MATH 219

Fall 2020

Lecture 18

Lecture notes by Özgür Kişisel

Content: Review of power series. Series solutions near an ordinary point.

Suggested Problems: (Boyce, Di Prima, 9th edition)

§5.1: 4,7,13,16,21,26

§5.2: 1,4,8,12,20,21

§5.3: 2,7,10,12,22,23

Let us assume that we have a second order linear differential equation whose coefficients are not necessarily constant:

$$P(x)y'' + Q(x)y' + R(x)y = F(x).$$

Previously discussed methods that use the characteristic equation are useless for such an equation unless $Q(x)/P(x)$ and $R(x)/P(x)$ are both constant. Indeed, it is a difficult to matter to find explicit solutions of such an equation. Instead, we will try to find the Taylor series expansions of the solutions around a given point x_0, of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

and try to determine what the coefficients a_n should be. The advantage of this method is that it is widely applicable and straightforward: It does not require any "tricks" that differ from one equation to another. On the other hand there are some disadvantages: First, it will not be easy to determine all a_n's at once. Instead, one determines each a_n in terms of the previous values of the sequence recursively, but it is in general difficult to find a closed form expression for a_n in terms of n. Second, the solution obtained may converge fairly slowly and even more so when x moves away from x_0. Furthermore, there often is a limitation on how far we can move away from x_0; most of the time the series will have a finite radius of convergence and will not be usable outside this range.

We will start by reviewing power series and their properties. Readers who are already comfortable with these notions may prefer to skip to section 2 and look at section 1 whenever necessary.

1 Review of Power Series

1.1 Definition, Region of Convergence

Definition 1.1 *A power series centered at* x_0 *is an infinite series*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Note that there are no negative power terms in the series and all terms are centered around x_0. By convention, we assume that $(x - x_0)^0 = 1$ for all x. A power series typically converges for certain values of x and diverges for others. There is a certain pattern for the set of values of x for which the series converges. This pattern is valid for all power series:

Theorem 1.1 *(Existence of Radius of Convergence) Say* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is a power series centered at* x_0*. Then there exists a nonnegative real number* ρ *(the cases* $\rho = 0$ *and* $\rho = \infty$ *are also allowed) such that*

1. *The series converges absolutely for all* x *such that* $|x - x_0| < \rho$*,*
2. *The series diverges for all* x *such that* $|x - x_0| > \rho$*.*

The number ρ is called the **radius of convergence** of the series. Basically anything can happen on the boundary between these two cases, which is the set $|x - x_0| = \rho$, namely the situation depends on the particular example. The **interval of convergence** of the series is the set of all x for which the series converges. By the theorem above, this set is an open, closed or half open interval centered around x_0.

Example 1.1 *Find the center, coefficients* a_n*, radius and interval of convergence of the series*

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}.$$

Solution: *Rewrite the series in the form*

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n(x-(-1/2))^n}{n^2}.$$

Therefore the center x_0 *of the series is* $-1/2$ *and*

$$a_n = \frac{2^n}{n^2}.$$

In order to find the radius of convergence, apply the ratio test: The series converges absolutely if

$$\begin{aligned} \lim_{n\to\infty} \frac{|2x+1|^{n+1}/(n+1)^2}{|2x+1|^n/n^2} &< 1 \\ |2x+1| &< 1 \\ x &\in (-1,0). \end{aligned}$$

Similarly, the series diverges if $|2x+1| > 1$. *We need to check the endpoints of the interval* $(-1,0)$ *separately. If* $x = 0$ *then*

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by the p-test. Similarly, for $x = -1$ *, the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges since it absolutely converges. Therefore, the radius of the power series is $1/2$ *and the interval of convergence is* $[-1,0]$.

1.2 Operations on Power Series

Suppose that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges to $f(x)$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converges to $g(x)$ for $|x-x_0| < \rho$. Then the following statements hold for $|x-x_0| < \rho$:

1. $f(x) + g(x) = \sum_{n=0}^{\infty}(a_n + b_n)(x-x_0)^n$

2. $f(x) - g(x) = \sum_{n=0}^{\infty} (a_n - b_n)(x - x_0)^n$

3. $f(x)g(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

4. $f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$

5. $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$

These statements will not be proven here, but they will be used freely.

Definition 1.2 *A function $f(x)$ is said to be* ***analytic*** *at a point x_0 if there exists an open neighborhood of x_0 in which the Taylor series of $f(x)$ converges to $f(x)$.*

Remark 1.1 By property 4 above, if a function is analytic at a point, then it is continuous there, furthermore it has derivatives of all orders. The converse is not true. There exist functions $f(x)$ having derivatives of all orders but yet $f(x)$ is not equal to its Taylor series in any open neighborhood of x_0. One example is the function having values $\mathrm{e}^{-1/x}$ for $x > 0$ and value 0 for $x \leq 0$. However, polynomials, trigonometric functions and exponentials are analytic everywhere. Rational functions are analytic at their points of continuity. Products and sums of analytic functions are analytic.

