MATH 219

Fall 2020

Lecture 5

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Content: Exact equations and integrating factors (section 2.6).

Suggested Problems: (Boyce, Di Prima, 9th edition)

§2.6: 3, 5, 9, 10, 13, 18, 21, 24, 30, 32

1 Exact equations

Let us consider the case of an arbitrary first order ODE once again. Say that our independent variable is x rather than t. Suppose for a moment that we found the solutions of the equation and that they can be written in the implicit form

$$F(x, y) = c.$$

by leaving the constant c alone. Of course, when c changes, the solution curve will change. We can easily write dy/dx in terms of x: Take the derivative of both sides with respect to x. By the chain rule,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

The question is whether we can reverse this process. Namely, given the ODE, can we recover such a function $F(x, y)$? One important remark at this point is that, even if we could, given the ODE we just know the *ratio* of $\partial F/\partial x$ to $\partial F/\partial y$, but F and the values of the partial derivatives themselves are not uniquely determined at all. In some favourable cases, the functions appearing in the particular way we write the ODE will a priori be equal to the derivatives of a certain function F. To better understand this, let us write the ODE in a more symmetric form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

An equivalent way of writing this equation is

$$M(x,y)dx + N(x,y)dy = 0.$$

Derivatives with respect to x are suppressed in this notation. Instead of $\frac{d}{dx}$ we write d etc. [1] With this notation, we have

$$d(F(x,y)) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

Definition 1.1 *The equation $M(x,y)dx + N(x,y)dy = 0$ is called* ***exact*** *on a domain D if there exists a differentiable function $F(x,y)$ on D such that the left hand side of the equation can be written as $d(F(x,y))$ on D. A function $F(x,y)$ satisfying this condition is called a* ***potential*** *for this equation.*

By the chain rule, the condition on $F(x,y)$ is equivalent to the pair of equations

$$\frac{\partial F}{\partial x} = M, \qquad \frac{\partial F}{\partial y} = N$$

We emphasize that the potential $F(x,y)$ must be defined as a single valued function on the whole domain D.

Example 1.1 *The equation $ydx+(x+2y)dy = 0$ is exact on $\mathbb{R}^2$. Indeed, if $F(x,y) = xy + y^2$ then*

$$\begin{aligned} d(F(x,y)) &= d(xy + y^2) \\ &= ydx + (x + 2y)dy \end{aligned}$$

Remark 1.1 Recall from multivariable calculus that a vector field $< M(x,y), N(x,y) >$ is called conservative if it can be written in the form ∇F for some function $F(x,y)$. It is clear that the equation $Mdx + Ndy = 0$ is exact if and only if the vector field $< M, N >$ is conservative.

[1] From a more advanced perspective, this is an equality of differential 1-forms. We will not pursue this viewpoint here

If the equation $Mdx + Ndy = 0$ is exact with potential F, then it can be rewritten as $dF = 0$. Consequently, the equations $F(x,y) = c$ for arbitrary values of c give us all solutions of the ODE in an implicit form.

Example 1.2 *Solve the initial value problem* $ydx + (x+2y)dy = 0$, $y(1) = 5$.

Solution: *We saw above that the equation is exact with* $F(x,y) = xy + y^2$ *a potential. Therefore the solutions of the equation are* $xy + y^2 = c$. *Using the intial condition, we find that* $c = 1 \times 5 + 5^2 = 30$. *Hence the solution is* $xy + y^2 = 30$ *(in implicit form).*

Suppose that $M(x,y)$ and $N(x,y)$ are themselves continuously differentiable on a common domain D. As in the case of conservative vector fields, a necessary condition for the existence of a potential function is

$$\frac{\partial M}{\partial y} = \frac{\partial F}{\partial x \partial y} = \frac{\partial F}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

This condition is not always sufficient for the existence of a potential F. However, if the domain is *simply connected*, then it is. A simply connected domain, intuitively, is a domain with no interior holes. An example of a simply connected domain is a rectangle. We will formulate and use the result in this particular case.

