MATH 320 SET THEORY

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ABSTRACT. These are the lecture notes I used for a 14-week introductory set theory class I taught at the Department of Mathematics of Middle East Technical University during Spring 2018. In order to determine the course content and prepare the lecture notes, I mainly used the textbook by Hrbacek and Jech [1], which I also listed as a supplementary resource for the course.

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Week 1

0. Prelude

0.1. **Some historical remarks.** If one examines the history of mathematics, one sees that towards the end of 19^{th}century, some mathematicians started to investigate the "nature" of mathematical objects. For example, Dedekind gave a construction for the real numbers, Peano axiomatized the natural numbers, Cantor established a rigorous way to deal with the notion of infinity. These works may be considered as first steps to understand *what* mathematical objects are.

In early 20^{th}century, arose what is known as *the foundational crisis of mathematics*. Mathematicians searched for a proper foundations of mathematics which is free of contradictions and is sufficient to carry out all traditional mathematical reasoning. There were several philosophical schools having different views on how mathematics should be done and what mathematical objects are. Among these philosophical schools, the leading one was Hilbert's formalist approach, according to which mathematics is simply an activity carried out in some formal system[1]. On the one hand, mathematics had already been done "axiomatically" since Euclid. On the other hand, Hilbert wanted to provide a **rigorous** axiomatic foundation to mathematics[2]. With the work of Dedekind and Cantor, the idea that mathematics can be founded on set theory became more common. This eventually led[3] to the development of the **Z**ermelo-**F**raenkel set theory with the axiom of **C**hoice, by Ernst Zermelo, with the later contributions of Abraham Fraenkel, Thoralf Skolem and John von Neumann.

Today, some mathematicians consider ZFC as *the* foundation of mathematics, in which one can formalize virtually all known mathematical reasoning. In this course, we aim to study the axioms of ZFC and investigate their consequences. That said, we should note there are many other set theories with different strengths introduced for various purposes, such as von Neumann-Gödel-Bernays set theory, Morse-Kelley set theory, New Foundations, Kripke-Platek set theory and the Elementary Theory of the Category of Sets.

0.2. **The language of set theory and well-formed formulas.** We shall work in first-order logic with equality symbol whose language consists of a single binary relation symbol $\in$. For those who are not familiar with first-order logic, we first review how the well-formed formulas in the language of set theory are constructed. Our basic symbols consist of the symbols

$$\in \; = \; \forall \; \exists \; \neg \; \wedge \; \vee \; \rightarrow \leftrightarrow \; (\;)$$

together with an infinite supply of variable symbols

$$a \; b \; c \; d \; e \; \ldots$$

The *(well-formed) formulas* in the language of set theory are those strings that can be obtained in finite numbers of steps by application of the following rules.

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[1]To illustrate this point,we should perhaps remind Hilbert's famous saying: "Mathematics is a game played according to certain simple rules with meaningless marks on paper."

[2]In fact, Hilbert wanted more than this. Those who wish to learn more should google the term "Hilbert's program".

[3]We refer reader to the web page https://plato.stanford.edu/entries/settheory-early/ for a detailed and more accurate historical description.

- Strings of the form $x \in y$ and $x = y$, where x and y are variable symbols, are formulas.
- If φ and ψ are formulas and x is any variable symbol, then the following strings are formulas

$$\neg\varphi \quad \exists x\varphi \quad \forall x\varphi \quad (\varphi \wedge \psi) \quad (\varphi \vee \psi) \quad (\varphi \rightarrow \psi) \quad (\varphi \leftrightarrow \psi)$$

For example, the string $\exists x\forall y\neg y \in x$ is a well-formed formula in the language of set theory, whereas, the string $\exists x\forall\neg x \rightarrow \exists\vee$ is not. A variable in a formula is said to be *bound* if it is in the scope of a quantifier; otherwise, it is said to be *free*. A formula with no free variables is called a *sentence*. For example, the string $\exists x\forall y\neg y \in x$ is a sentence, and the string $\exists z\exists t((\neg z = t \wedge z \in x) \wedge t \in y)$ is a formula with two free variables x and y and hence not a sentence.

