Week 2

2. From Pairs to Products

2.1. **Relations.** One of the most fundamental concepts in mathematics is the concept of a relation and we begin this section by introducing the set-theoretic definition of a relation.

Definition 6. *A set R is said to be a (binary) relation if it consists of ordered pairs, i.e.* $\forall z \in R\ \exists x\ \exists y\ z = (x, y)$

Given a relation R, using the union and separation axioms, we can form the sets

$$dom(R) = \{a : \exists b\ (a, b) \in R\} \text{ and } ran(R) = \{b : \exists a\ (a, b) \in R\}$$

These sets are called the *domain* of R and the *range* of R respectively. Intuitively speaking, one can think of the relation R as a "rule" that *relates* certain sets in $dom(R)$ to certain sets in $ran(R)$. If R is a relation and $(a, b) \in R$, then one says that "a is in relation R with b" or "a is related to b under the relation R". It is common practice to write aRb instead of $(a, b) \in R$.

Definition 7. *Let A be a set and R be a relation. The image of the set A under the relation R is the set*

$$\{y : \exists x \in A\ xRy\}$$

and is denoted by $R[A]$.

Definition 8. *Let B be a set and R be a relation. The inverse image of the set B under the relation R is the set*

$$\{x : \exists y \in B\ xRy\}$$

and is denoted by $R^{-1}[B]$.

Exercise 6. *Let a, b, c be sets. Show that the set*

$$\{(a, b), (a, a), (c, a), (b, b)\}$$

is a relation and find its domain and range. Then find the image and the inverse image of the set $\{a, c\}$ under this relation.

We will not list many exercises regarding these basic notions and refer the reader who wish to practice to any elementary textbook on set theory.

Definition 9. *Let R be a relation. The inverse relation of R is the set*

$$\{(b, a) : (a, b) \in R\}$$

and is denoted by R^{-1}.

At this point, one may object that our notation creates an ambiguity since it is not clear whether the set $R^{-1}[A]$ denotes the image of A under R^{-1} or the inverse image of A under R. This objection is resolved by the following exercise which justifies our usage of the notation $R^{-1}[A]$ to denote both sets.

Lemma 2. *Let R be a relation and A be a set. Show that the image of A under the relation R^{-1} is the same as the inverse image of A under R.*

Proof. Left to the reader as an exercise. □

Next comes the definition of the cartesian product of two sets. One can easily check that if $a \in A$ and $b \in B$, then $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. Thus, given two sets A and B, using the axioms introduced so far, we can form the set of all ordered pairs whose first components belong to A and whose second components belong to B.

Definition 10. *Let A and B be sets. The cartesian product of A and B is the set*

$$\{(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B)) : a \in A \wedge b \in B\}$$

and is denoted by $A \times B$.

Definition 11. *Let R be a relation and A, B be sets. The relation R is said to be*

- *a relation from A to B if $R \subseteq A \times B$;*
- *a relation on A if $R \subseteq A \times A$.*

In particular, every relation R is a relation from $dom(R)$ to $ran(R)$. However, notice that a relation R being from the set A to the set B does not necessarily mean that $A = dom(R)$ and $B = ran(R)$.

Definition 12. *Let R and S be relations. Then the composition of S and R is the relation*

$$\{(a, b) : \exists c \; (a, c) \in R \; \wedge \; (c, b) \in S\}$$

and is denoted by $S \circ R$.

The notion of composition of two relations is most frequently used when both relations are a special type of relations called *functions*. On the other hand, some useful properties of the operation $\circ$ still hold for arbitrary relations.

Exercise 7. *Let R and S be relations. Prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.*

Exercise 8. *Let R, S and T be relations. Prove that $T \circ (S \circ R) = (T \circ S) \circ R$.*

Before introducing the notion of a function, we would like to mention two relations defined on an arbitrary set, which will be useful in later sections.

Definition 13. *Let A be a set. The membership relation on A is the relation*

$$\{(a, b) \in A \times A : a \in b\}$$

and is denoted by $\in_A$.