Theorem 1.1 *(Test for exactness) Suppose that* $M, N, \partial M/\partial y$ *and* $\partial N/\partial x$ *are continuous on a rectangle* R. *Then* $Mdx + Ndy = 0$ *is exact if and only if* $\partial M/\partial y = \partial N/\partial x$ *at each point of* R.

Proof: Fix x_0. The functions $F(x,y)$ satisfying the equation $\partial F/\partial x = M$ can be found by integrating M along a line segment from (x_0, y) to (x, y), since each such line segment remains in the rectangle R:

$$F(x,y) = \int_{(x_0,y)}^{(x,y)} M(s,y)ds$$

The result is any antiderivative of M with respect to x plus a function of y to be determined. Namely, it is of the form $F(x,y) = R(x,y) + h(y)$ where $\partial R/\partial x = M$.

The question is whether or not we can always choose $h(y)$ so that the equation $\partial F/\partial y = N$ is also satisfied. We need

$$\begin{aligned} \frac{\partial F}{\partial y} &= \frac{\partial R}{\partial y} + h'(y) = N(x,y) \\ h'(y) &= N(x,y) - \frac{\partial R}{\partial y} \end{aligned}$$

The last equation has a solution for $h'(y)$ (and consequently for $h(y)$) if and only if its right hand side is independent of x. In order to test whether this is true or not, let us look at its partial derivative with respect to x:

$$\begin{aligned} \frac{\partial}{\partial x}\left(N(x,y) - \frac{\partial R}{\partial y}\right) &= \frac{\partial N}{\partial x} - \frac{\partial^2 R}{\partial x \partial y} \\ &= \frac{\partial N}{\partial x} - \frac{\partial^2 R}{\partial y \partial x} \\ &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \\ &= 0. \end{aligned}$$

Therefore we can solve for $h(y)$, and the equation is exact. This completes the proof. □

Example 1.3 *Find the value of the constant a for which the ODE*

$$3e^y dx + (2y + axe^y)dy = 0$$

is exact. Solve the equation for this value of a.

Solution: *We have* $M(x,y) = 3e^y$ *and* $N(x,y) = 2y + axe^y$*. Since*

$$\frac{\partial M}{\partial y} = 3e^y \qquad\qquad \frac{\partial N}{\partial x} = ae^y$$

the equality holds if and only if $a = 3$*. Since* M, N *and their partial derivatives are all continuous on* $\mathbb{R}^2$*, we can apply the theorem and conclude that the ODE is exact for* $a = 3$*. Now,*

$$\begin{aligned} \frac{\partial F}{\partial x} &= 3e^y \\ F(x,y) &= 3xe^y + h(y) \\ \frac{\partial F}{\partial y} &= 3xe^y + h'(y) = 2y + 3xe^y \\ h'(y) &= 2y. \end{aligned}$$

Therefore $h(y) = y^2$ *is a solution and* $F(x,y) = 3xe^y + y^2$ *is a potential. The solutions of the ODE are*

$$3xe^y + y^2 = c$$

where $c \in \mathbb{R}$ *is a constant.* $\square$

Example 1.4 *Solve the initial value problem*

$$\left(\frac{2xy}{x^2+1} - 2x\right)dx - (2 - \ln(x^2+1))dy = 0, \qquad y(5) = 0.$$

Determine the largest interval on which the solution is valid.

Solution: *Here,* $M(x,y) = \dfrac{2xy}{x^2+1} - 2x$ *and* $N(x,y) = -2 + \ln(x^2+1)$*. We compute*

$$\frac{\partial M}{\partial y} = \frac{2x}{x^2+1} = \frac{\partial N}{\partial x}$$

Both M, N *and their partial derivatives are continuous on* $\mathbb{R}^2$ *(which can be viewed as an infinite rectangle). Therefore, by the test for exactness, the equation is exact. Let us find a potential.*

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{2xy}{x^2+1} - 2x \\ F(x,y) &= y\ln(x^2+1) - x^2 + h(y) \\ \frac{\partial F}{\partial y} &= \ln(x^2+1) + h'(y) = -2 + \ln(x^2+1) \\ h'(y) &= -2 \\ h(y) &= -2y. \end{aligned}$$