"Officially", we work in an axiomatic system that consists of the axioms of ZFC and the standard logical axioms (in the language of set theory) together with a sound and complete proof system[4]. "Unofficially", we are going to work in natural language and carry out our mathematical arguments informally, as is the case in any other branch of mathematics. Nevertheless, if necessary, the reader should be able to convert arguments in natural language to formal proofs in first-order logic and vice versa.

0.3. **What are sets anyway?** Up to this point, we have not mentioned anything related to the *meaning* of the formulas in the language of set theory. For example, what does $x \in y$ really *mean*?

On the one hand, we note that it is perfectly possible to take a purely formalist approach and simply derive theorems in the aforementioned axiomatic system with attaching no meaning to symbols. On the other hand, we believe that this approach is pedagogically inappropriate for students who are exposed to set theory for the first time; and that it fails to acknowledge the role of mathematical intuition, which not only manipulates symbols but also understands what they *refer to*. Consequently, we shall adopt a Platonist point of view that we think is better-suited for teaching purposes[5]. Back to the question... What does $x \in y$ really *mean*?

A long time ago in a galaxy far, far away.... existed the universe of mathematical objects called *sets* which is denoted by $\mathbf{V}$. We shall not try to define what a set is. You should think of sets as primitive objects, perhaps by comparing it to points of Euclid's Elements. Sets are to us like points are to Euclid. Sets are simply the objects in the universe of sets.

Between certain sets holds the *membership* relation which we denote by $x \in y$. Our intuitive interpretation of the relation $\in$ is that $x \in y$ holds if the set y contains the set x as its element. In this sense, sets are objects that contain certain other sets as their members.

Quantifiers ranging over the universe of sets and logical connectives having their usual intended meanings, a sentence in the language of set theory is simply an assertion about the universe of sets that is either true or false, depending on how the membership relation holds between sets.

[4]Details of our proof system are not really relevant for this course, since most of our arguments are going to be done informally. Moreover, there are many (essentially equivalent) proof systems that are sufficient for our purposes. Those students who wish to learn how a sound and complete proof system for first-order logic may be set up should google the term *Hilbert(-style) proof system.*

[5]However, I personally do not consider myself as a follower of mathematical Platonism.

We assume that the axioms of ZFC are true sentences about the universe of sets, whose truth is self-evident and dictated by our mathematical intuition[6]. In this course, we shall study the logical consequences of the axioms of ZFC and try to understand the structure of the universe of sets $\mathbf{V}$.

0.4. **Classes vs. Sets.** A *class* is simply a collection of sets and hence is a subcollection of the universe of sets. We remark that classes are not (necessarily) objects in the universe of sets according to this definition. Consequently, we cannot directly talk about them in our axiomatic system by referring to them via variable symbols[7]. However, there is a way to get around this problem and make assertions about classes in a meaningful manner.

Let $\varphi(x)$ be a *property* of sets, i.e. a formula in the language of set theory with one free variable. The collection C of sets satisfying the formula $\varphi(x)$ is a class and is denoted by

$$\{x : \varphi(x)\}$$

In this case, the class C is said to be *defined* by the formula $\varphi(x)$. We also allow multiple free variables to appear in the defining formula, in which case the class

$$\{x : \psi(x, p, q, \ldots, t)\}$$

is said to be defined by ψ with parameters $p, q, \ldots, t$, where $p, q, \ldots, t$ are fixed sets.

For the rest of this course, we shall restrict our attention to those classes that are defined by some formula in the language of set theory possibly via some parameters. As such, we can meaningfully make assertions about classes in our axiomatic system by identifying formulas with the corresponding classes. For example, if C and D are classes that are defined by the formulas $\varphi(x)$ and $\psi(x)$ respectively, then the assertion $C = D$ can be stated by the sentence $\forall x(\varphi(x) \leftrightarrow \psi(x))$. We can also "quantify" over a class C defined by the formula $\varphi(x)$ using the formulas

$$\forall x(\varphi(x) \rightarrow \psi) \text{ and } \exists x(\varphi(x) \wedge \psi)$$

which would intuitively correspond to $\forall x \in C\ \psi$ and $\exists x \in C\ \psi$ respectively if we could have quantified over the classes in the first place. One can similarly define quantification over classes defined via parameters.

