MATH 219

Fall 2020

Lecture 14

Lecture notes by Özgür Kişisel

Content: Homogeneous equations with constant coefficients.

Suggested Problems: (Boyce, Di Prima, 9th edition)

§4.2: 14, 16, 17, 20, 22, 31, 32, 36, 37

1 Homogenous equations with constant coefficients

Consider now an nth order, linear ODE of the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$$

where $a_1, a_2 \ldots, a_n \in \mathbb{R}$ are constants. By using the procedure that was described in the previous lecture, we can convert the ODE into a system $\mathbf{x}' = A\mathbf{x}$ where A is the constant matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots \\ & & & \ldots & \\ 0 & \ldots & & 0 & 1 \\ -a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \end{bmatrix}.$$

Recall from the theory of first order linear systems discussed in the previous lectures that

$$\mathbf{x} = e^{At}\mathbf{c} = Pe^{Jt}P^{-1}\mathbf{c},$$

or alternatively, we can use another fundamental matrix (and another set of constants) to write

$$\mathbf{x} = \Psi(t)\mathbf{c} = Pe^{Jt}\mathbf{c},$$

where J is the Jordan form of A and the columns of P are eigenvectors or generalized eigenvectors. Therefore, this formula will also lead us to the solution of the ODE for y, since $y = x_1$. In a sense we already know how to solve this problem then. But, instead of going through the whole lengthy procedure and solving the full system, we will find some shortcuts which will let us compute $y(t)$ in a quicker way. We start by finding the characteristic polynomial of A.

<u>2×2 case:</u>

$$\begin{vmatrix} -\lambda & 1 \\ -a_2 & -a_1 - \lambda \end{vmatrix} = \lambda^2 + a_1\lambda + a_2$$

<u>3×3 case:</u>

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_3 & -a_2 & -a_1 - \lambda \end{vmatrix} &= (-\lambda) \begin{vmatrix} -\lambda & 1 \\ -a_2 & -a_1 - \lambda \end{vmatrix} + (-a_3) \begin{vmatrix} 1 & 0 \\ -\lambda & 1 \end{vmatrix} \\ &= -(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) \end{aligned}$$

In this computation, the determinant was expanded with respect to the first column. We make a guess for the $n \times n$ case:

$$\det(A - \lambda I) = p_n(\lambda) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \ldots + a_n).$$

Let us prove that this guess is correct, by induction. The assertion is true for $n = 2$ as seen above. Suppose that it is true up to n. For the $(n+1) \times (n+1)$ case, again by expanding the determinant with respect to the first column, we have:

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & 0 & \ldots \\ & & & \ldots & \\ 0 & \ldots & 0 & -\lambda & 1 \\ -a_{n+1} & -a_n & \ldots & -a_2 & -a_1 - \lambda \end{vmatrix} &= (-\lambda)p_n(\lambda) + (-1)^n(-a_{n+1}) \\ &= (-1)^{n+1}(\lambda^{n+1} + a_1\lambda^n + \ldots + a_n\lambda + a_{n+1}) \end{aligned}$$

This proves the result.

Definition 1.1 *The equation $p_n(\lambda) = 0$ is called the* ***characteristic equation*** *of the system.*

This result lets us compute the eigenvalues without actually writing down a matrix: Simply copy the coefficients of the ODE into a polynomial as its coefficients and find the roots of this resulting polynomial. On the other hand, we will see that computing the eigenvectors of the matrix will not be necessary at all. Let us investigate the possibilities for the roots of the characteristic equation.

1.1 Distinct eigenvalues

Suppose that the n roots (real or complex) of the equation $\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$ are distinct. Then A is diagonalizable. We have

$$\Psi(t) = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \ldots \\ & & \ldots & \\ 0 & \ldots & 0 & e^{\lambda_n t} \end{bmatrix}$$

A solution $y = x_1$ is the first entry of $\Psi(t)\mathbf{c}$ for a constant vector c. In particular, it is a linear combination of the functions in the set $\{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\}$. There are precisely n linearly independent functions in this set, therefore this set must be a basis for the space of solutions. It follows that all solutions of the ODE are

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. If some of the λ's are complex conjugate pairs, then the corresponding complex solutions should be replaced by their real and imaginary parts as in the case of systems with complex eigenvalues.

Example 1.1 *Solve the equation $y'' - y = 0$.*

Solution: *The characteristic equation is $\lambda^2 - 1 = 0$ whose roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Therefore all solutions of the equation are*

$$y = c_1 e^t + c_2 e^{-t}$$

with $c_1, c_2 \in \mathbb{R}$.