We deduce that $F(x,y) = y\ln(x^2+1) - x^2 - 2y$ *is a potential. All solutions of the ODE are* $y\ln(x^2+1) - x^2 - 2y = c$*. Using the initial condition* $y(5) = 0$*, we find that* $0\ln(5^2+1) - 5^2 - 2 \times 0 = c$*, therefore* $c = -25$*. So,*

$$\begin{aligned} y\ln(x^2+1) - x^2 - 2y &= -25 \\ y(\ln(x^2+1) - 2) &= x^2 - 25 \\ y = \frac{x^2-25}{\ln(x^2+1)-2}. \end{aligned}$$

This function is defined if and only if $\ln(x^2+1) - 2 \neq 0$*, namely for* $x^2 + 1 \neq e^2$*. The interval of definition, which must be a connected interval, could then be either of* $(-\infty, -\sqrt{e^2-1})$*,* $(-\sqrt{e^2-1}, \sqrt{e^2-1})$ *or* $(\sqrt{e^2-1}, \infty)$ *but since the initial point* $x = 5$ *belongs to the last one, the answer is* $(\sqrt{e^2-1}, \infty)$*.*

2 Integrating Factors

Recall from the lecture on first order linear equations that an ODE of the form $y' + p(t)y = q(t)$ can be solved by multiplying the equation by an integrating factor $\mu(t)$. In this case, the equation for $\mu(t)$ turned out to be easy to solve and we even got a formula $\mu(t) = exp(\int p(t)dt)$.

Let us now suppose that we have an ODE of the form $M(x, y)dx + N(x, y)dy = 0$. If the equation is exact, then we know what to do. If it is not exact, we may try to find an integrating factor $\mu(x, y)$ such that, after multiplication with μ, the new ODE

$$\mu M dx + \mu N dy = 0$$

is exact. Let us assume that all of these functions and their partial derivatives are continuous on a rectangle R, so that we can use the test for exactness. Then the new equation is exact if and only if

$$\begin{aligned} \frac{\partial(\mu M)}{\partial y} &= \frac{\partial(\mu N)}{\partial x} \\ \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} &= \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}. \end{aligned}$$

The problem that we encounter here is that this new differential equation for μ is terribly difficult to solve. It is not even an ODE, it is a PDE. Therefore, finding an integrating factor in this very general setting is a hopelessly difficult task. Only when there is some additional information that tells us something about the form of the integrating factor, this method could be useful.

Example 2.1 *Show that $\mu(x, y) = (x^2 + y^2)^{-1}$ is an integrating factor for the ODE*

$$(3x^2 + x + 3y^2)dx + (7x^2 + y + 7y^2)dy = 0$$

and use it to find all solutions of this ODE.

Solution: *The original equation is not exact (please check this). If we multiply the ODE throughout by $\mu(x, y)$, we get*

$$\left(3 + \frac{x}{x^2 + y^2}\right) dx + \left(7 + \frac{y}{x^2 + y^2}\right) dy = 0.$$

The functions $3 + \dfrac{x}{x^2+y^2}$ *and* $7 + \dfrac{y}{x^2+y^2}$ *are defined on* $\mathbb{R}^2 - \{(0,0)\}$. *Since this set is not a rectangle (in fact, it is not a simply connected domain), we cannot use the test for exactness here. We should directly show that a potential function exists:*

$$\begin{aligned} \frac{\partial F}{\partial x} &= 3 + \frac{x}{x^2+y^2} \\ F(x,y) &= 3x + \frac{1}{2}\ln(x^2+y^2) + h(y) \\ \frac{\partial F}{\partial y} &= \frac{y}{x^2+y^2} + h'(y) = 7 + \frac{y}{x^2+y^2} \\ h'(y) &= 7 \\ h(y) &= 7y. \end{aligned}$$

Therefore $F(x,y) = 3x + 7y + \frac{1}{2}\ln(x^2+y^2)$ *is a potential. Existence of a potential implies that* $\mu(x,y)$ *is indeed an integrating factor. The solutions of the ODE are*

$$3x + 7y + \frac{1}{2}\ln(x^2+y^2) = c$$

where $c \in \mathbb{R}$ *is a constant.*