It is clear that every set, being a collection of sets, is a class. More precisely, given a set x, we can simply define it by the formula $y \in x$ using the set x itself as a parameter, i.e. $x = \{y : y \in x\}$. On the other hand, not every class is a set.

Theorem 1 (Russell's paradox)**.** *The class $R = \{x : \neg x \in x\}$ is not a set. More precisely,*

$$\neg\exists x \forall y(y \in x \leftrightarrow \neg y \in y)$$

Proof. Assume to the contrary that there exists x such that $\forall y(y \in x \leftrightarrow \neg y \in y)$. Then, letting y be the set x, we have $\neg x \in x \leftrightarrow x \in x$, which is a contradiction. □

Classes that are not sets are called *proper classes*. For example, the class R defined above is a proper class. As we shall see later, another example of a proper class is the universe of sets $\mathbf{V}$ which can be defined by the formula $x = x$.

[6]Those students with philosophical tendencies may read Penelope Maddy's famous articles *Believing the Axioms, I*, *Believing the Axioms, II* and her book *Defending the Axioms* after completing this course.

[7]We note that some of the set theories we mentioned earlier are capable of talking about classes directly. For example, this can be done in NBG and MK.

0.5. **Notational remarks.** In what follows, our assertions about sets should ideally be written in the language of set theory, having only $\in$ as a non-logical symbol. However, this approach is cumbersome and for convenience we will often expand our language by introducing new non-logical symbols that are abbreviations for certain formulas of set theory. For example, the formula $\neg x \in y$ is abbreviated as $x \notin y$. The reader is expected to keep track of introductions of such abbreviations.

Another notational convenience we shall adopt is to write $\forall z \in x\ \varphi$ instead of $\forall z(z \in x \to \varphi)$ and to write $\exists z \in x\ \varphi$ instead of $\exists z(z \in x \wedge \varphi)$ where φ is a formula in the language of set theory. Finally, we note that parentheses are usually omitted whenever there is no ambiguity.

1. Some axioms of ZFC and their elementary consequences

1.1. **And G said, "Let there be sets"; and there were sets.** We begin our discussion by introducing the axiom which asserts that the universe of sets is not void.

Axiom 1 (The axiom of empty set)**.** *There exists a set with no elements.*

$$\exists x \forall y\ y \notin x$$

A set with no elements will be referred to as an *empty set*. One can ask whether there may be more than one empty set. Unfortunately, we cannot answer this question without additional axioms.

Intuitively speaking, the only feature of sets is to contain certain other sets. Thus one may argue that a set should be completely determined by its elements. This suggests the following axiom.

Axiom 2 (The axiom of extensionality)**.** *Two sets are equal if and only if they have the same elements.*

$$\forall x \forall y\ (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$$

Now we are in a position to prove our first theorem in set theory.

Theorem 2. *There exists a unique set with no elements.*

Proof. Assume to the contrary that x and y are sets with no elements such that $x \neq y$. Then, by the axiom of extensionality, there exists z such that either that $z \in x$ and $z \notin y$, or that $z \notin x$ and $z \in y$. In both cases, we have a contradiction since x and y have no elements. □

From now on, the (unique) empty set with no elements will be denoted by $\emptyset$. At this point, we cannot prove the existence of sets other than the empty set without further axioms. For all we know, the universe of sets could consist of only the empty set.

1.2. **Constructing more sets.** In this section, we introduce several axioms that enable us to define some elementary operations on the universe of sets and construct sets other than the empty set.

Axiom 3 (The axiom of pairing)**.** *For any sets x and y, there exists a set z which consists of the elements x and y.*

$$\forall x \forall y \exists z \forall t(t \in z \leftrightarrow (t = x \vee t = y))$$

In other words, for any sets x and y, the collection $\{x, y\}$ is indeed a set. We shall call this set the *unordered pair* of x and y. Here are two applications of the axiom of pairing.

