Week 3

2.4. **To choose or not to choose.** In this section, we shall introduce the axiom of choice, one of the most famous axioms of ZFC. For historical reasons, the axiom of choice became so famous that the letter **C** of ZFC stands for this axiom.

There are literally dozens of equivalent formulations of the axiom of choice. Below, we introduce the formulation which states that the product of an indexed system of non-empty sets is non-empty. Some equivalent formulations of this axiom will be mentioned in later sections.

Axiom 7 (The axiom of choice). *For all sets I and for all indexed systems of sets $\{A_i\}_{i\in I}$ with $A_i \neq \emptyset$ for all $i \in I$, the product $\prod_{i\in I} A_i$ is non-empty*[3].

Recall that an element f of the product $\prod_{i\in I} A_i$ is a function with $f(i) \in A_i$ for all $i \in I$. Loosely speaking, the function f *chooses* one element from each A_i. In this sense, the axiom of choice allows us to "simultaneously choose" an element from each set in a set of non-empty sets. The reader who does not feel comfortable with indexed systems may find the following lemma more intuitive.

Lemma 6. *Let M be a set whose elements are non-empty sets. Then there exists a function $f : M \to \bigcup M$ such that $f(x) \in x$ for all $x \in M$.*

Proof. Notice that every set can be indexed by itself through identity function. More precisely, let $M = I$ and $\{M_i\}_{i\in I}$ be the indexed system of sets with $M_i = i$. Then, since $M_i \neq \emptyset$ for all $i \in I$, by the axiom of choice, there exists $f : I \to \bigcup M$ such that $f(i) \in i$ for all $i \in I$, which is precisely what we wanted to prove. □

It is easily seen that the axiom of choice is implied by the statement of the lemma above together with Axioms 1-6.

3. Equivalence Relations and Order Relations

In this section, we will learn several important types of relations. In order to provide nice examples, we shall assume throughout this chapter that the set of natural numbers $\mathbb{N}$ is already constructed together with its usual arithmetical operations and relations. The reader who wish to see a precise construction of these may read Section 4.

3.1. **Equivalence relations, partitions and transversals.** We begin by introducing the notion of an equivalence relation, which is frequently used when different mathematical objects are needed to be considered "the same" for various purposes.

Definition 22. *Let X be a set and E be a relation on X, i.e. $E \subseteq X \times X$. The relation E is said to be an equivalence relation if it is*

- *reflexive, i.e. for all $x \in X$ we have xEx,*
- *symmetric, i.e. for all $x, y \in X$ we have $xEy \to yEx$, and*
- *transitive, i.e. for all $x, y, z, \in X$ we have $(xEy \wedge yEz) \to xEz$.*

[3]Unlike the other axioms introduced up to now, we shall not attempt to write the axiom of choice in the language of set theory since it will require **too much** space. A curious reader may attempt to do this in his or her free time!

For example, the identity relation Δ_X is an equivalence relation on X for any set X. Indeed, we have $\Delta_X \subseteq E$ for all equivalence relations E on X. Note that the empty set is an equivalence relation on itself and, indeed, is the unique equivalence relation on the empty set, which will be referred to as the *empty relation*.

Exercise 14. *Let X be any set and define the relation $E \subseteq X \times X$ by*

$$xEy \leftrightarrow \text{ There exists a bijection between } x \text{ and } y$$

for all $x, y \in X$. Show that E is an equivalence relation on X.

Exercise 15. *Let X be a non-empty set such that the elements of X are equivalence relations on some fixed set Y. Show that if we have $E \subseteq F$ or $F \subseteq E$ for all $E, F \subseteq X$, then $\bigcup X$ is an equivalence relation on Y.*

We next introduce some terminology to talk about elements that are related to each other under some equivalence relation.

Definition 23. *Let X be a set, E be an equivalence relation on X and $x \in X$. The equivalence class of x modulo E is the set*

$$\{y \in X : yEx\}$$

and is denoted by $[x]_E$.

Given an equivalence relation E on some set X, the sets of the form $[x]_E$ for some $x \in X$ are referred to as *E-equivalence classes*. Two elements in the same E-equivalence class are said to be *E-equivalent*. Observe that, according to this definition, the empty relation on the empty set has no equivalence classes, since we require each equivalence class of E to be of the form $[x]_E$ for *some* $x \in X$.

The following lemma shows that equivalence classes of an equivalence relation on a non-empty set are either identical or disjoint.

