# MATH 219

Fall 2020

Lecture 12

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**Content:** Nonhomogenous linear systems (variation of parameters only).

**Suggested Problems: (Boyce, Di Prima, 9th edition)**

**§7.9:** 2, 5, 7, 10, 11, 13

## 1 Variation of Parameters

In the previous lecture, we outlined a method to solve any constant coefficient homogenous linear system. Suppose now that we have a **nonhomogenous** linear system:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}.$$

Recall that a fundamental matrix $\Psi(t)$ is any matrix satisfying

$$\frac{d\Psi}{dt} = A\Psi$$
$$\det(\Psi) \neq 0.$$

Provided that we can find such a matrix $\Psi(t)$, we can write down all solutions of the homogenous system $\mathbf{x}' = A\mathbf{x}$ as

$$\mathbf{x} = \Psi(t)\mathbf{c}$$

where $\mathbf{c}$ is a vector of constants. In particular if $A$ is a constant matrix, then $e^{At}$ or $Pe^{Jt}$ that were found in the previous lecture are fundamental matrices.

We will use a method called **variation of parameters** in order to solve the nonhomogenous system. The idea of variation of parameters is to replace the constant vector $\mathbf{c}$ in the formula $\mathbf{x} = \Psi(t)\mathbf{c}$ by a nonconstant vector $\mathbf{v}(t)$ and hope that we can extract a solution of the nonhomogenous system of the form $\Psi(t)\mathbf{v}(t)$. In fact,

**Theorem 1.1** *All solutions of* $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ *are of the form* $\mathbf{x} = \Psi(t)\mathbf{v}(t)$ *where* $\mathbf{v}(t) = \int \Psi^{-1}(t)\mathbf{b}(t)dt$.

*Proof:* Plug $\mathbf{x} = \Psi(t)\mathbf{v}$ into the differential equation and use product rule to differentiate:

$$\begin{aligned} \mathbf{x}' &= \frac{d\Psi}{dt}\mathbf{v} + \Psi\frac{d\mathbf{v}}{dt} \\ &= A\Psi\mathbf{v} + \Psi\frac{d\mathbf{v}}{dt} \end{aligned}$$

We want the right hand side of this equation to be equal to $A\mathbf{x} + \mathbf{b}$, namely to $A\Psi\mathbf{v} + \mathbf{b}$. This equality holds if and only if

$$\begin{aligned} \Psi\frac{d\mathbf{v}}{dt} &= \mathbf{b} \\ \frac{d\mathbf{v}}{dt} &= \Psi^{-1}\mathbf{b} \\ \mathbf{v} &= \int \Psi^{-1}\mathbf{b}dt \end{aligned}$$

Therefore the expression $\mathbf{x} = \Psi \int \Psi^{-1}\mathbf{b}dt$ in the statement is really a solution. How can we be sure that there are no other solutions? We can write the indefinite integral above as $\int = \int_0^t + \mathbf{c}$ where $\mathbf{c}$ is a vector of constants. Then the solutions obtained above are of the form $\mathbf{x} = \Psi\mathbf{c} + \Psi \int_0^t \Psi^{-1}(\tau)\mathbf{b}(\tau)d\tau$. Then

$$\mathbf{x}_p = \Psi \int_0^t \Psi^{-1}(\tau)\mathbf{b}(\tau)d\tau$$

is a particular solution of the nonhomogenous system. If $\mathbf{x}$ is any other solution, then by the principle of superposition $\mathbf{x} - \mathbf{x}_p$ must be a solution of the corresponding homogenous system, therefore it must be of the form $\Psi\mathbf{c}$. This proves the claim. □

**Example 1.1** *Solve the system* $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^{2t} \\ t \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda).$$

*Therefore the eigenvalues are* $\lambda_1 = 2$ *and* $\lambda_2 = 1$. *Let us find the eigenvectors for* $\lambda_1$*:*

$$\begin{aligned} \left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -1 & 0 \end{array}\right] &\xrightarrow{R_1/3 \to R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \\ &\xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

*So the eigenvectors are of the form* $k\begin{bmatrix}1\\0\end{bmatrix}$*. Next, let us find the eigenvectors for* $\lambda_2$*. The matrix*

$$\begin{bmatrix}1 & 3 & | & 0\\0 & 0 & | & 0\end{bmatrix}$$

*is already in row echelon form. So the eigenvectors are of the form* $k\begin{bmatrix}-3\\1\end{bmatrix}$*. Therefore we can write down two linearly independent solutions* $\mathbf{x}^{(1)} = \begin{bmatrix}e^{2t}\\0\end{bmatrix}$ *and* $\mathbf{x}^{(2)} = \begin{bmatrix}-3e^t\\e^t\end{bmatrix}$*. So a fundamental matrix is*