Example 1.2 *Solve the differential equation $y^{(4)} + y = 0$.*

Solution: *The characteristic equation is $\lambda^4 + 1 = 0$. Its roots are the fourth roots of $-1 = e^{i\pi}$. These four roots are*

$$\lambda_1 = e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \qquad \lambda_2 = e^{-i\pi/4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$
$$\lambda_3 = e^{i3\pi/4} = \frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \qquad \lambda_4 = e^{-i3\pi/4} = \frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

For the pair of complex conjugate roots λ_1 and λ_2 we get

$$Re(e^{(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})t}) = e^{\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right), \qquad Im(e^{(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})t}) = e^{\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right)$$

Similarly for the pair λ_3 and λ_4 we get

$$Re(e^{(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})t}) = e^{-\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right), \qquad Im(e^{(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})t}) = e^{-\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right)$$

Therefore, all solutions of the equation are

$$y = c_1e^{\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)+c_2e^{\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right)+c_3e^{-\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right)+c_4e^{-\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right)$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Example 1.3 *Solve the initial value problem $y'' + 3y' + 2y = 0$, $y(0) = 2$, $y'(0) = 1$.*

Solution: *The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$. It factorizes as $(\lambda + 2)(\lambda + 1) = 0$, therefore $\lambda_1 = -2, \lambda_2 = -1$. These roots are distinct. Therefore the general solution is*

$$y = c_1e^{-2t} + c_2e^{-t},$$

Note that $y' = -2c_1e^{-2t} - c_2e^{-t}$. The initial conditions give us the linear system

$$\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 - c_2 &= 1. \end{aligned}$$

The solution is $c_1 = -3, c_2 = 5$. Hence,

$$y = -3e^{-2t} + 5e^{-t}.$$

Example 1.4 *Find all solutions of $4y''' + y' + 5y = 0$.*

Solution: *The characteristic equation is $4\lambda^3 + \lambda + 5 = 0$. It is a third order polynomial and in general it may not be straightforward to find the roots of such a* *polynomial by hand. In the current example, let us observe that $\lambda = -1$ is a root. To find the other roots, divide the polynomial $4\lambda^3 + \lambda + 5$ by $\lambda + 1$. We get*

$$\begin{aligned} 4\lambda^3 + \lambda + 5 &= (\lambda + 1)(4\lambda^2 - 4\lambda + 5) \\ &= 4(\lambda + 1)\left(\lambda - \left(\frac{1}{2} + i\right)\right)\left(\lambda - \left(\frac{1}{2} - i\right)\right). \end{aligned}$$

Therefore the roots are $-1, \frac{1}{2} + i, \frac{1}{2} - i$. They are all distinct. We can deduce that all solutions of the ODE are

$$y = c_1 e^{-t} + c_2 e^{t/2} \cos t + c_3 e^{t/2} \sin t.$$

1.2 Repeated roots

Let us now consider the case where some of the roots of the characteristic equation are repeated. We will first discuss the case when the nth degree polynomial has an n-fold repeated root λ. The general case is obtained just by putting together the solutions for different eigenvalues.

We claim that corresponding to the n-fold repeated root λ, there must be only one Jordan block

$$J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \dots & 1 \\ & & & \lambda \end{bmatrix}$$

and not multiple, smaller Jordan blocks. The reason for this is as follows: For a single Jordan block, the functions that we obtain from e^{Jt} are linear combinations of $e^{\lambda t}, te^{\lambda t}, \dots, t^{n-1}e^{\lambda t}$. There are precisely n of them, and this is barely enough. If there were multiple Jordan blocks, we wouldn't get all the way up to $t^{n-1}e^{\lambda t}$ and there would be less than n linearly independent solutions, a contradiction. Therefore all solutions of the system are

$$y = c_1 e^{\lambda t} + c_2 t e^{\lambda t} + \ldots + c_n t^{n-1} e^{\lambda t}.$$

If there are other eigenvalues as well, the solution is just the linear combination of the solutions obtained from different eigenvalues. For complex conjugate pairs, we take the real and imaginary parts of the corresponding exponentials.

Example 1.5 *Solve the initial value problem $y'' - 2y' + y = 0$, $y(0) = 1, y'(0) = -1$.*

Solution: *The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ whose roots are $\lambda_1 = \lambda_2 = 1$. Therefore the general solution is*

$$y = c_1 e^t + c_2 t e^t.$$

We compute $y' = c_1 e^t + c_2 e^t + c_2 t e^t$. Using the initial values,

$$\begin{aligned} c_1 + 0 &= 1 \\ c_1 + c_2 &= -1 \end{aligned}$$

Therefore $c_1 = 1$ and $c_2 = -2$. We get

$$y = e^t - 2te^t.$$

Example 1.6 *Solve the ODE $y^{(4)} + 6y^{(2)} + 9y = 0$.*

Solution: *The characteristic equation is*

$$\begin{aligned} \lambda^4 + 6\lambda^2 + 9 &= 0 \\ (\lambda^2 + 3)^2 &= 0 \\ (\lambda - i\sqrt{3})^2(\lambda + i\sqrt{3})^2 &= 0 \end{aligned}$$

Therefore the roots are $\lambda_1 = \lambda_2 = i\sqrt{3}$ and $\lambda_3 = \lambda_4 = -i\sqrt{3}$. For one pair of complex conjugate roots, we get $\cos(\sqrt{3}t)$ and $\sin(\sqrt{3}t)$. For the repeated pair, we get $t\cos(\sqrt{3}t)$ and $t\sin(\sqrt{3}t)$. The general solution is

$$y = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t) + c_3 t \cos(\sqrt{3}t) + c_4 t \sin(\sqrt{3}t)$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.