MATH 219

Fall 2020

Lecture 16

Lecture notes by Özgür Kişisel

Content: The method of variation of parameters.

Suggested Problems: (Boyce, Di Prima, 9th edition)

§4.4: 3, 4, 5, 8, 11, 16

Suppose that we have a non-homogenous linear ODE with constant coefficients:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = b(t).$$

In the previous lecture, we saw that if $b(t)$ has a polynomial annihilator, then we can use the method of undetermined coefficients to solve the problem. However, this condition is rather demanding. For instance, familiar functions such as $1/t$, $\sqrt{t}$ or $\tan t$ do not have polynomial annihilators. We still have a method that we can use for such cases: variation of parameters. We can convert the ODE into a first order $n \times n$ system and use variation of parameters for this system. However, since we will only need to find the first component of the resulting solution vector, some shortcuts can be taken. We will discuss these issues now.

1 The 2×2 case

Suppose that y_1 and y_2 are two linearly independent solutions of the homogenous equation

$$y'' + a_1 y' + a_2 y = 0.$$

Recall that we can convert this ODE into a first order linear system by setting $x_1 = y, x_2 = y'$. Then

$$\Psi(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

is a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ associated to the ODE. The system itself, by the way, can be written explicitly as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b(t) \end{bmatrix}.$$

In order to find $\mathbf{x}$, use the variation of parameters formula:

$$\begin{aligned}
\mathbf{x} &= \Psi \int \Psi^{-1}\mathbf{b}dt \\
&= \Psi \int \frac{1}{W(y_1,y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ b(t) \end{bmatrix} dt \\
&= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \int \frac{1}{W(y_1,y_2)} \begin{bmatrix} -y_2 b(t) \\ y_1 b(t) \end{bmatrix} dt.
\end{aligned}$$

Recall that $y = x_1$. Therefore,

$$y = x_1 = y_1 \int \frac{-y_2 b(t)}{W(y_1,y_2)} dt + y_2 \int \frac{y_1 b(t)}{W(y_1,y_2)} dt.$$

We obtained a nice formula for the 2×2 case which is applicable for any (continuous or piecewise continuous) function $b(t)$.

Example 1.1 *Solve the ODE*

$$4y'' + y = 2\sec(t/2)$$

where $-\pi < t < \pi$.

Solution: *The characteristic equation is* $4\lambda^2 + 1 = 0$ *whose roots are* $\pm i/2$. *Therefore* $y_h = c_1 y_1 + c_2 y_2 = c_1 \cos(t/2) + c_2 \sin(t/2)$.

$$W(y_1,y_2) = \begin{vmatrix} \cos(t/2) & \sin(t/2) \\ -\frac{1}{2}\sin(t/2) & \frac{1}{2}\cos(t/2) \end{vmatrix} = \frac{1}{2}.$$

Let us now use the formula. But note that $b(t) = \sec(t/2)/2$ *(the initial coefficient of the ODE needs to be* 1 *if we wish to use the formula).*

$$\begin{aligned}
y &= \cos(t/2) \int \frac{-\sin(t/2)\sec(t/2)/2}{1/2} dt + \sin(t/2) \int \frac{\cos(t/2)\sec(t/2)/2}{1/2} dt \\
&= \cos(t/2)(2\ln|\cos(t/2)| + c_1) + \sin(t/2)(t + c_2) \\
&= c_1 \cos(t/2) + c_2 \sin(t/2) + 2\cos(t/2)\ln|\cos(t/2)| + t\sin(t/2).
\end{aligned}$$

2 The $n \times n$ case

More generally, let us consider an nth order linear ODE

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = b(t).$$

Suppose that $y_1, y_2, \ldots, y_n$ are n linearly independent solutions of the associated homogenous equation. Then the fundamental matrix takes the form

$$\Psi(t) = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix}.$$

We again have $\mathbf{x} = \Psi \int \Psi^{-1}\mathbf{b}dt$. It would be a lengthy computation to find all entries of the vector $\mathbf{x}$. On the other hand, all we need is its first entry x_1. We need to quote one result from linear algebra in order to find x_1 without finding all of $\mathbf{x}$:

Theorem 2.1 *(Cramer's rule) Suppose that $B\mathbf{z} = \mathbf{l}$ where B is an $n \times n$ invertible matrix, $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ is a vector of unknowns and $\mathbf{l} = \begin{bmatrix} l_1 \\ \ldots \\ l_n \end{bmatrix}$ is a given vector. Let B_i be the matrix obtained by replacing the ith column of B by the vector $\mathbf{l}$. Then*

$$z_i = \frac{\det(B_i)}{\det(B)}.$$

The proof will be omitted. It can be found in texts on linear algebra. (Actually, the reader can reconstruct the proof without much difficulty: Since B is invertible, the system must have a unique solution. Plug the proposed solution into the system and see if it works. You will need to make one observation about determinants). Let us apply Cramer's rule to our case. Recall that $\mathbf{x} = \Psi\mathbf{v}$. Then

$$\Psi\mathbf{v}' = \mathbf{b}.$$

Therefore if Ψ_i is the matrix obtained by replacing the ith column of Ψ by $\mathbf{b}$ then $v_i' = \dfrac{\det(\Psi_i)}{\det(\Psi)}$. This gives us

$$y = x_1 = \sum_{i=1}^{n} y_i \int \frac{\det(\Psi_i)}{\det(\Psi)} dt.$$

There is one more small simplification that we can make. Notice that

$$\det(\Psi_i) = \begin{vmatrix} y_1 & y_2 & \dots & 0 & \dots & y_n \\ y_1' & y_2' & \dots & 0 & \dots & y_n' \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & b(t) & \dots & y_n^{(n-1)} \end{vmatrix} = b(t) \begin{vmatrix} y_1 & y_2 & \dots & 0 & \dots & y_n \\ y_1' & y_2' & \dots & 0 & \dots & y_n' \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & 1 & \dots & y_n^{(n-1)} \end{vmatrix}.$$

Let us denote the last determinant above by W_i, namely W_i is the determinant of the matrix obtained by replacing the ith column of $\Psi(t)$ by the vector $\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^T$. Also, let us write $W = \det(\Psi)$. Then the variation of parameters formula takes the form:

$$y = x_1 = \sum_{i=1}^{n} y_i \int b(t) \frac{W_i}{W} dt.$$

Note that for $n = 2$ this agrees with the formula found before.

Example 2.1 *Solve the ODE*

$$y^{(4)} + 2y'' + y = \sin t.$$

by using variation of parameters.

Solution: *First, let us find the roots of the characteristic equation:*

$$\begin{aligned} \lambda^4 + 2\lambda^2 + 1 &= 0 \\ (\lambda^2 + 1)^2 &= 0 \\ \lambda_1 = \lambda_2 = i, \quad \lambda_3 &= \lambda_4 = -i. \end{aligned}$$

Therefore a basis for the set of solutions of the associated homogenous equation is $y_1 = \cos t, y_2 = \sin t, y_3 = t\cos t, y_4 = t \sin t$.

$$\begin{aligned} W = \det(\Psi) &= \begin{vmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t + \cos t & t\cos t + \sin t \\ -\cos t & -\sin t & -t\cos t - 2\sin t & -t\sin t + 2\cos t \\ \sin t & -\cos t & t\sin t - 3\cos t & -t\cos t - 3\sin t \end{vmatrix} \\ {}_{R_1+R_3\to R_3, R_2+R_4\to R_4} &= \begin{vmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t + \cos t & t\cos t + \sin t \\ 0 & 0 & -2\sin t & 2\cos t \\ 0 & 0 & -2\cos t & -2\sin t \end{vmatrix} \\ &= 4. \end{aligned}$$