Definition 14. *Let A be a set. The identity relation on A is the relation*

$$\{(a, b) \in A \times A : a = b\}$$

and is denoted by Δ_A.

The notion of a binary relation can be generalized to that of an n-ary relation, which is a relation that holds or not holds between n many sets. However, the most convenient way to define n-ary relations requires the construction of natural numbers and the n-fold cartesian product of sets. Consequently, we postpone the definition of an n-ary relation until Section 3.

2.2. **Functions.** Recall that one can think of a relation R as a "rule" that *relates* certain sets in $dom(R)$ to certain sets in $ran(R)$. If this "rule" happens to *uniquely assign* each set $dom(R)$ to a certain set in $ran(R)$, then the corresponding relation is said to be a function. More precisely,

Definition 15. *Let R be a relation. The relation R is said to be a function if*

$$\forall a \forall b \forall c\ (aRb \wedge aRc \to b = c)^{1}$$

The simplest example of a function is the empty set $\emptyset$. Notice that the definition of a function vacuously holds for the empty set for it has not elements.

Definition 16. *Let R be a relation and A, B be sets. The relation R is said to be a function from A to B if R is a function, $dom(R) = A$ and $ran(R) \subseteq B$. In this case, R is said to have domain A and codomain B.*

An important point to realize is that, according to this definition, the *very same set* can be considered as a function from the same domain to different codomains. For this reason, whenever it is necessary, we shall always specify the codomain of a function.

Definition 17. *Let R be a function and $x \in dom(R)$. The (necessarily) unique element $y \in ran(R)$ for which $(x, y) \in R$ is called the value of R at x.*

Before we proceed, we introduce some notation regarding functions. From now on, we shall write $R : A \to B$ whenever we need to denote a set R which is a function from the set A to the set B. The value of R at a will be denoted by $R(a)$.

We would also like to emphasize that functions *are* relations and hence all notions introduced for relations so far are applicable to functions as well. We next introduce the notion of a bijective function, which will be central to our study of infinite sets.

Definition 18. *Let $f : A \to B$ be a function with domain A and codomain B. Then f is said to be*

- *one-to-one (or injective) if for all $x, y \in A$ we have $f(x) = f(y) \to x = y$.*
- *onto (or surjective) if $ran(f) = f[A] = B$.*
- *one-to-one correspondence (or bijection) if it is both one-to-one and onto.*

Observe that surjectivity and bijectivity of a function both depend on the specified codomain, unlike injectivity. Consequently, the *very same set* can be surjective for some codomain and not surjective for some other codomain. The following exercise illustrates this fact.

Exercise 9. *Prove that the empty set $\emptyset$ is a bijection as a function from $\emptyset$ to $\emptyset$ and not a surjection as a function from $\emptyset$ to $\{\emptyset\}$.*

The notion of injectivity can be generalized to arbitrary relations. More specifically, a relation R is said to be injective if and only if $\forall x \forall y \forall z\ (xRz \wedge yRz \to x = y)$. It is easily seen that a relation being injective is equivalent to its inverse relation being a function and vice versa. Consequently, we have the following fact.

Lemma 3. *Let R be a relation. Then the relation R is an injective function if and only if the inverse relation R^{-1} is an injective function.*

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[1]Some authors call a relation satisfying this property *well-defined*. In this terminology, functions are simply relations that are well-defined.

Proof. Let R be a relation that is an injective function. Since R is injective, $\forall x \forall y \forall z\ (xRz\ \wedge yRz \rightarrow x = y)$ and hence $\forall x \forall y \forall z\ (zR^{-1}x\ \wedge zR^{-1}y \rightarrow x = y)$, which is exactly what it means for R^{-1} to be a function. Since R is a function, $\forall x \forall y \forall z\ (xRy\ \wedge xRz \rightarrow y = z)$ and hence $\forall x \forall y \forall z\ (yR^{-1}x\ \wedge zR^{-1}x \rightarrow y = z)$, which is exactly what it means for R^{-1} to be injective. By changing the roles of R and R^{-1}, the proof of the right-to-left direction can be done similarly. □

One can easily verify that any subset of a function is itself a function. This observation suggests the following definition.