Example 1.2 *Check that $(\mathrm{e}^x)' = \mathrm{e}^x$ by looking at the Taylor series of e^x.*

Solution: *Recall that the Taylor series of e^x is*

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Differentiate this series term by term:

$$\begin{aligned}
(\mathrm{e}^x)' &= \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} \\
&= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{m=0}^{\infty} \frac{x^m}{m!} \\
&= e^x.
\end{aligned}$$

In the third step, we substituted $m = n - 1$. This type of "index shifting" is pretty common when one deals with power series, so it will be considered in the next subsection.

1.3 Index Shifting

Suppose we have an infinite series

$$\sum_{n=l}^{\infty} a_n x^{n-k}.$$

Put $m = n - k$. Then the series takes the form

$$\sum_{m=l-k}^{\infty} a_{m+k} x^m.$$

It is customary to use n for the dummy index again and write

$$\sum_{n=l-k}^{\infty} a_{n+k} x^n.$$

Such a notational change in the index, which actually does not change anything about the series, is called an **index shift**. Such shifts are especially useful when one needs to compare two power series term by term.

Example 1.3 *If the equality*

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n$$

holds for every x and $a_0 = 1, a_1 = 0$, then find a_n for each n.

Solution: *Shift index in the first sum such that the main term contains x^n rather than x^{n-2}:*

$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} a_{n+2} x^n = \sum_{n=0}^{\infty} a_n x^n.$$

In order for the equality to hold for all x, the coefficients of x^n on both sides must be equal. Therefore we obtain the recursive relation

$$\frac{(n+2)(n+1)}{2} a_{n+2} = a_n$$

for all $n \geq 0$. Using $a_0 = 1$ and $a_1 = 0$ let us compute the first few terms:

$$\begin{aligned}
a_2 &= \frac{2}{2 \cdot 1} a_0 = 1, \\
a_3 &= \frac{2}{3 \cdot 2} a_1 = 0, \\
a_4 &= \frac{2}{4 \cdot 3} a_2 = \frac{2^2}{4!}, \\
a_5 &= \frac{2}{5 \cdot 4} a_3 = 0, \\
a_6 &= \frac{2}{6 \cdot 5} a_4 = \frac{2^3}{6!}, \\
&\cdots
\end{aligned}$$

It is clear that $a_n = 0$ for all odd values of n. It is reasonable to guess that $a_{2k} = 2^k/(2k)!$. Let us prove this by induction. It is true for $n = 2k = 2$. Assume it holds up to $n = 2k$ and check for $n = 2k+2$:

$$\begin{aligned}
a_{2k+2} &= \frac{2}{(2k+2)(2k+1)} a_{2k} \\
&= \frac{2}{(2k+2)(2k+1)} \frac{2^k}{(2k)!} \\
&= \frac{2^{k+1}}{(2k+2)!}.
\end{aligned}$$

So, our guess was correct. Therefore the series is

$$\sum_{k=0}^{\infty} \frac{2^k x^{2k}}{(2k)!}.$$

We note in passing that the series can be rewritten in terms of familiar functions:

$$\sum_{k=0}^{\infty} \frac{2^k x^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(\sqrt{2}x)^{2k}}{(2k)!} = \frac{e^{\sqrt{2}x} + e^{-\sqrt{2}x}}{2} = \cosh(\sqrt{2}x).$$

2 Solving ODE's Near an Ordinary Point

Suppose that we have an ODE of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

The equation can be rewritten as

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0.$$

Definition 2.1 *We say that x_0 is an **ordinary point** for the ODE, if the functions $Q(x)/P(x)$ and $R(x)/P(x)$ are both analytic in a neighborhood of x_0. Otherwise, x_0 is called a **singular point**.*

Around an ordinary point, the strategy for finding power series solutions is straightforward: Substitute $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ and solve for a_n in terms of a_0 and a_1.

Example 2.1 *Solve the ODE $y'' - xy = 0$ around $x_0 = 0$.*

Solution: *The functions $Q(x)/P(x) = 0$ and $R(x)/P(x) = -x$ are analytic at all points. In particular, x_0 is an ordinary point. Set*

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and put it in the equation. First of all,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Now, write the equation and modify until the general terms of all summands have the same power of x in their general terms:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

The coefficient of x^0 on the left hand side is $2a_2$. Hence $2a_2 = 0$, therefore $a_2 = 0$.

By looking at the coefficient of x^n on both sides for $n \geq 1$ we get

$$\begin{aligned}(n+2)(n+1)a_{n+2} - a_{n-1} &= 0\\ a_{n+2} &= \frac{a_{n-1}}{(n+2)(n+1)}\end{aligned}$$

Since $a_2 = 0$, we get $a_5 = a_8 = a_{11} = \ldots = 0$. Starting from a_0, we have

$$a_3 = \frac{a_0}{3 \cdot 2}, \quad a_6 = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \quad a_9 = \frac{a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}, \ldots$$

Similarly, starting from a_1 we get

$$a_4 = \frac{a_1}{4 \cdot 3}, \quad a_7 = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}, \quad a_{10} = \frac{a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}, \ldots$$

Therefore

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_0\left(1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \frac{x^9}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} + \ldots\right)\\ &+ a_1\left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \frac{x^{10}}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} + \ldots\right)\end{aligned}$$

One can check by ratio test that this series converges for all values of $x \in \mathbb{R}$. Note that the solution is a linear combination of two solutions y_1, y_2. We can check that y_1 and y_2 are linearly independent as follows: Suppose $c_1y_1 + c_2y_2 = 0$. Evaluate at 0 and use $y_1(0) = 1, y_2(0) = 0$ in order to get

$$c_1 = 0$$

Now $c_2y_2 = 0$. But y_2 is not identically 0, therefore $c_2 = 0$. Hence y_1 and y_2 are linearly independent.

