MATH 219

Fall 2020

Lecture 19

Lecture notes by Özgür Kişisel

Content: Regular singular points. Euler equations.

Suggested Problems: (Boyce, Di Prima, 9th edition)

§5.4: 7,10,16,20,26,37

1 Regular Singular Points

Suppose that we have a second order linear ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0$$

Definition 1.1 *A point x_0 is called a* ***regular singular point*** *for the ODE above if it is*

- *a singular point (in other words, not an ordinary point),*
- *the functions $\dfrac{(x-x_0)q(x)}{p(x)}$ and $\dfrac{(x-x_0)^2 r(x)}{p(x)}$ are analytic in a neighborhood of x_0.*

Strictly speaking, the two functions above are not defined at x_0 if x_0 is a singular point. However, the condition above should be interpreted as follows: The limits of the two functions above are both finite and if we declare the value of each function to be equal to the relevant limit, then the resulting function is analytic.

Example 1.1 *Consider the ODE*

$$x^3(x-1)y'' + (x+2)y' + 4y = 0.$$

Since the functions $\dfrac{x+2}{x^3(x-1)}$ *and* $\dfrac{4}{x^3(x-1)}$ *are both analytic at all points other than* 0 *and* 1*, all points other than these two are ordinary points. Both* 0 *and* 1 *are singular points. Since the functions*

$$\frac{(x-1)(x+2)}{x^3(x-1)}, \qquad \frac{(x-1)^2 4}{x^3(x-1)}$$

are both analytic near 1*, the point* 1 *is a regular singular point. On the other hand, the function*

$$\frac{x(x+2)}{x^3(x-1)}$$

is not analytic at 0*, so the point* 0 *is not a regular singular point (it is a "worse" type of singularity).*

A regular singular point is in a sense a "well-behaved" singularity and it turns out that power series methods are useful near regular singular points. We will first look at some of the simplest ODE's which have regular singularities in the next section, these are called Euler equations. Euler equations can be explicitly solved, and their solutions will then be used as prototypes to guess the form of the power series solutions of an arbitrary ODE with regular singular points in the next lecture.

2 Euler Equations

Definition 2.1 *An equation of the form*

$$(x-x_0)^2 y'' + \alpha(x-x_0)y' + \beta y = 0$$

where α, β *are constants, is called an* ***Euler equation*** *(or a Cauchy-Euler equation).*

It is easy to check that the point $x = x_0$ is a regular singular point for an Euler equation. The two intervals $x > x_0$ and $x < x_0$ should be analyzed separately. First, assume that $x > x_0$. In order to solve an Euler equation, use the substitution

$x - x_0 = e^t$ which makes sense since $x - x_0$ is positive. Since $\frac{dx}{dt} = e^t$, we have

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{dy}{dt}\frac{dt}{dx} \\ &= e^{-t}\frac{dy}{dt} \end{aligned}$$

$$\begin{aligned} y'' &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx} \\ &= -e^{-2t}\frac{dy}{dt} + e^{-2t}\frac{d^2y}{dt^2} \end{aligned}$$

Substitute these expressions into the ODE:

$$\begin{aligned} e^{2t}\left(-e^{-2t}\frac{dy}{dt} + e^{-2t}\frac{d^2y}{dt^2}\right) &+ \alpha e^t\left(e^{-t}\frac{dy}{dt}\right) + \beta y = 0 \\ \frac{d^2y}{dt^2} + (\alpha - 1)\frac{dy}{dt} + \beta y &= 0 \end{aligned}$$

Therefore, we obtain a constant coefficient linear ODE in y and t, which can be solved by using the characteristic equation $r^2 + (\alpha - 1)r + \beta = 0$. There are three cases, depending on the sign of the discriminant $\Delta = (\alpha - 1)^2 - 4\beta$:

Case 1, $\Delta > 0$: In this case, the characteristic equation has two real, distinct roots r_1, r_2. The general solution is

$$\begin{aligned} y &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ &= c_1(x - x_0)^{r_1} + c_2(x - x_0)^{r_2} \end{aligned}$$

Case 2, $\Delta = 0$: In this case, the characteristic equation has repeated real roots, $r_1 = r_2 = r$. The general solution is

$$\begin{aligned} y &= c_1 e^{rt} + c_2 t e^{rt} \\ &= c_1(x - x_0)^r + c_2(\ln(x - x_0))(x - x_0)^r \\ &= (x - x_0)^r(c_1 + c_2 \ln(x - x_0)) \end{aligned}$$

Case 3, $\Delta < 0$: In this case, the roots of the characteristic equation are two complex numbers that are conjugates of each other, namely $r_1 = k + il$, $r_2 = k - il$ with $l \neq 0$. The general solution is

$$\begin{aligned} y &= c_1 e^{kt}\cos(lt) + c_2 e^{kt}\sin(lt) \\ &= c_1 (x - x_0)^k \cos(l \ln(x - x_0)) + c_2 (x - x_0)^k \sin(l \ln(x - x_0)) \end{aligned}$$

What about the interval $x < x_0$? This case can be handled by making the substitution $x - x_0 = -e^t$ and repeating the same steps as above. One can easily check that the effect of this change is replacing $x - x_0$ by $|x - x_0|$ in each of the formulas above. For instance, in Case 1 above, the solution becomes

$$y = c_1 |x - x_0|^{r_1} + c_2 |x - x_0|^{r_2}$$

etc.