Definition 19. *Let f be a function and A be a set. The restriction of f to A is the function*

$$\{(a, b) \in f : a \in A\}$$

and is denoted by $f \restriction A$.

Definition 20. *Two functions f and g are said to be compatible if $f(x) = g(x)$ for all $x \in dom(f) \cap dom(g)$.*

In other words, two functions are compatible if the values they take agree at every element in the intersection of their domains. The following lemma shows that two functions being compatible is equivalent to their union being a function.

Lemma 4. *Let f and g be functions. Then f and g are compatible if and only if $f \cup g$ is a function.*

Proof. For the left-to-right direction, assume that f and g are compatible functions. Clearly $f \cup g$ is a relation. We want to show that for all x, y, z if $(x, y) \in f \cup g$ and $(x, z) \in f \cup g$, then $y = z$. Let $(x, y) \in f \cup g$ and $(x, z) \in f \cup g$. There are four cases.

- If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$ since f is a function.
- If $(x, y) \in g$ and $(x, z) \in g$, then $y = z$ since g is a function.
- If $(x, y) \in f$ and $(x, z) \in g$, then $y = z$ since f and g are compatible.
- If $(x, y) \in g$ and $(x, z) \in f$, then $y = z$ since f and g are compatible.

Thus, $f \cup g$ is a function. For the converse direction, assume that $f \cup g$ is a function. Let $x \in dom(f) \cap dom(g)$. Since f and g are functions, there exist y and z such that $f(x) = y$ and $g(x) = z$. Then, clearly we have $(x, y), (x, z) \in f \cup g$. However, by assumption $f \cup g$ is a function and hence $f(x) = y = z = g(x)$. Thus, f and g are compatible. □

The following exercise shows that the lemma above can be generalized to arbitrary collections of compatible functions.

Exercise 10. *Let S be a set such that elements of S are functions which are pairwise compatible. Show that $\bigcup S$ is a function with domain $\bigcup\{dom(f) : f \in S\}$.*

The next lemma shows that the class of functions are closed under the operation of composition.

Lemma 5. *Let f and g be functions. Then the composition $g \circ f$ is a function.*

Proof. Let x, y, z be sets such that $(x, y) \in g \circ f$ and $(x, z) \in g \circ f$. We want to show that $y = z$. By definition of composition, there exist y' and z' such that $(x, y') \in f$ and $(y', y) \in g$; and $(x, z') \in f$ and $(z', z) \in g$. Since f is a function, $(x, y') \in f$ and $(x, z') \in f$ implies that $y' = z'$. Since g is a function and $y' = z'$, $(y', y) \in g$ and (z', z) implies that $y = z$. □

Exercise 11. *Let f and g be functions. Show that the domain of the function $g \circ f$ is $dom(f) \cap f^{-1}[dom(g)]$ and that $(g \circ f)(x) = g(f(x))$ for all x in this domain.*

Given two sets x and y, a function f from x to y is an element of $\mathcal{P}(x \times y)$ and hence we can form the set of all functions from x to y

$$\{f \in \mathcal{P}(x \times y) : \forall a \forall b \forall c\ ((a, b) \in f \wedge (a, c) \in f) \rightarrow b = c\}$$

using the axioms introduced so far. From now on, the set of all functions from the set x to the set y will be denoted by ${}^{x}y$. Some authors use the notation y^x to denote this set, however, we reserve this notation for exponentiation on ordinal and cardinal numbers in order to avoid ambiguities.

2.3. **Products and sequences.** Next will be discussed how to define the product of an arbitrary collection of sets.

Recall that when we defined the cartesian product $A \times B$ of two sets, the order of the sets A and B mattered. Even though the cartesian product $B \times A$ is in a natural bijection with the cartesian product $A \times B$, these are different objects in the universe of sets. Therefore, in order to generalize the concept of cartesian product to arbitrarily many sets, we first need to *label* the sets whose product is to be taken. This labeling can be done through some function.