Lemma 7. *Let X be a non-empty set and E be an equivalence relation X. Then for all $x, y \in X$ we have that either $[x]_E = [y]_E$ or $[x]_E \cap [y]_E = \emptyset$.*

Proof. Let $x, y \in X$ and assume that $[x]_E \cap [y]_E \neq \emptyset$. Then there exists $z \in X$ such that zEx and zEy and hence xEy by symmetry and transitivity of E. Now pick $w \in [x]_E$, then wEx and xEy and hence $w \in [y]_E$. This shows that $[x]_E \subseteq [y]_E$. By a symmetric argument, one can show that $[y]_E \subseteq [x]_E$ and hence $[x]_E = [y]_E$. This shows that $[x]_E = [y]_E$ or $[x]_E \cap [y]_E = \emptyset$. Since $[x]_E$ and $[y]_E$ are both non-empty, both cases cannot occur simultaneously. Therefore, either $[x]_E = [y]_E$ or $[x]_E \cap [y]_E = \emptyset$. □

Exercise 16. *Let E be the equivalence relation on $\mathbb{N} \times \mathbb{N}$ defined by*

$$(p, q)E(r, s) \leftrightarrow p + s = q + r$$

Show that E is an equivalence relation and find the equivalence class $[(2, 0)]_E$.

Next, we define a partition of a set and the quotient set of an equivalence relation, notions which will be related through a fundamental theorem.

Definition 24. *Let X be a set and E be an equivalence relation on X. The quotient set of X with respect to E is the set*

$$\{[x]_E : x \in X\}$$

which consists of the equivalence classes of E and is denoted by X/E.

Definition 25. *Let X be a set. Then a subset $S \subseteq \mathcal{P}(X)$ is said to be a partition of X if*

- *elements of S are non-empty, i.e. for all $A \in S$ we have $A \neq \emptyset$.*
- *distinct elements of S are disjoint, i.e. for all $A, B \in S$ if $A \neq B$ then $A \cap B = \emptyset$, and*
- *the union of S is the set X, i.e. $\bigcup S = X$*

According to this definition, the empty set has a unique partition, which is the empty set itself. We remark that, in some textbooks, the notion of a partition of a set may not be defined for the empty set. The following theorem is an easy consequence of Lemma 7.

Theorem 4. *Let X be a set and E be an equivalence relation on X. Then X/E is a partition of X.*

Proof. Left to the reader as an exercise. □

In other words, every equivalence relation on a set induces a partition. It turns out that the converse is also true, i.e. every partition is induced by some equivalence relation.

Lemma 8. *Let S be a partition of a set X. Then the relation E_S defined by*

$$xE_Sy \leftrightarrow \exists D \in S\ (x \in D \wedge y \in D)$$

is an equivalence relation on X.

Proof. The claim clearly holds when X is the empty set. Now, assume that X is non-empty and let $x, y, z \in X$ be such that xE_Sy and yE_Sz. Then, by definition, there exist $C, D \in S$ such that $x \in C$ and $y \in C$; and, $y \in D$ and $z \in D$. However, since S is a partition, $C \cap D \neq \emptyset$ implies that $C = D$. But then $x \in C$ and $z \in C$, which implies that $(x, z) \in E_S$. Hence, E_S is transitive.

Showing that E_S is reflexive and symmetric is left as an exercise to the reader. □

It is not difficult to check that different partitions of the same set induce different equivalence relations, i.e. if $S \neq D$ are partitions of a set X, then $E_S \neq E_D$. Consequently, we have the following theorem, which some authors refer to as the fundamental theorem of equivalence relations.

Theorem 5. *Let X be any set. The map f from the set of partitions of X to the set of equivalence relations on X given by $f(S) = E_S$ where*

$$E_S = \{(x, y) \in X \times X : \exists D \in S\ x \in D \wedge y \in D\}$$

is a bijection.

Next will be discussed the notion of a transversal of an equivalence relation.

Definition 26. *Let X be a set and E be an equivalence relation on X. A subset $T \subseteq X$ is said to be a transversal (or, a set of representatives) for the equivalence relation E if for every $x \in X$ there exists $y \in X$ such that $[x]_E \cap T = \{y\}$.*

In other words, a transversal for an equivalence relation is a set that contains exactly one element from each equivalence class. Given a transversal $T \subseteq X$ for an equivalence relation $E \subseteq X \times X$, the unique set in $[x]_E \cap T$ is called the *representative* of the equivalence class $[x]_E$.

By the axiom of choice, since we can simultaneously pick one element from each set in a set of non-empty sets, transversals exist for non-empty equivalence relations. More precisely, we have the following theorem.