- By pairing $\emptyset$ with itself, we can now prove that the set $\{\emptyset\}$ exists.
- By pairing the set $\{\emptyset\}$ with $\emptyset$, we can also construct the set $\{\emptyset, \{\emptyset\}\}$.

Next follows an important application of the axiom of pairing. Let x and y be sets. Then, by the axiom of pairing, the sets $\{x\}$ and $\{x, y\}$ both exist. By pairing these sets, we obtain the set $\{\{x\}, \{x, y\}\}$.

Definition 1 (Kuratowski). *The set $\{\{x\}, \{x, y\}\}$ is called the ordered pair of x and y and is denoted by (x, y).*

The reason (x, y) is called the *ordered* pair is easily seen from the next lemma.

Lemma 1. *Let x, y, x', y' be sets. $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$.*

Proof. Left to the reader as an exercise. □

We next introduce an axiom that allows us to collect the elements of elements of a set into a single set.

Axiom 4 (The axiom of union). *For any set x, there exists a set y which consists of exactly the elements of elements of x.*

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists s (s \in x \wedge z \in s))$$

We are used to thinking of union as an operation applied to a collection of sets instead of a single set. In the axiom above, you should think of the set x as the collection of sets whose union is to be taken. In this case, the set y is the union of elements of x. We shall call y simply the *union* of x and denote it by $\bigcup x$. In other words,

$$\bigcup x = \{z : \exists s \in x \ z \in s\}$$

Next follows the definition of the union of two sets. Let x and y be sets. Then, by pairing, the set $\{x, y\}$ exists.

Definition 2. *The set $\bigcup\{x, y\}$ is called the union of x and y and is denoted by $x \cup y$.*

Exercise 1. *Let x and y be sets. Prove that for all z, we have that $z \in x \cup y$ if and only if $z \in x$ or $z \in y$.*

The dual notion of the union of a set is the *intersection* of a set x, which can be defined as follows.

$$\bigcap x = \{z : \forall s \in x \ z \in s\}$$

Exercise 2. *Show that every set belongs to the class $\bigcap \emptyset$. In other words, $\bigcap \emptyset = \mathbf{V}$.*

Note that we do not know yet whether or not the class $\bigcap x$ is indeed a set for every non-empty set x. In order to show this, we shall need the following axiom.

Axiom 5 (The axiom of separation). *Let $\varphi(z, p)$ be a formula in the language of set theory with two variables z and p. For any p and for any x, there exists a set y that consists of elements of x satisfying the property $\varphi(\cdot, p)$.*

$$\forall p \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \varphi(z, p)))$$

We would like to emphasize that the axiom of separation is an axiom *schema* that consists of infinitely many axioms, one for each formula in the language of set theory with two free variables. In each such axiom, you should think of the variable p as a parameter which, when fixed, defines a property $\varphi(\cdot, p)$ of sets. In some textbooks, the axiom of separation is stated for formulas that may have an arbitrary number of parameters. Together with other axioms, one can prove that our formulation of the axiom of separation implies this formulation and vice versa.

One consequence of the axiom of separation is the existence of the intersection of a non-empty set. Let A be a non-empty set, B be an element of A and $\varphi(x, y)$ be the formula $\forall s(s \in y \to x \in s)$. Then, by an instance of the axiom of separation, the class $\{x : x \in B \wedge \varphi(x, A)\}$ forms a set. But this set is precisely $\bigcap A$. Having shown that the intersection of a non-empty set exists, we now define the intersection of two sets.

Definition 3. *The set* $\bigcap\{x, y\}$ *is called the intersection of* x *and* y *and is denoted by* $x \cap y$.

Exercise 3. *Let* x *and* y *be sets. Prove that for all* z, *we have that* $z \in x \cap y$ *if and only if* $z \in x$ *and* $z \in y$.

