MATH 219

Fall 2020

Lecture 23

Lecture notes by Özgür Kişisel

Content: Differential equations with discontinuous forcing functions.

Suggested Problems (Boyce, Di Prima, 9th edition):

§6.4: 1, 3, 5, 10, 13, 16

In this lecture, we will use previously developed techniques in order to solve linear, constant coefficient ODE's with a piecewise continuous right hand side:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = g(t).$$

The main strategy is as before: Take Laplace transforms of both sides, solve for $Y(s)$ from the resulting equation and recover $y(t)$ from $Y(s)$. If $g(t)$ is expressed in terms of unit step functions then the theorem of the last lecture will be of great help for the computations. Let us directly start working on examples.

Example 0.1 *Solve the initial value problem*

$$y'' + 3y' + 2y = u_2(t) - u_3(t), \quad y(0) = y'(0) = 0.$$

Solution: *Take Laplace transforms of both sides. Then*

$$\begin{aligned} s^2Y(s) + 3sY(s) + 2Y(s) &= \frac{e^{-2s} - e^{-3s}}{s} \\ Y(s) &= \frac{e^{-2s} - e^{-3s}}{s(s+1)(s+2)} \end{aligned}$$

It remains to find the inverse transform $y(t)$. Let us first invert $\frac{1}{s(s+1)(s+2)}$. We will need its partial fraction expansion:

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$
$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Putting $s = 0$ we get $A = 1/2$, likewise putting $s = -1$ gives $B = -1$ and putting $s = -2$ gives $C = 1/2$. Therefore,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{2s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{2(s+2)}\right\} \\ &= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\end{aligned}$$

By the theorem of the last lecture,

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)(s+2)}\right\} = u_2(t)\left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right)$$

and

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s+1)(s+2)}\right\} = u_3(t)\left(\frac{1}{2} - e^{-(t-3)} + \frac{1}{2}e^{-2(t-3)}\right),$$

therefore the answer is the difference between these two expressions.

Example 0.2 *Suppose that we have an undamped spring-mass system, initially at rest, where the object has mass equal to 1 unit, and the spring constant is 4 units. Initially there is no external force on the system. Starting from time $t = 3$ upto time $t = 6$, an external force linearly increasing from 0 to 9 units is applied to the system. Afterwards, the external force decreases linearly from 9 units to 0 from time $t = 6$ upto $t = 15$. After $t = 15$, the system is free again. Write a differential equation that models the system and solve it.*

Solution: *The equation of motion is*

$$y'' + 4y = F(t)$$

where $F(t)$ is the external force applied to the object. Furthermore, $y(0) = y'(0) = 0$ since the system is initially at rest. The external force is

$$F(t) = \begin{cases} 0, & t < 3, \\ 3t - 9, & 3 \le t < 6, \\ 15 - t, & 6 \le t < 15, \\ 0, & 15 \le t. \end{cases}$$

Let us express $F(t)$ in terms of step functions:

$$F(t) = (3t - 9)u_3(t) + (24 - 4t)u_6(t) + (t - 15)u_{15}(t).$$

Therefore, we can rewrite the initial value problem in the form:

$$y'' + 4y = (3t-9)u_3(t) + (24-4t)u_6(t) + (t-15)u_{15}(t), \quad y(0) = y'(0) = 0.$$

Take Laplace transforms of both sides:

$$\begin{aligned}(s^2+4)Y(s) &= \frac{3e^{-3s}}{s^2} - \frac{4e^{-6s}}{s^2} + \frac{e^{-15s}}{s^2}, \\ Y(s) &= \frac{3e^{-3s}}{s^2(s^2+4)} - \frac{4e^{-6s}}{s^2(s^2+4)} + \frac{e^{-15s}}{s^2(s^2+4)}.\end{aligned}$$

We will need the partial fraction expansion of $1/(s^2(s^2+4))$*:*

$$\begin{aligned}\frac{1}{s^2(s^2+4)} &= \frac{A}{s^2} + \frac{B}{s^2+4} \\ 1 &= (A+B)s^2 + 4A \\ A &= \frac{1}{4}, \quad B = -\frac{1}{4}.\end{aligned}$$

(Note: We did not need the s *terms in the partial fraction expansion since the original fraction can be written in terms of* s^2 *only. If this is confusing, substitute* $u = s^2$ *first.) We have*

$$\begin{aligned}h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{4s^2} - \frac{1}{4(s^2+4)}\right\} \\ &= \frac{t}{4} - \frac{\sin(2t)}{8}\end{aligned}$$

