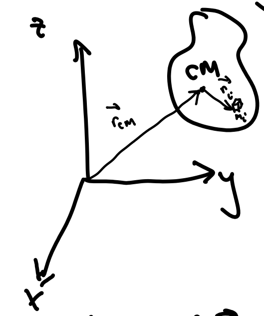
Consider any point and any axis



$$\vec p_i = m_i \vec v_i = m_i \frac{d}{dt}\left(\vec r_{cm} + \vec r_i'\right)$$

$$= m_i \vec V_{cm} + m_i \frac{d\vec r_i'}{dt}$$

$$\vec L = \sum_i m_i \left(\vec r_{cm} + \vec r_i'\right) \times \vec V_{cm} + \sum_i m_i \left(\vec r_{cm} + \vec r_i'\right) \times \frac{d\vec r_i'}{dt}$$

$$\vec L = \left(\sum_i m_i\right) \vec r_{cm} \times \vec V_{cm} + \left(\sum m_i \vec r_i'\right) \times \vec V_{cm} + \vec r_{cm} \times \frac{d}{dt}\left(\sum_i m_i \vec r_i'\right) + \sum_i m_i \vec r_i' \times \frac{d\vec r_i'}{dt}$$

Angular Momentum With Respect to Any Point

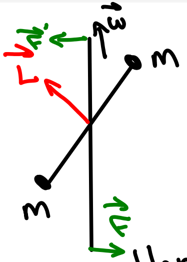
$$\vec{L} = \left(\sum_i m_i\right)\vec{r}_{cm}\times\vec{v}_{cm} + \cancelto{0}{\left(\sum_i m_i\vec{r}_i\right)}\times\vec{v}_{cm}$$
$$+\,\vec{r}_{cm}\times\frac{d}{dt}\cancelto{0}{\left(\sum_i m_i\vec{r}_i\right)} + \sum m_i\,\vec{r}_i\times\frac{d\vec{r}_i}{dt}$$

$$\vec{L} = M\vec{r}_{cm}\times\vec{v}_{cm} + \sum_i \vec{r}_i\times\left(m_i\frac{d\vec{r}_i}{dt}\right)$$

$$\vec{L} = \vec{r}_{cm}\times\vec{P}_{cm} + \vec{L}_{cm}$$

The first term is the angular momentum of CM, the second term is the ang. mom relative to the CM

Rotational Imbalance



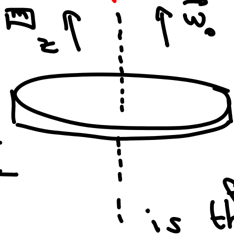
If the system is rotating around the shown axis, $\vec{L}$ is also rotating

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

Hence we need a torque. At the shown instant, $\vec{\tau}$ should be out of the board. The necessary torque can be provided by the forces shown in green

Q: Why is it necessary to adjust the balance of car wheels?

Example



Q: A mass m is dropped on a disk that is initially rotating with angular velocity $\vec{\omega}_0$. If it rests at a distance d from the rotation access, what is the final angular speed?

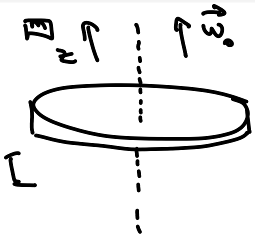
A

initial angular momentum: $\vec{L}_i = I\vec{\omega}_0$

final angular momentum: $\vec{L}_f = I\vec{\omega} + mvd\hat{z}$

$$\vec{L}_f = [I\omega + m(\omega d)d]\hat{z} = (I+md^2)\omega\hat{z}$$

$$\vec{L}_i = \vec{L}_f \Rightarrow I\omega_0 = (I+md^2)\omega \Rightarrow \omega = \frac{I}{(I+md^2)}\omega_0$$



The change in the KE of the system:

$$(KE)_i = \frac{1}{2} I \omega_0^2$$

$$(KE)_f = \frac{1}{2} I \omega^2 + \frac{1}{2} m (\omega d)^2$$

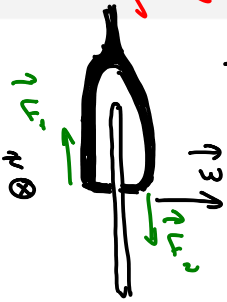
$$= \frac{1}{2} \cancel{(I+md^2)} \frac{I^2}{(I+md^2)^{\cancel{2}}} \omega_0^2$$

$$= \frac{1}{2} I \omega_0^2 \left[\frac{I}{I+md^2} \right]$$

$$\Delta(KE) = \frac{1}{2} I \omega_0^2 \left[\frac{I}{I+md^2} - 1 \right] = \frac{1}{2} I \omega_0^2 \left(\frac{-md^2}{I+md^2} \right)$$

This energy is converted into internal energy

Bicycle



Q: Consider the shown bicycle wheel. If the rides leans to one side, which direction will the wheel rotate?

A: the forces acting on the wheel are shown in green

Both of the forces will create a torque inside the screen (in the $\hat{z}$ direction). If the rider is leaning to the left, the steering wheel will also turn towards the left