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\begin{aligned}
W_1 &= \begin{vmatrix} 0 & \sin t & t\cos t & t\sin t \\ 0 & \cos t & -t\sin t+\cos t & t\cos t+\sin t \\ 0 & -\sin t & -t\cos t-2\sin t & -t\sin t+2\cos t \\ 1 & -\cos t & t\sin t-3\cos t & -t\cos t-3\sin t \end{vmatrix} \\
{}_{R_1+R_3\to R_3, R_2+R_4\to R_4} &= \begin{vmatrix} 0 & \sin t & t\cos t & t\sin t \\ 0 & \cos t & -t\sin t+\cos t & t\cos t+\sin t \\ 0 & 0 & -2\sin t & 2\cos t \\ 1 & 0 & -2\cos t & -2\sin t \end{vmatrix} \\
&= -\begin{vmatrix} \sin t & t\cos t & t\sin t \\ \cos t & -t\sin t+\cos t & t\cos t+\sin t \\ 0 & -2\sin t & 2\cos t \end{vmatrix} \\
&= -\sin t \begin{vmatrix} -t\sin t+\cos t & t\cos t+\sin t \\ -2\sin t & 2\cos t \end{vmatrix} + \cos t \begin{vmatrix} t\cos t & t\sin t \\ -2\sin t & 2\cos t \end{vmatrix} \\
&= -2\sin t + 2t\cos t.
\end{aligned}
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\begin{aligned}
W_2 &= \begin{vmatrix} \cos t & 0 & t\cos t & t\sin t \\ -\sin t & 0 & -t\sin t+\cos t & t\cos t+\sin t \\ -\cos t & 0 & -t\cos t-2\sin t & -t\sin t+2\cos t \\ \sin t & 1 & t\sin t-3\cos t & -t\cos t-3\sin t \end{vmatrix} \\
{}_{R_1+R_3\to R_3, R_2+R_4\to R_4} &= \begin{vmatrix} \cos t & 0 & t\cos t & t\sin t \\ -\sin t & 0 & -t\sin t+\cos t & t\cos t+\sin t \\ 0 & 0 & -2\sin t & 2\cos t \\ 0 & 1 & -2\cos t & -2\sin t \end{vmatrix} \\
&= \begin{vmatrix} \cos t & t\cos t & t\sin t \\ -\sin t & -t\sin t+\cos t & t\cos t+\sin t \\ 0 & -2\sin t & 2\cos t \end{vmatrix} \\
&= \cos t \begin{vmatrix} -t\sin t+\cos t & t\cos t+\sin t \\ -2\sin t & 2\cos t \end{vmatrix} + \sin t \begin{vmatrix} t\cos t & t\sin t \\ -2\sin t & 2\cos t \end{vmatrix} \\
&= 2\cos t + 2t\sin t.
\end{aligned}
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\begin{aligned}
W_3 &= \begin{vmatrix} \cos t & \sin t & 0 & t\sin t \\ -\sin t & \cos t & 0 & t\cos t + \sin t \\ -\cos t & -\sin t & 0 & -t\sin t + 2\cos t \\ \sin t & -\cos t & 1 & -t\cos t - 3\sin t \end{vmatrix} \\
&\overset{R_1+R_3\to R_3, R_2+R_4\to R_4}{=} \begin{vmatrix} \cos t & \sin t & 0 & t\sin t \\ -\sin t & \cos t & 0 & t\cos t + \sin t \\ 0 & 0 & 0 & 2\cos t \\ 0 & 0 & 1 & -2\sin t \end{vmatrix} \\
&= -2\cos t.
\end{aligned}
$$

$$
\begin{aligned}
W_4 &= \begin{vmatrix} \cos t & \sin t & t\cos t & 0 \\ -\sin t & \cos t & -t\sin t + \cos t & 0 \\ -\cos t & -\sin t & -t\cos t - 2\sin t & 0 \\ \sin t & -\cos t & t\sin t - 3\cos t & 1 \end{vmatrix} \\
&\overset{R_1+R_3\to R_3, R_2+R_4\to R_4}{=} \begin{vmatrix} \cos t & \sin t & t\cos t & 0 \\ -\sin t & \cos t & -t\sin t + \cos t & 0 \\ 0 & 0 & -2\sin t & 0 \\ 0 & 0 & -2\cos t & 1 \end{vmatrix} \\
&= -2\sin t.
\end{aligned}
$$

$$
\begin{aligned}
y &= \sum_{i=1}^{n} y_i \int b(t)\frac{W_i}{W}dt \\
&= \frac{\cos t}{2}\int \sin^2 t + t\sin t\cos t dt + \frac{\sin t}{2}\int \sin t\cos t + t\sin^2 t dt \\
&+ \frac{t\cos t}{2}\int -\sin t\cos t dt + \frac{t\sin t}{2}\int -\sin^2 t dt \\
&= \frac{\cos t}{2}\left(\frac{t}{2} - \frac{t\cos 2t}{4} - \frac{\sin 2t}{8} + c_1\right) + \frac{\sin t}{2}\left(\frac{t^2}{4} - \frac{t\sin 2t}{4} - \frac{3\cos 2t}{8} + c_2\right) \\
&+ \frac{t\cos t}{2}\left(\frac{\cos 2t}{4} + c_3\right) + \frac{t\sin t}{2}\left(\frac{\sin 2t}{4} - \frac{t}{2} + c_4\right)
\end{aligned}
$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

We note that this particular example can also be solved by the method of undetermined coefficients, which is a much faster way of finding the solution. The power of *variation of parameters is that it can be applied for an arbitrary right hand side. In real life applications, the integrals can often be numerically evaluated and this gives an effective procedure to find* y *once* $b(t)$ *is given.*