MATH 219

Fall 2020

Lecture 21

Lecture notes by Özgür Kişisel

Content: Laplace transform. Definition and first properties.

Suggested Problems (Boyce, Di Prima, 9th edition):

§6.1: 1, 2, 10, 17, 23, 26, 27

1 Improper Integrals, Piecewise Continuity

Definition 1.1 *Let $a \in \mathbb{R}$. The **improper integral** from a to ∞ of $f(t)$ is defined to be*

$$\int_a^\infty f(t)dt = \lim_{M\to\infty} \int_a^M f(t)dt.$$

*We say that the integral **converges** if the limit exists and **diverges** if it does not exist.*

Definition 1.2 *Let $f(t)$ be a function defined on $[a,b]$. We say that $f(t)$ is **piecewise continuous** on $[a,b]$ if there exists finitely many points a_i having the property $a = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = b$ such that*

- *$f(t)$ is continuous on each interval (a_{i-1}, a_i),*
- *the right and left limits of $f(t)$ at a_i exist for each $0 < i < n$, the right limit of $f(t)$ exists at a_0 and the left limit of $f(t)$ exists at a_n.*

We say that $f(t)$ is piecewise continuous on an infinite interval if it is piecewise continuous on every finite subinterval of this infinite interval.

Example 1.1 *Suppose that*

$$f(t) = \begin{cases} 1, & 2n \le t < 2n+1, \\ 0, & 2n+1 \le t < 2n+2 \end{cases}$$

for all $n \in \mathbb{Z}$. Then $f(t)$ is piecewise continuous on $\mathbb{R}$.

Example 1.2 *Say*

$$g(t) = \begin{cases} 1, & \dfrac{1}{2n} \leq t < \dfrac{1}{2n-1}, \\ 0, & \dfrac{1}{2n+1} \leq t < \dfrac{1}{2n} \end{cases}$$

for n a positive integer, and $g(0) = 0$. Then $g(t)$ is not piecewise continuous on $[0, 1]$ since it has infinitely many jump discontinuities in a finite interval.

The main result that we need about piecewise continuous functions is the following:

Theorem 1.1 *Say $f(t)$ is piecewise continuous on a finite closed interval $[a, b]$. Then $f(t)$ is integrable on $[a, b]$.*

We will also need the following comparison theorem, which we also state without proof:

Theorem 1.2 *Suppose that*

- *$f(t)$ and $g(t)$ are piecewise continuous on $[a, \infty)$,*
- *$|f(t)| \leq g(t)$ on $[a, \infty)$,*
- *$\int_a^\infty g(t)dt$ converges.*

Then $\int_a^\infty f(t)dt$ is convergent.

2 Laplace transform

Laplace transform is an example of an integral transform. Let us first define integral transforms in general.

Definition 2.1 *Say $K(s, t)$ is a given function of s and t, and $\alpha < \beta$ are two fixed real numbers. Then*

$$F(s) = \int_\alpha^\beta K(s, t) f(t) dt$$

*is called an **integral transform** with kernel $K(s, t)$.*

An integral transform produces a function $F(s)$ of s from a function $f(t)$ of t. With careful choices of $K(s,t)$, one can hope that certain nice properties of $f(t)$ transform into some other nice properties of $F(s)$.

Definition 2.2 *Suppose that $f(t)$ is piecewise continuous on $[0,\infty)$. Then*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

*is called the **Laplace transform** of $f(t)$.*

The Laplace transform of $f(t)$ typically converges for certain values of s and diverges for others.

Example 2.1 *Let us find the Laplace transform of $f(t) = 1$.*

$$\begin{aligned}
\mathcal{L}(1) &= \int_0^\infty e^{-st} 1 dt \\
&= \lim_{M\to\infty} \int_0^M e^{-st} dt \\
&= \lim_{M\to\infty} -\frac{e^{-st}}{s}\bigg|_0^M \\
&= \lim_{M\to\infty} \frac{1}{s} - \frac{e^{-sM}}{s}.
\end{aligned}$$

The limit exists and equals $\frac{1}{s}$ if $s > 0$. It does not exist if $s < 0$. For $s = 0$, the integration step above is not valid. For $s = 0$ the integral instead gives $\lim_{M\to\infty} M$ which diverges. Hence,

$$\mathcal{L}\{1\} = \frac{1}{s}$$

and it converges if and only if $s > 0$.