Theorem 6. *Let X be a non-empty set and E be an equivalence relation on X. Then there exists a transversal T for E.*

Proof. Since X is non-empty, the partition X/E is a set of non-empty sets. By Lemma 6, there exists a function $f : X/E \to X$ such that $f(x) \in x$ for all $x \in X/E$. Notice that f picks exactly one element from each equivalence class. Thus, the range $f[X]$ of this function is a transversal for E. □

In the next subsection, we will solve a seemingly-impossible-to-solve puzzle using the existence of transversals. For this puzzle, we will assume the familiarity of the reader with *finite* and *countably infinite* sets. The reader who does not feel comfortable using these concepts at this point may skip the next subsection and read it after completing Section 5.

3.2. **A Game of Thrones, Prisoners and Hats.** After the battle of the Blackwater, King Joffrey of Westeros captured countably infinitely many soldiers of Stannis Baratheon as his prisoners and put the set of prisoners in a bijection with the set of natural numbers. In other words, every prisoner is uniquely labeled by some natural number.

King Joffrey, who has been known for his cruel games, explained to the prisoners that they would be executed the next morning, unless they succeed in the following game that will take place before the execution:

The prisoners will be standing in a straight line in such a way that every prisoner will be able to see the infinitely many prisoners whose labels are greater than his label, i.e. the prisoners are standing on the number line facing the positive direction.

Then each prisoner will be *randomly* given a hat that is either red or blue. The prisoners can see *all* the hats in front of them but cannot see their own hats. Moreover, they are not allowed to move or communicate in any way. After all the hats are distributed, each prisoner will be asked to guess the color of his own hat and write his guess in a piece of paper.

The rules of the game are as follows: If there are only finitely many prisoners who guess wrong, then all the prisoners are set free. Otherwise, they all are executed.

Once the rules are explained to the prisoners, they immediately think that it is impossible to succeed since they are in no position to obtain information about the colors of their own hats by looking at the colors of other prisoners' hats.

Tyrion Lannister, who is not fond of King Joffrey and who has studied set theory in his youth, decides to help the prisoners. Soldiers of Stannis are so smart that they have been known to memorize infinite amount of information if necessary. Knowing this fact, Tyrion realizes that he can set the prisoners free.

Theorem 7. *There exists a survival strategy for the prisoners.*

Proof. Let E be the equivalence relation on the set ${}^{\mathbb{N}}2$ defined by

$$fEg \leftrightarrow \exists m \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n \geq m \to f(n) = g(n))$$

In other words, two functions from $\mathbb{N}$ to 2 are E-equivalent if and only if they take the same values at sufficiently large natural numbers. We skip the details of checking that E is indeed an equivalence relation and leave this as an exercise to the reader.

By Theorem 6, there exists a transversal $T \subseteq {}^{\mathbb{N}}2$ for the equivalence relation E. Let $F : {}^{\mathbb{N}}2 \to {}^{\mathbb{N}}2$ be the function defined by

$$F(f) = \bigcup [f]_E \cap T$$

for all $f \in {}^{\mathbb{N}}2$, that is, the function F sends each f to the unique element of T which is E-equivalent to f. The survival strategy of the prisoners is as follows. When asked the color of his hat, Prisoner n first constructs the function $f : \mathbb{N} \to 2$ defined by

- $f(i) = 0$ for all $i \leq n$,
- $f(i) = 0$ if prisoner i has red hat and $i > n$, and
- $f(i) = 1$ if prisoner i has blue hat and $i > n$

In other words, Prisoner n first "encodes" the colors of the hats into a function from $\mathbb{N}$ to 2, assuming that the colors of the hats he does not see are all red. Then he guesses red if $(F(f))(n) = 0$ and guesses blue if $(F(f))(n) = 1$.

We claim that the prisoners survive if they use this strategy. To see this, let $g : \mathbb{N} \to 2$ be the function that encodes the actual state of the hats after the game starts, i.e. $g(i) = 0$ if and only if the hat of Prisoner i is red.

Let $h : \mathbb{N} \to 2$ be the unique function such that $h \in T$ and hEg. By construction, Prisoner i will guess the color of his hat based on the value $h(i)$, i.e. he guesses red if $h(i) = 0$ and blue if $h(i) = 1$. However, by definition, there exists $m \in \mathbb{N}$ such that for all $n \geq m$ we have $h(n) = g(n)$. This means that Prisoner n will guess the color of his hat correctly for all $n \geq m$. Therefore, the prisoners survive. □

3.3. **Order relations.** In this subsection, we will learn about order relations, which are frequently used in mathematics when different mathematical objects are needed to be "compared" for various purposes.

Definition 27. *Let X be a set and E be a relation on X, i.e. $E \subseteq X \times X$. The relation E is said to be a (partial) order relation if it is*

- *reflexive, i.e. for all $x \in X$ we have xEx,*
- *anti-symmetric, i.e. for all $x, y \in X$ we have $(xEy \wedge yEx) \to x = y$, and*
- *transitive, i.e. for all $x, y, z, \in X$ we have $(xEy \wedge yEz) \to xEz$.*

We shall often use the symbols $\leq$ or $\preccurlyeq$ to denote various partial order relations and read $x \leq y$ as "x is less than or equal to y". The reader should keep in mind that, depending on the context, relations denoted by these symbols may have nothing to do with their usual intended meaning on various number systems.