$$\Psi(t) = \begin{bmatrix}e^{2t} & -3e^t\\0 & e^t\end{bmatrix}$$

*Its inverse can be easily computed to be* $\Psi^{-1} = \begin{bmatrix}e^{-2t} & 3e^{-2t}\\0 & e^{-t}\end{bmatrix}$*. Now use the formula* $\mathbf{v} = \int \Psi^{-1}\mathbf{b}dt$*:*

$$\begin{aligned}\mathbf{v} &= \int \begin{bmatrix}e^{-2t} & 3e^{-2t}\\0 & e^{-t}\end{bmatrix}\begin{bmatrix}e^{2t}\\t\end{bmatrix}dt\\ &= \begin{bmatrix}\int 1+3te^{-2t}dt\\ \int te^{-t}dt\end{bmatrix}\\ &= \begin{bmatrix}t-\frac{3}{2}te^{-2t}-\frac{3}{4}e^{-2t}+c_1\\ -te^{-t}-e^{-t}+c_2\end{bmatrix}\end{aligned}$$

*(The integrals above can be found by employing integration by parts.) Finally we can find the general solution for* $\mathbf{x}$*:*

$$\begin{aligned}\mathbf{x} &= \Psi\mathbf{v}\\ &= \begin{bmatrix}e^{2t} & -3e^t\\0 & e^t\end{bmatrix}\begin{bmatrix}t-\frac{3}{2}te^{-2t}-\frac{3}{4}e^{-2t}\\ -te^{-t}-e^{-t}\end{bmatrix} + \begin{bmatrix}e^{2t} & -3e^t\\0 & e^t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}\\ &= \begin{bmatrix}te^{2t}+\frac{3}{2}t+\frac{9}{4}\\ -t-1\end{bmatrix} + c_1\begin{bmatrix}e^{2t}\\0\end{bmatrix} + c_2\begin{bmatrix}-3e^t\\e^t\end{bmatrix}\end{aligned}$$

*where* $c_1, c_2 \in \mathbb{R}$ *are arbitrary constants.*

**Example 1.2** *Consider the system*

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}' = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}k_1\\k_2\end{bmatrix}$$

*where $a, b, c, d, k_1, k_2$ are constants. Suppose that the coefficient matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two distinct negative real eigenvalues. Show that the limits $\lim_{t\to+\infty} x_1(t)$ and $\lim_{t\to+\infty} x_2(t)$ exist and do not depend on the initial values of $x_1$ and $x_2$. Compute these limits in terms of $A, k_1$ and $k_2$.*

***Solution:*** *Let the eigenvalues of $A$ be $\lambda_1$ and $\lambda_2$. Since they are not equal to each other, the matrix $A$ must be diagonalizable. So there exists an invertible matrix $P$ (which we will not attempt to compute) such that*

$$A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

*Consequently, we have*

$$\Psi(t) = Pe^{Jt} = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}, \quad \Psi^{-1}(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{bmatrix} P^{-1}.$$

*In order to apply the variation of parameters formula, we will need to look at $\Psi^{-1}(t) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{bmatrix} P^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. The last product in this formula will again give us some vector of constants. So we can write*

$$\Psi^{-1}(t) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} l_1 e^{-\lambda_1 t} \\ l_2 e^{-\lambda_2 t} \end{bmatrix}$$

*for certain constants $l_1, l_2$. Now, let us apply the variation of parameters formula:*

$$\begin{aligned} \mathbf{x} &= \Psi(t) \int \Psi^{-1}(t) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\ &= \Psi(t) \int \begin{bmatrix} l_1 e^{-\lambda_1 t} \\ l_2 e^{-\lambda_2 t} \end{bmatrix} dt \\ &= \Psi(t) \left( \begin{bmatrix} -\frac{l_1}{\lambda_1} e^{-\lambda_1 t} \\ -\frac{l_2}{\lambda_2} e^{-\lambda_2 t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right) \\ &= P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \left( \begin{bmatrix} -\frac{l_1}{\lambda_1} e^{-\lambda_1 t} \\ -\frac{l_2}{\lambda_2} e^{-\lambda_2 t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right) \\ &= P \begin{bmatrix} -\frac{l_1}{\lambda_1} \\ -\frac{l_2}{\lambda_2} \end{bmatrix} + P \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}. \end{aligned}$$

*When $t$ tends to infinity, the second summand above goes to 0 since both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are decaying exponentials by assumption. The first summand is a constant.*

*Therefore, the limit exists and it is independent of the initial values because it is independent of the values of the constants* $c_1, c_2$*. In order to compute the limiting values* $x_1(\infty)$ *and* $x_2(\infty)$*, notice that the derivatives of the functions* $x_1$ *and* $x_2$ *will tend to* $0$ *at infinity (to see this, we may for instance use the formula for* $\mathbf{x}$ *obtained above). Therefore, by considering the original system of differential equations, we must have*

$$\begin{aligned} \mathbf{0} &= A \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\ \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} &= -A^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}. \end{aligned}$$