Example 2.2 *Find the first 5 nonzero terms of the power series solution of the initial value problem*

$$(1-x)y'' + xy' - y = 0, \qquad y(0) = -3, y'(0) = 2$$

around $x_0 = 0$.

Solution: *The functions $Q(x)/P(x) = x/(1-x)$ and $R(x)/P(x) = -1/(1-x$ are both analytic at $x_0 = 0$. Therefore $x_0 = 0$ is an ordinary point. Set $y = \sum_{n=0}^{\infty} a_n x^n$. Then*

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

We have

$$\begin{aligned}
(1-x)\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) + x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n &= 0.
\end{aligned}$$

Notice that $y(0) = a_0 = -3$ and $y'(0) = a_1 = 2$. Looking at the coefficients of x^0 on both sides, we find

$$2a_2 + a_0 = 0 \Rightarrow a_2 = 3/2.$$

Now look at the coefficients of x^n on both sides for $n \geq 1$:

$$\begin{aligned}
(n+2)(n+1)a_{n+2} - (n+1)n a_{n+1} + (n-1)a_n &= 0 \\
a_{n+2} &= \frac{n}{n+2} a_{n+1} - \frac{n-1}{(n+1)(n+2)} a_n
\end{aligned}$$

We get

$$\begin{aligned}
a_3 &= \frac{1}{3}a_2 - \frac{0}{2\cdot 3}a_1 = \frac{1}{2} \\
a_4 &= \frac{2}{4}a_3 - \frac{1}{3\cdot 4}a_2 = \frac{1}{8} \\
a_5 &= \frac{3}{5}a_4 - \frac{2}{4\cdot 5}a_3 = \frac{1}{40}
\end{aligned}$$

Therefore

$$y = -3 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \frac{x^5}{40} + \dots$$

How can we guarantee that this method always works when x_0 is an ordinary point? First, notice that if $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, then

$$\begin{aligned} y^{(n)}(x) &= n!a_n + (n+1)n\dots 2a_{n+1}(x-x_0) + \dots \\ y^{(n)}(x_0) &= n!a_n \\ a_n &= \frac{y^{(n)}(x_0)}{n!}. \end{aligned}$$

Therefore, given the initial conditions $y(0) = a_0$ and $y'(0) = a_1$, we can solve for all a_n in terms of a_0 and a_1 if and only if we can solve for all $y^{(n)}(x_0)$ recursively in terms of $y(0)$ and $y'(0)$. We can use the equation to solve for $y^{(n)}(x_0)$ as follows: Since $P(x)y'' + Q(x)y' + R(x)y = 0$, we have

$$y''(x_0) = -\frac{Q(x_0)}{P(x_0)}y'(x_0) - \frac{R(x_0)}{P(x_0)}y(x_0).$$

We already know $y'(x_0)$ and $y(x_0)$. Furthermore, $Q(x_0)/P(x_0)$ and $R(x_0)/P(x_0)$ can be computed since x_0 is an ordinary point. Hence we can find $y''(x_0)$.

The other values $y^{(n)}(x_0)$ can be recursively found in a similar manner. For instance, in order to find $y'''(x_0)$, first differentiate the ODE once and leave the y''' term alone. From the resulting equation one can solve for $y'''(x_0)$. For this computation and the computation of the higher derivatives of y at x_0 one needs the fact that $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ have derivatives of all orders at x_0, which is a consequence of their analyticity.

We saw that we can find a power series solution $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$. Of course, this solution would be useless if it did not converge in an open set containing x_0. We quote the following theorem about the radius of convergence of such a solution without proof. This theorem guarantees the convergence of the series in such a neighborhood:

Theorem 2.1 *Let x_0 be an ordinary point and let x_1 be the nearest singular point to x_0. Let ρ be the distance from x_0 to x_1 (take $\rho = \infty$ if there are no singular points). Then the radius of convergence of the power series solution constructed above is at least ρ.*

Now, it is a fact that analytic functions cannot have infinitely many singular points which converge towards an ordinary point (this weird phenomenon could occur if we relaxed the assumption of analyticity). For this reason, the ρ in the theorem will always be positive. The last thing that one must check is that the series obtained is actually a solution of the ODE. This again requires the analyticity of $Q(x)/P(x)$ and $R(x)/P(x)$ which implies that the series computations agree with what one would get by computations with the original functions. This finishes the outline of the proof that the method always works at an ordinary point.