Exercise 17. *Let $\preccurlyeq$ be the relation on $\mathbb{N}^+ = \{k \in \mathbb{N} : k \neq 0\}$ defined by*

$$x \preccurlyeq y \leftrightarrow \exists k \in \mathbb{N}^+ \; y = k \cdot x$$

for all $x, y \in \mathbb{N}$. Show that $\preccurlyeq$ is a partial order relation.

Given a partial order relation $\leq$ on some set X, it will sometimes be more convenient to work with the relation $<$ defined by $x < y \to x \leq y \; \wedge x \neq y$ for all $x, y \in X$. It turns out that those relations that are obtained from partial order relations in this way are exactly those that are transitive and asymmetric.

Definition 28. *Let X be a set and E be a relation on X, i.e. $E \subseteq X \times X$. The relation E is said to be a strict (partial) order relation if it is*

- *asymmetric, i.e. for all $x, y \in X$ we have $xEy \to \neg yEx$, and*
- *transitive, i.e. for all $x, y, z, \in X$ we have $(xEy \wedge yEz) \to xEz$.*

Lemma 9. *Let F be a strict partial order relation on X and E be the relation on X defined by $xEy \leftrightarrow xFy \ \vee \ x = y$. Then E is a partial order relation on X.*

Proof. E is clearly reflexive since $x = x$ for all $x \in X$. Let $x, y \in X$ such that xEy and yEx. Then, by definition, we have $xFy \ \vee \ x = y$ and $yFx \ \vee \ y = x$. Since F is asymmetric, it cannot be that xFy and yFx. Thus, $x = y$ and hence E is anti-symmetric. To see that E is transitive, let $x, y, z \in X$ such that xEy and yEz. Then, by definition, we have $xFy \ \vee \ x = y$ and $yFz \ \vee \ y = z$. If $x = y$, then xEz. Otherwise, xFy and hence xFz by transitivity of F, which implies that xEz. Thus, E is transitive. This completes the proof that E is a partial order. □

Lemma 10. *Let E be a partial order relation on X and F be the relation on X defined by $xFy \leftrightarrow xEy \ \wedge \ x \neq y$. Then F is a strict partial order relation on X.*

Proof. Let $x, y \in X$ such that xFy. Then, by definition, $xEy \ \wedge \ x \neq y$. If it were the case that yFx, then we would have yEx which would imply $x = y$ by anti-symmetry of E, which gives a contradiction. Thus, $\neg yFx$ and hence F is asymmetric. To see that F is transitive, let $x, y, z \in X$ such that xFy and yFz. Then, by definition, $xEy \wedge x \neq y$ and $yEz \wedge y \neq z$. By transitivity of E, we have xEz. If it were the case that $x = z$, then xEy and yEz would imply $x = y$, which is a contradiction. Thus, $x \neq z$ and hence xFz, which completes the proof that F is a strict partial order. □

From now on, whenever we mention the *induced* strict partial order relation $<$ of a partial order relation $\leq$ or the *induced* partial order relation $\leq$ of a strict partial order relation $<$, the reader should understand that $x \leq y \leftrightarrow x < y \ \vee x = y$ and $x < y \leftrightarrow x \leq y \ \wedge \ x \neq y$.

Given a partial order relation $\leq$ on some set, we say that two elements a and b are said to be *comparable* (with respect to $\leq$) if $a \leq b$ or $b \leq a$. Similarly, given a strict partial order relation $<$, two elements a and b are said to be *comparable* (with respect $<$) if $a = b$ or $a < b$ or $b < a$. If two elements are not comparable, then they are called *incomparable*. Partial orders in which any two elements are comparable will be of special importance to us.

Definition 29. *Let $\leq$ be a partial order relation on a set X. The relation $\leq$ is said to be a linear order relation if for all $a, b \in X$, the elements a and b are comparable (with respect to $\leq$).*

Definition 30. *Let $<$ be a strict partial order relation on a set X. The relation $<$ is said to be a strict linear order relation if for all $a, b \in X$, the elements a and b are comparable (with respect to $<$).*

Exercise 18. *Let X be a set that contains at least two elements. Show that the relation E on $\mathcal{P}(X)$ given by*

$$xEy \leftrightarrow x \subseteq y$$

for all $x, y \in \mathcal{P}(X)$ is a partial order relation which is not a linear order.

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