By using the theorem from the last lecture, this immediately gives us,

$$y(t) = 3u_3(t)h(t-3) - 4u_6(t)h(t-6) + u_{15}(t)h(t-15).$$

Example 0.3 *Suppose that*

$$y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 1 - u_{100}(t), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Find $y(t)$ *and the approximate value of* $y(101)$*.*

Solution: *Take Laplace transforms of both sides. We get:*

$$\begin{aligned}
(s^4 + 2s^3 + 3s^2 + 2s + 1)Y(s) &= \frac{1}{s}(1 - e^{-100s}) \\
(s^2 + s + 1)^2 Y(s) &= \frac{1}{s}(1 - e^{-100s}) \\
Y(s) &= \frac{1}{s(s^2 + s + 1)^2}(1 - e^{-100s}).
\end{aligned}$$

We need the partial fraction expansion of $1/(s(s^2 + s + 1)^2)$*. Notice that* $s^2 + s + 1$ *has negative discriminant. So we have doubled complex conjugate roots involved.*

$$\begin{aligned}
\frac{1}{s(s^2 + s + 1)^2} &= \frac{A}{s} + \frac{Bs + C}{s^2 + s + 1} + \frac{Ds + E}{(s^2 + s + 1)^2} \\
1 &= A(s^2 + s + 1)^2 + (Bs + C)s(s^2 + s + 1) + (Ds + E)s \\
1 &= (A + B)s^4 + (2A + B + C)s^3 + (3A + B + C + D)s^2 + (2A + C + E)s + A
\end{aligned}$$

Solving this system, we get

$$A = 1, B = -1, C = -1, D = -1, E = -1.$$

The next step is to find the inverse Laplace transforms of the summands of the fraction.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} &= 1, \\
\mathcal{L}^{-1}\left\{\frac{-s - 1}{s^2 + s + 1}\right\} &= \mathcal{L}^{-1}\left\{-\frac{s + \frac{1}{2}}{(s + \frac{1}{2})^2 + \frac{3}{4}} - \frac{\frac{1}{2}}{(s + \frac{1}{2})^2 + \frac{3}{4}}\right\} \\
&= -e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)
\end{aligned}$$

Before finding the inverse Laplace transform of the last summand, note that

$$\begin{aligned}
\mathcal{L}\{t\sin(at)\} &= -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right) = \frac{2as}{(s^2 + a^2)^2} \\
\mathcal{L}\{t\cos(at)\} &= -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) = \frac{s^2 - a^2}{(s^2 + a^2)^2} = \frac{1}{s^2 + a^2} - \frac{2a^2}{(s^2 + a^2)^2}
\end{aligned}$$

Incidentally, the last equation shows that

$$\mathcal{L}^{-1}\left\{\frac{2a^2}{(s^2 + a^2)^2}\right\} = -t\cos(at) + \frac{1}{a}\sin(at).$$

Now,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{-s-1}{(s^2+s+1)^2}\right\} &= \mathcal{L}^{-1}\left\{-\frac{s+\frac{1}{2}}{((s+\frac{1}{2})^2+\frac{3}{4})^2}-\frac{\frac{1}{2}}{((s+\frac{1}{2})^2+\frac{3}{4})^2}\right\}\\ &= -\frac{1}{\sqrt{3}}te^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)+\frac{4}{3}te^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{8}{3\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\end{aligned}$$

Combining all of this, we have

$$\begin{aligned}h(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+s+1)^2}\right\}\\ &= 1-e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}te^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\\ &\quad +\frac{4}{3}te^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{8}{3\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\end{aligned}$$

Finally,

$$y(t) = h(t) - u_{100}(t)h(t-100).$$

In order to find the approximate value of $y(101)$*, first notice that* $u_{100}(101) = 1$*. Therefore,*

$$y(101) = h(101) - h(1).$$

The second observation is that $h(101)$ *is approximately equal to* 1*: All other summands have an* $e^{-50.5}$ *factor which cannot possibly be balanced by any other terms, so they can be safely ignored. Therefore,*

$$y(101) \simeq 1 - h(1) = -\frac{1}{3}e^{-0.5}\cos\left(\frac{\sqrt{3}}{2}\right)+\frac{14}{3\sqrt{3}}e^{-0.5}\sin\left(\frac{\sqrt{3}}{2}\right) \simeq 1.13$$