Let J be a set which contains the sets whose product is to be taken and possibly other sets. Let $F : I \rightarrow J$ be an arbitrary function. We will refer to the function F an *indexed system of sets* with the *index set* I. Here we think of the set $i \in I$ as the *label* of the set $F(i)$ for all $i \in I$. While talking about indexed systems of sets, it is customary to write F_i instead of $F(i)$ and write $\{F_i\}_{i \in I}$ instead of $F[I]$, which we will also refer to as an indexed system of sets.

Definition 21. *Let $\{F_i\}_{i \in I}$ be an indexed system of sets with the index set I. The product of the indexed system $\{F_i\}_{i \in I}$ is the set*

$$\{f : I \rightarrow \bigcup \{F_i\}_{i \in I} \mid \forall i \in I\ f(i) \in F_i\}$$

and is denoted by $\prod_{i \in I} F_i$.

In other words, the product $\prod_{i \in I} F_i$ is the set of all functions f with domain I such that $f(i) \in F_i$ for all $i \in I$. One usually denotes a set $f \in \prod_{i \in I} F_i$ using the sequence notation $(f(i))_{i \in I}$ since f can be considered as a *sequence* which takes values in F_i at each component i.

Indeed, this is exactly how we define sequences over arbitrary sets. Let $\{S_i\}_{i \in I}$ be an indexed family of sets for some index set I such that $S_i = S$ for all $i \in I$. An element f of the product $\prod_{i \in I} S$ is called a *sequence* over S with the index set I and is denoted by $(f(i))_{i \in I}$.

We have not constructed the natural numbers yet. For the following exercises, the reader should assume[2] that $0 = \emptyset$, $1 = \{0\}$ and $2 = \{0, 1\}$.

Exercise 12. *Let $\{A_i\}_{i \in 2}$ be an indexed system of set with the index set 2. Show that the map $f : \prod_{i \in 2} A_i \rightarrow A_0 \times A_1$ given by $f(g) = (g(0), g(1))$ is a bijection.*

[2]Since the notions introduced so far are enough to carry out the construction of natural numbers, the curious reader may read the first subsection of Section 4 for a precise construction at this point.

Consequently, the notion of cartesian product can be considered as a special case of the product of an indexed system of sets. As the reader may guess, once we define natural numbers, the cartesian product of sets $A_1, \ldots, A_n$ will simply be defined as the product of the indexed family $\{A_i\}_{i \in n}$.

The next exercise shows that there is a natural bijection between the power set of any set X and the product of an appropriately chosen system with index set X.

Exercise 13. *Let X be any set. Show that ${}^X 2 = \prod_{i \in X} 2$ and that the map f from $\prod_{i \in X} 2$ to $\mathcal{P}(X)$ given by $f(g) = \{x \in X : g(x) = 1\}$ is a bijection.*

We next focus on a seemingly simple question. Assume that $\{A_i\}_{i \in I}$ is an indexed system of sets such that $A_i \neq \emptyset$ for all $i \in I$. Is the product $\prod_{i \in I} A_i$ necessarily non-empty?

Given a fixed finite set I such as $I = \{0, 1, 2\}$, the reader can prove as an exercise that the answer is affirmative. As can be seen from the exercise above, the answer is also affirmative when $A_i = 2$.

However, it is not clear whether or not $\prod_{i \in I} A_i \neq \emptyset$ for all indexed system of sets $\{A_i\}_{i \in I}$ with $A_i \neq \emptyset$ for all $i \in I$. It turns out that this statement cannot be proven or disproven from Axioms 1-6 plus the axioms of Infinity, Replacement and Foundation. In the next section, we shall introduce an axiom that settles this question.

Before we conclude this subsection, we would like to mention two notations regarding indexed systems of sets. From now on, given an indexed system of set $\{A_i\}_{i \in I}$, we shall denote the sets $\bigcup \{A_i\}_{i \in I}$ and $\bigcap \{A_i\}_{i \in I}$ by $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ respectively.

References

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[2] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513

[3] Kenneth Kunen, *Set theory*, Studies in Logic (London), vol. 34, College Publications, London, 2011. MR 2905394

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