Theorem 2.1 *Suppose that $f(t)$ is piecewise continuous on $[0,\infty)$. Say that there exist constants K, a such that $|f(t)| \leq Ke^{at}$ for all t. Then $\mathcal{L}\{f(t)\}$ converges for $s > a$.*

Proof Set $g(t) = Ke^{at}$. First, let us look at the Laplace transform of $g(t)$:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t) dt \\
&= \lim_{M\to\infty} \int_0^M e^{-st} K e^{at} dt \\
&= \lim_{M\to\infty} \int_0^M K e^{(a-s)t} dt \\
&= \lim_{M\to\infty} \left. \frac{Ke^{(a-s)t}}{a-s} \right|_0^M \\
&= \lim_{M\to\infty} \frac{K}{s-a} - \frac{Ke^{(a-s)M}}{s-a}.
\end{aligned}$$

The limit exists if $s > a$. Since $|f(t)| \le g(t)$ by assumption, $|f(t)e^{-st}| \le g(t)e^{-st}$. Now, apply the comparison theorem to $f(t)e^{-st}$ and $g(t)e^{-st}$. This shows that $\mathcal{L}\{f(t)\}$ converges for $s > a$. $\square$

Remark 2.1

1. *There exist functions for which it is impossible to find any such* K, a. *For instance, consider the double exponential function* $f(t) = e^{e^t}$. *One can directly check that the Laplace transform of* $f(t)$ *does not converge for any value of* s.

2. *Using the comparison theorem, one can show that the region of convergence for the Laplace transform of any function is of the form* $s > a$, *or the empty set or all of* $\mathbb{R}$.

3. *It turns out that it is more natural to consider the values of* s *in* $\mathbb{C}$ *rather than* $\mathbb{R}$. *In this case, the region of convergence in the theorem above becomes* $Re(s) > a$.

Let us now compute the Laplace transforms of some familiar functions.

Example 2.2 *Suppose that* $f(t) = e^{at}$.

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} dt \\
&= \lim_{M\to\infty} \int_0^M e^{(a-s)t} dt \\
&= \lim_{M\to\infty} \left. \frac{e^{(a-s)t}}{a-s} \right|_0^M \\
&= \lim_{M\to\infty} \frac{1}{s-a} - \frac{e^{(a-s)M}}{s-a}.
\end{aligned}$$

This limit exists when $Re(s) > Re(a)$ *and is equal to* $\dfrac{1}{s-a}$.

Lemma 2.1 *Suppose that* $\mathcal{L}\{f(t)\} = F(s)$ *for* $Re(s) > a$ *and* $\mathcal{L}\{g(t)\} = G(s)$ *for* $Re(s) > b$. *Then*

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s)$$

for $Re(s) > \max(a, b)$.

Proof

$$\begin{aligned}
\mathcal{L}\{c_1 f(t) + c_2 g(t)\} &= \int_0^\infty e^{-st}(c_1 f(t) + c_2 g(t)) dt \\
&= c_1 \int_0^\infty e^{-st} f(t) dt + c_2 \int_0^\infty e^{-st} g(t) dt \\
&= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}.
\end{aligned}$$

These operations are valid in the region where both summands are convergent, in particular they are valid for $Re(s) > \max(a, b)$. $\square$

Example 2.3 *Say* $f(t) = \cos at$. *By Euler's formula,* $e^{iat} = \cos at + i \sin at$. *From* *here, we deduce*

$$\begin{aligned}\cos at &= \frac{1}{2}(e^{iat} + e^{-iat}) \\ \mathcal{L}\{\cos at\} &= \frac{1}{2}(\mathcal{L}\{e^{iat}\} + \mathcal{L}\{e^{-iat}\}) \\ &= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) \\ &= \frac{s}{s^2+a^2}.\end{aligned}$$

Since $Re(ia) = Re(-ia) = 0$*, the Laplace transform of* $\cos at$ *converges to* $\frac{s}{s^2+a^2}$ *for* $Re(s) > 0$*.*

Example 2.4 *Say* $f(t) = \sin at$*. As in the previous example, we can write*

$$\begin{aligned}\sin at &= \frac{1}{2i}(e^{iat} - e^{-iat}) \\ \mathcal{L}\{\sin at\} &= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) \\ &= \frac{a}{s^2+a^2}.\end{aligned}$$

Again, the Laplace transform converges for $Re(s) > 0$*.*

Example 2.5 *Suppose now that* $f(t) = t^n$ *where* n *is a nonnegative integer. Denote the Laplace transform of* t^n *by* $F_n(s)$*. When* $n = 0$*,* $f(t) = 1$ *therefore* $F_0(s) = 1/s$ *for* $Re(s) > 0$*. Now let* $n > 0$ *and* $Re(s) > 0$ *:*

$$
\begin{aligned}
F_n(s) = \mathcal{L}\{t^n\} &= \int_0^\infty t^n e^{-st} dt \\
&= \lim_{M\to\infty} \int_0^M t^n e^{-st} dt \\
&= \lim_{M\to\infty} -\frac{t^n e^{-st}}{s}\Big|_0^M + \frac{1}{s}\int_0^M n t^{n-1} e^{-st} dt \\
&= \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt \\
&= \frac{n}{s} F_{n-1}(s)
\end{aligned}
$$

The first limit on the third line above is zero by L'Hospital's rule. We therefore obtain the recursive relation $F_n(s) = nF_{n-1}(s)/s$. Using this relation and the fact that $F_0(s) = 1/s$, one easily sees that

$$\mathcal{L}(t^n) = F_n(s) = \frac{n!}{s^{n+1}}$$

The methods of computation used in two of the examples above can be significantly generalized:

Proposition 2.1 *Suppose that $\mathcal{L}\{f(t)\} = F(s)$ for $Re(s) > k$. Then*

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

for $Re(s) > k + Re(a)$.

