MATH 219

Fall 2020

Lecture 13

Lecture notes by Özgür Kişisel

Content: General theory of nth order linear equations.

Suggested Problems: (Boyce, Di Prima, 9th edition)

§4.1: 2, 4, 6, 9, 10, 13, 15

1 Higher Order Linear ODE's

We now wish to study higher order differential equations. As in the first order case, linear equations have more structure than non-linear ones, hence they are comparatively easier to study. Because of their relative simplicity, linear equations tend to occur often in applications. Furthermore, studying linear equations provides us with valuable insight about non-linear equations as well.

Suppose that y is a variable depending on t. An **nth order linear ODE** for $y(t)$ is a differential equation that can be written in the form

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n(t)y = b(t).$$

Of course, another notation for the same equation is

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = b(t)$$

where $y^{(j)}$ denotes the j-th derivative of y with respect to t. If $b(t)$ is identically zero, then the equation is said to be **homogenous**. Otherwise, it is said to be **non-homogenous**. If the coefficients $a_i(t)$ are all independent of t, then we say that the equation has **constant coefficients**.

2 Converting a Higher Order ODE into a First Order System

Let us consider the ODE

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = b(t). \tag{1}$$

We can replace this ODE by an equivalent $n \times n$ system of first order ODE's by applying a simple procedure that we will describe now. Define auxilliary variables $x_1, x_2, \ldots, x_n$ (all depending on t) as follows:

$$\begin{aligned} x_1 &= y \\ x_2 &= x_1' = y' \\ x_3 &= x_2' = y^{(2)} \\ &\ldots \\ x_n &= x_{n-1}' = y^{(n-1)} \end{aligned}$$

Then, we can compute x_n' by using equation (1):

$$\begin{aligned} x_n' &= y^{(n)} = -a_n(t)y - a_{n-1}(t)y' - \ldots - a_1(t)y^{(n-1)} + b(t) \\ &= -a_n(t)x_1 - a_{n-1}(t)x_2 - \ldots - a_1(t)x_n + b(t). \end{aligned}$$

Therefore, the following system of linear first order ODE's must be satisfied:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\ldots \\ x_{n-1}' &= x_n \\ x_n' &= -a_n(t)x_1 - a_{n-1}(t)x_2 - \ldots - a_1(t)x_n + b(t). \end{aligned} \tag{2}$$

Conversely, if the system above is satisfied, then the equations $x_1 = y, \ldots, x_n = y^{(n-1)}, x_n' = y^{(n)} = -a_n(t)y - \ldots - a_1(t)y^{(n-1)} + b(t)$ show that the original ODE is satisfied. Hence there is a one to one correspondence between solutions of the ODE (1) and the x_1 component of the solutions of the system (2).

The system that we obtained can be written in matrix form as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ \dots & 0 & 1 & 0 & \dots & 0 \\ & & & \dots & & \\ & & & & \dots & \\ 0 & 0 & \dots & \dots & 0 & 1 \\ -a_n(t) & -a_{n-1}(t) & \dots & \dots & -a_2(t) & -a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ b(t) \end{bmatrix}$$

Example 2.1 *Let us convert the third order ODE*

$$y''' + 3y'' - ty' + 2y = \cos t$$

into a first order system. According to the procedure described above, $x_1 = y, x_1' = x_2 = y', x_2' = x_3 = y''$ *and*

$$\begin{aligned} x_3' &= y''' = -2y + ty' - 3y'' + \cos t \\ &= -2x_1 + tx_2 - 3x_3 + \cos t \end{aligned}$$

Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & t & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

3 Structure of the solution set

Let us first consider the case of a linear homogenous ODE, namely an equation of the form

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = 0.$$

Then the corresponding system $\mathbf{x}' = A(t)\mathbf{x}$ will also be homogenous. By the general theory of nth order linear systems, there exists a basis of n linearly independent solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ so that the general solution of the system is

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \ldots + c_n\mathbf{x}^{(n)}.$$

Since the first component of a solution of the system gives us a solution of the ODE, set y_1 to be the first coordinate of $\mathbf{x}^{(1)}$, y_2 to be the first coordinate of $\mathbf{x}^{(2)}$ etc.

