}(\vec{r}), \quad \vec{J} = \vec{J}(\vec{r})$$

Special case : $\vec{J} = 0$



$$\vec{J} = I\,\delta(x)\delta(y)\hat{z}$$

$\vec{\nabla}\times\vec{B} = 0 \Rightarrow$ a magnetostatic potential, V_m

Ch6

★) Magnetization (density)

~ Magnetic dipoles in matter

Force on a magnetic dipole $\vec{m}$:

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}), \quad \vec{B} : \text{external field}$$

$$\vec{B} = -\vec{\nabla}V_m$$

$$-\vec{m}\cdot\vec{\nabla}V_m = -\vec{\nabla}\cdot(\vec{m}V_m)$$

★) Potential energy of a dipole in ext. field, $\vec{B}$:

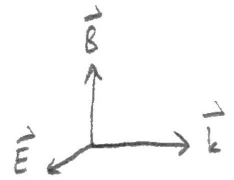
$$U = -\vec{m}\cdot\vec{B} \quad \text{s.t} \quad \vec{F} = -\vec{\nabla}U = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

May 2

Monochromatic plane waves :

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$$

$$\vec{B}(z,t) = \vec{B}_0 e^{i(kz-\omega t)}$$



From Maxwell with no source ($\rho = 0$, $\vec{J} = 0$) and above, we have

$$\vec{B} = \frac{1}{v}\vec{k}\times\vec{E}$$

Then

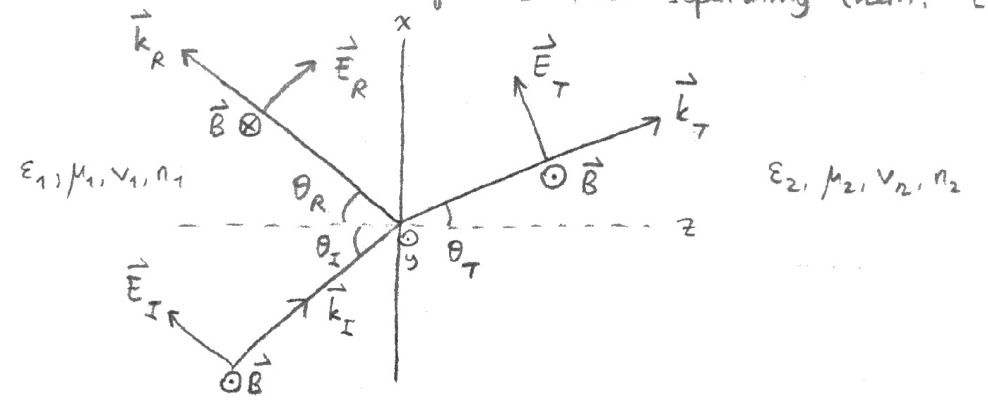
$$B_0 = \frac{1}{c}E_0$$

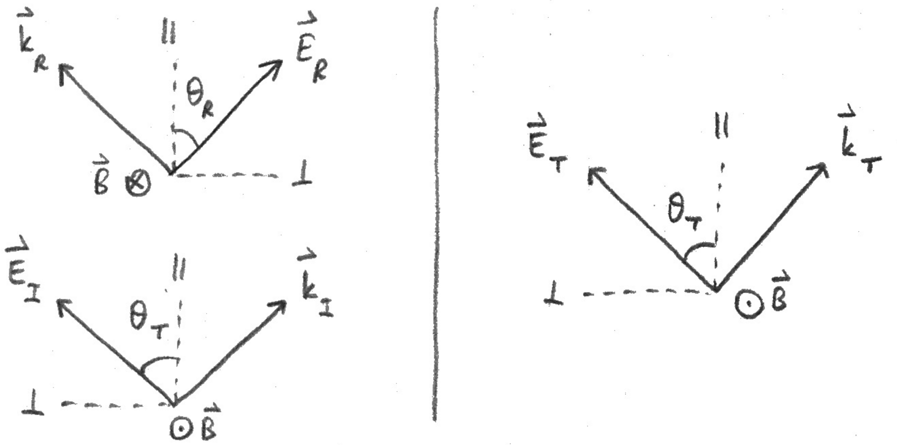
Recall boundary conditions with $\rho = 0$, $\vec{J} = 0$:

$$\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} \qquad E_1^{\parallel} = E_2^{\parallel}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}B_1^{\parallel} = \frac{1}{\mu_2}B_2^{\parallel}$$

consider two media with a free surface separating them : $c = vn$





Boundary conditions:

$$\varepsilon_1(-E_{0I}\sin\theta_I + E_{0R}\sin\theta_R) = -\varepsilon_2 E_{0T}\sin\theta_T$$

$$E_{0I}\cos\theta_I + E_{0R}\cos\theta_R = E_{0T}\cos\theta_T$$

$$\frac{1}{\mu_1 v_1}(E_{0I} - E_{0R}) = \frac{1}{\mu_2 v_2}E_{0T}$$

No energy lost passing thru the other medium, therefore ω's are preserved:

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega$$

$$(k_I)_x = (k_R)_x = (k_T)_x$$

$$(k_I)_y = (k_R)_y = (k_T)_y$$

On the boundary:

$$(k_I)_x = (k_R)_x$$

$$\left.\begin{aligned} k_I\sin\theta_I &= k_R\sin\theta_R \\ k_I v_1 &= k_R v_1 \end{aligned}\right\} \Rightarrow \boxed{\theta_I = \theta_R}$$

$$(k_I)_x = (k_T)_x$$

$$\left.\begin{aligned} k_I\sin\theta_I &= k_T\sin\theta_T \\ k_I v_1 = k_T v_2 \quad &\text{or} \quad k_I\frac{c}{n_1} = k_T\frac{c}{n_2} \end{aligned}\right\} \boxed{n_1\sin\theta_I = n_2\sin\theta_T}$$

Snell's law

Substitute all these into boundary conditions:

$$\begin{cases} E_{0I} - E_{0R} = \beta E_{0T}, & \beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2} \\ E_{0I} + E_{0R} = \alpha E_{0T}, & \alpha \equiv \dfrac{\cos\theta_T}{\cos\theta_I} \end{cases}$$

$$\boxed{E_{0T} = \underbrace{\frac{2}{\alpha+\beta}}_{\text{transmission coeff's}} E_{0I} \qquad E_{0R} = \underbrace{\frac{\alpha-\beta}{\alpha+\beta}}_{\text{reflection coeff.}} E_{0I}}$$

Fresnel eq's

Assumption: $\mu_1 \simeq \mu_2$ (which is indeed the case usually)

$\alpha = \beta$ Condition for no reflection $\Rightarrow$ polarization

$$\frac{\cos\theta_T}{\cos\theta_I} = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{v_1}{v_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1}$$

With a little algebra,

$$\boxed{\tan\theta_B = \frac{n_2}{n_1}}$$ Brewster angle