Two sets x and y are said to be *disjoint* if $x \cap y = \emptyset$. It is trivial to observe[1] that the axiom of separation tells us that the subclass of a set consisting of elements satisfying a certain property is indeed a set, i.e. if a is a set and $\varphi(x)$ is a property of sets, then the class

$$\{x : x \in a \wedge \varphi(x)\} = \{x \in a : \varphi(x)\}$$

is a set. An important consequence of this observation is that the universe of sets $\mathbf{V}$ is a proper class.

Theorem 3. *There does not exist a set which contains all sets, i.e.* $\neg\exists x \forall y\ y \in x$.

Proof. Assume to the contrary that there exist a set U which contains all sets. Then, by separation, there exists a set R such that

$$R = \{x \in U : x \notin x\}$$

But then, since $R \in U$, we have $R \in R \leftrightarrow R \notin R$, which is a contradiction. □

We next introduce some standard operations between sets. Note that for any x and y, the set $\{z \in x : z \notin y\}$ exists by the axiom of separation.

Definition 4. *The set* $\{z \in x : z \notin y\}$ *is called the difference of* x *and* y *and is denoted by* $x - y$.

By taking the union of the sets $x - y$ and $y - x$, we obtain the operation known as the symmetric difference.

Definition 5. *The set* $(x - y) \cup (y - x)$ *is called the symmetric difference of* x *and* y *and is denoted by* $x \triangle y$.

We shall not include here the list of basic properties of the operations introduced so far and refer the reader to any elementary textbook on set theory.

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[1] To derive this from our formulation of the axiom of separation, given a formula $\varphi(x)$ and a set x, apply the axiom of separation to a with using formula $\psi(x, y) :\ \varphi(x) \wedge y = y$.

Before introducing the next axiom, we will need the notion of a subset of a set. Let x and y be sets. The set x is said to be a *subset* of y if every element of x belongs to y. More precisely, x is a subset of y if we have $\forall z(z \in x \rightarrow z \in y)$. We shall write $x \subseteq y$ if x is a subset of y; and write $x \subsetneq y$ if $x \subseteq y$ and $x \neq y$. In the latter case, x is said to be a *proper subset* of y. The reader can easily verify that for all x, y, z and non-empty w, we have that

- $\emptyset \subseteq x$ and $x \subseteq x$,
- $\{t : t \in x \wedge \varphi(t)\} \subseteq x$ for any property φ,
- $(x \subseteq y \wedge y \subseteq x) \leftrightarrow x = y$,
- $(x \subseteq y \wedge y \subseteq z) \rightarrow x \subseteq z$,
- $\bigcap w \subseteq \bigcup w$
- $y \in x \rightarrow \bigcap x \subseteq y \subseteq \bigcup x$

The next axiom guarantees the existence of the set of all subsets of a set.

Axiom 6 (The axiom of power set). *For any set x there exists a set y that consists of all subsets of x.*

$$\forall x \exists y \forall z(z \subseteq x \leftrightarrow z \in y)$$

The set $\{z : z \subseteq x\}$ is called the *power set* of x and is denoted by $\mathcal{P}(x)$. When we introduce infinite sets, the power set of an infinite set will be a central object to study, some fundamental properties of which cannot be decided[2] via the axioms of ZFC.

Exercise 4. *Prove that for any set x, the set $\mathcal{P}(x)$, together with the binary operation $\triangle$, forms an abelian group in which every non-identity element has order 2.*

Exercise 5. *Prove that for any non-empty set x, the set $\mathcal{P}(x)$ forms a commutative ring in which every element equals its square, where the binary operations for addition and multiplication are $\triangle$ and $\cap$ respectively.*

Axioms 1-6 are far from being complete to serve as a foundation of mathematics. For once, we cannot prove the existence of an "infinite" set without further axioms. Before introducing more axioms, in the next section, we are going to study how various mathematical concepts can be represented by sets.

[2]The proper term for this phenomenon is *independence*. A sentence φ is said to be *independent* of ZFC in the case that neither φ nor $\neg\varphi$ can be proven from ZFC.

References

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