Notice that

$$\mathbf{x}^{(1)} = \begin{bmatrix} y_1 \\ y_1' \\ y_1'' \\ \dots \\ y_1^{(n-1)} \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} y_2 \\ y_2' \\ y_2'' \\ \dots \\ y_2^{(n-1)} \end{bmatrix}, \dots, \mathbf{x}^{(n)} = \begin{bmatrix} y_n \\ y_n' \\ y_n'' \\ \dots \\ y_n^{(n-1)} \end{bmatrix}$$

Since there is a one-to-one correspondence between solutions of (1) and solutions of (2), we deduce that all solutions of the homogenous ODE are of the form

$$y = c_1y_1 + c_2y_2 + \dots + c_ny_n.$$

Lemma 3.1 *The set of functions* $\{y_1, y_2, \dots, y_n\}$ *is linearly independent.*

Proof: Suppose that $c_1y_1 + c_2y_2 + \dots + c_ny_n = 0$. By taking derivatives, we obtain $c_1y_1^{(j)} + c_2y_2^{(j)} + \dots + c_ny_n^{(j)} = 0$ for every j. Therefore $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)} = \mathbf{0}$. But the $\mathbf{x}^{(i)}$'s are linearly independent, hence $c_1 = c_2 = \dots = c_n = 0$. □

The discussion above gives us a proof of the following theorem:

Theorem 3.1 *The set of solutions of a linear homogenous nth order ODE* $y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = 0$ *is*

$$y = c_1y_1 + c_2y_2 + \dots + c_ny_n$$

where $c_1, \dots, c_n$ *are arbitrary constants and* $\{y_1, y_2, \dots, y_n\}$ *is a linearly independent set of functions.*

In particular, the solution space has dimension n. Since any linearly independent set of n solutions of the ODE will form a basis, the problem is reduced to finding a set of n linearly independent solutions of the equation.

We can test a set of functions for linear independence by looking at their Wronskian:

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}.$$

If $W(y_1, y_2, \dots, y_n)$ is not identically zero, then the functions must be linearly independent.

Example 3.1 *Check that $y_1 = t$, $y_2 = t^2$ and $y_3 = 1/t$ are solutions of the ODE*

$$t^3 y''' + t^2 y'' - 2ty' + 2y = 0$$

for $t > 0$, and write down all solutions of the equation.

Solution:

$$t^3 y_1''' + t^2 y_1'' - 2ty_1' + 2y_1 = 0 + 0 - 2t + 2t = 0$$
$$t^3 y_2''' + t^2 y_2'' - 2ty_2' + 2y_2 = 0 + 2t^2 - 4t^2 + 2t^2 = 0$$
$$t^3 y_3''' + t^2 y_3'' - 2ty_3' + 2y_3 = -\frac{6}{t} + \frac{2}{t} + \frac{2}{t} + \frac{2}{t} = 0$$

therefore y_1, y_2, y_3 are solutions. Let us find their Wronskian:

$$W(y_1, y_2, y_3) = \begin{vmatrix} t & t^2 & \frac{1}{t} \\ 1 & 2t & -\frac{1}{t^2} \\ 0 & 2 & \frac{2}{t^3} \end{vmatrix} = \frac{6}{t} \neq 0.$$

Since their Wronskian is nonzero, the set of functions $\{y_1, y_2, y_3\}$ is linearly independent. We have three linearly independent solutions for a third order linear equation, hence by the basic theory, they must form a fundamental set. Therefore, the general solution must be:

$$y(t) = c_1 t + c_2 t^2 + \frac{c_3}{t}$$

where $c_1, c_2, c_3 \in \mathbb{R}$.