Proof

$$
\begin{aligned}
\mathcal{L}\{e^{at} f(t)\} &= \int_0^\infty e^{-st} e^{at} f(t) dt \\
&= \int_0^\infty e^{-(s-a)t} f(t) dt \\
&= F(s-a).
\end{aligned}
$$

Since the effect of this operation is to replace s by $s-a$, the latter integral converges for $Re(s-a) > k$, namely for $Re(s) > k + Re(a)$. $\square$

Proposition 2.2 *Suppose that $f(t)$ is piecewise continuous and $|f(t)| \leq Ke^{kt}$ for all t, so that $\mathcal{L}\{f(t)\} = F(s)$ converges for $Re(s) > k$. Then*

$$\mathcal{L}\{tf(t)\} = -\frac{dF}{ds}$$

for $Re(s) > k$.

Sketch of proof:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \\
-\frac{dF}{ds} &= -\frac{d}{ds}\int_0^\infty e^{-st} f(t)dt \\
&= -\int_0^\infty \frac{d}{ds}\left(e^{-st}f(t)\right)dt \\
&= \int_0^\infty e^{-st} tf(t)dt \\
&= \mathcal{L}\{tf(t)\}.
\end{aligned}$$

The reason that this is a sketch of proof rather than an honest proof is the step at which we switched the places of the d/ds operation and the integral sign. Such changes in two limiting processes are certainly not automatic in mathematics, one can easily construct examples where they fail. The condition needed is that the function in question decays "quickly enough", which turns out to work here because of the bound on $f(t)$. We refer the reader to books of advanced calculus for a more careful discussion of such matters. $\square$

Example 2.6 *Let us revisit the example of $f(t) = t^n$. Setting $F_n(s) = \mathcal{L}\{t^n\}$ as before, the proposition above implies,*

$$F_n(s) = \mathcal{L}\{t^n\} = \mathcal{L}\{t \cdot t^{n-1}\} = -\frac{dF_{n-1}}{ds}.$$

Indeed, one can directly check the equality

$$\frac{n!}{s^{n+1}} = -\frac{d}{ds}\left(\frac{(n-1)!}{s^n}\right).$$

Example 2.7 *Let us find the Laplace transform of* $f(t) = e^{-2t}\sin\sqrt{3}t$. *Since* $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, *we have*

$$\mathcal{L}\{\sin\sqrt{3}t\} = \frac{\sqrt{3}}{s^2+3}.$$

Now, using the first proposition above, the effect of multiplication by e^{-2t} *is to replace* s *by* $s+2$. *So, we have*

$$\mathcal{L}\{e^{-2t}\sin\sqrt{3}t\} = \frac{\sqrt{3}}{(s+2)^2+3}.$$

Example 2.8 *Let us find the Laplace transform of* $f(t) = t^2\cos 2t$. *First of all,*

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}.$$

By using the second proposition above,

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{s^2-4}{(s^2+4)^2}$$

Using the proposition once again,

$$\mathcal{L}\{t^2\cos 2t\} = -\frac{d}{ds}\left(\frac{s^2-4}{(s^2+4)^2}\right) = \frac{2s^3-24s}{(s^2+4)^3}.$$

Our final result in this lecture concerns the Laplace transform of the function obtained from $f(t)$ by scaling the variable t.

Proposition 2.3 *Suppose that* $\mathcal{L}\{f(t)\} = F(s)$ *for* $Re(s) > k$. *Say* $c > 0$ *is a constant. Then*

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right)$$

for $Re(s) > ck$.

Proof: Making the substitution $\tau = ct$ in the first integral below, we get

$$\begin{aligned}\mathcal{L}\{f(ct)\} &= \int_0^\infty e^{-st}f(ct)dt \\ &= \int_0^\infty e^{-\frac{s}{c}\tau}f(\tau)\frac{1}{c}d\tau \\ &= \frac{1}{c}F\left(\frac{s}{c}\right).\end{aligned}$$

The integral on the second line converges for $Re(s/c) > k$, namely for $Re(s) > ck$. $\square$

This result tells us the following: If we scale the domain of $f(t)$ so that the graph is contracted ($c > 1$ case), the opposite happens to the graph of $F(s)$ which is expanded along its domain. Furthermore, the amplitude of $F(s)$ is scaled by $1/c$. Taking $c < 1$ has the reverse effect.