# MATH 219

Fall 2020

Lecture 25

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**Content:** Heat Equation: Separation of Variables

**Suggested Problems: (Boyce, Di Prima, 9th edition)**

**§10.1 :** 2, 5, 6, 11, 13, 14, 15, 16, 20

In the last three lectures of the course, we will use some of the previously developed techniques together with some new ones in order to solve a certain partial differential equation. This will only be a first introduction to the vast subject of partial differential equations.

## 1 Heat Conduction on a Rod

Suppose that we have a uniform rod of length $L$, with an ignorable thickness. We will assume that there is an initial temperature distribution on the rod which evolves in time. We would like to understand how exactly it evolves. Let us place an axis labeled by the variable $x$, parallel to the rod and assume that the rod is placed between $x = 0$ and $x = L$. Let $u(x, t)$ denote the temperature of point $x$ at time $t$. We will assume that $u(0, t) = u(L, t) = 0$ for all $t$. In other words, the two ends are kept at the constant temperature 0. Heat can escape or enter from the ends, but not through the lateral surface.

In order to proceed mathematically, we need a differential equation which models how the temperature at each point changes with time. This is governed by the **heat equation:**

$$u_t = \alpha^2 u_{xx},$$

where $\alpha$ is a constant that depends on the material that the rod is made up of.

Here is a partial justification of why heat equation holds: Ignoring the boundary conditions for now, at an equilibrium we expect the temperature of each point to be equal to the average temperature of the points neighboring it. In 1 dimension, the only functions that satisfy this property at each point are linear functions. If $u_{xx} > 0$ at some point $p$, then the graph of $u$ with respect to $x$ is concave up at $p$ and the temperature at $p$ is less than the average of the temperatures of its neighboring points. Therefore, the temperature of the point $p$ will increase ($u_t > 0$) which is in accordance with the equation. Likewise, if $u_{xx} < 0$ at $p$, then the graph of $u$ with respect to $x$ is concave down at $p$ and the temperature at $p$ is greater than the average temperature in its neighborhood. In this case $u(p)$ should decrease, namely $u_t(p) < 0$. This partially justifies the heat equation by showing that $u_t$ and $u_{xx}$ must have the same sign for physical reasons. The heat equation says something stronger; it says that $u_t$ is directly proportional to $u_{xx}$. Alternatively one could view this equation as the linear approximation to a more complicated setup; the simplest possibility where $u_t$ and $u_{xx}$ have the same sign.

Our mathematical problem then is to solve the PDE

$$u_t = \alpha^2 u_{xx}$$

subject to the boundary conditions

$$u(0, t) = u(L, t) = 0$$

and an initial condition

$$u(x, 0) = f(x).$$

The domain of $u(x, t)$ is the subset $0 \leq x \leq L, t \geq 0$ of $\mathbb{R}^2$ which has the shape of an infinite rectangular strip. On the other hand, if we wish to *graph* the solution $u(x, t)$, we need one more dimension to place it. So the graph of $u(x, t)$ will be a surface in $\mathbb{R}^3$ over the rectangular strip; the boundary conditions and the initial condition tell us what happens on the boundary of this surface.

## 2 Strategy for Solving the Problem

We break the problem into two major steps:

1. Find as many solutions of the problem

$$u_t = \alpha^2 u_{xx}, \quad u(0, t) = 0, \;\; u(L, t) = 0$$

as possible. (The condition $u(x, 0) = f(x)$ is forgotten at this step.)

2. Among these solutions, find one that satisfies $u(x, 0) = f(x)$.

The first part is about separation of variables and solving the resulting boundary problem. The second part is about finding a Fourier series expansion. We will discuss all of these one by one now.

# 3 Separation of Variables

Let us look for solutions of the problem

$$u_t = \alpha^2 u_{xx}, \quad u(0,t) = 0, \;\; u(L,t) = 0$$

of the form

$$u(x,t) = X(x)T(t),$$

namely in the product form of a function of $x$ and a function of $t$. It might be unclear whether or not the problem has any interesting solutions of this form at all. There is the trivial solution 0, but it is unclear whether there are any nontrivial solutions or not. At the end of our analysis, we will see that there are many. Rewriting the PDE in terms of these functions, we have

$$\begin{aligned} X(x)T'(t) &= \alpha^2 X''(x)T(t) \\ \frac{T'(t)}{\alpha^2 T(t)} &= \frac{X''(x)}{X(x)}. \end{aligned}$$

The left hand side of the last equation depends only on $t$. The right hand side depends only on $x$. But they are equal, so this quantity must be independent of both $x$ and $t$. Since we have only two independent variables, this quantity must be a constant.

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives us two ordinary differential equations

$$\begin{aligned} T'(t) + \lambda\alpha^2 T(t) = 0 \\ X''(x) + \lambda X(x) = 0. \end{aligned}$$

In addition, we have

$$\begin{aligned} u(0,t) = X(0)T(t) = 0 \\ u(L,t) = X(L)T(t) = 0. \end{aligned}$$

Then, either $T(t) = 0$ for all $t$, or $X(0) = X(L) = 0$. The first of these would imply that $u(x,t) = X(x)T(t) = 0$, so we would have a trivial solution. Therefore we assume from now on that $X(0) = X(L) = 0$. Together with the ODE for $X(x)$, we have the following boundary value problem:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0.$$

The characteristic equation for the ODE is $r^2 + \lambda = 0$. Depending on the sign of $\lambda$, we have three possibilities:

(i) $\lambda < 0$: Set $\lambda = -k^2$ for convenience. Then, $r_{1,2} = \pm k$ are two distinct, real roots.

$$X(x) = c_1 e^{kx} + c_2 e^{-kx}.$$

The two boundary conditions give us

$$\begin{aligned} X(0) &= c_1 + c_2 = 0 \\ X(L) &= c_1 e^{kL} + c_2 e^{-kL} = 0. \end{aligned}$$

In matrix form, these equations can be written as

$$\begin{bmatrix} 1 & 1 \\ e^{kL} & e^{-kL} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This linear system has a nontrivial solution if and only if the determinant of the coefficient matrix $A$ is zero. However,

$$\det(A) = e^{-kL} - e^{kL} \neq 0.$$

Therefore, no nontrivial solutions arise from this case.

(ii)$\lambda = 0$: In this case, $r_1 = r_2 = 0$. Therefore,

$$X(x) = c_1 + c_2 x.$$

Plugging in the boundary conditions, we get

$$X(0) = c_1 = 0, \quad X(L) = c_1 + c_2 L = 0$$

from which we easily get $c_1 = c_2 = 0$. Again, no nontrivial solutions arise from this case.

(iii) $\lambda > 0$ Set $\lambda = k^2$. Then, $r_{1,2} = \pm ik$ are two complex conjugate roots. Therefore,

$$X(x) = c_1 \cos(kx) + c_2 \sin(kx).$$

The boundary conditions give

$$\begin{aligned} X(0) &= c_1 = 0 \\ X(L) &= c_1 \cos(kL) + c_2 \sin(kL) = 0. \end{aligned}$$

Write these equations in matrix form:

$$\begin{bmatrix} 1 & 0 \\ \cos(kL) & \sin(kL) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0. \end{bmatrix}$$

Again, let $A$ be the coefficient matrix. The system has nontrivial solutions if and only if the following equivalent conditions hold:

$$\begin{aligned} \det(A) = 0 \quad &\Leftrightarrow \quad \sin(kL) = 0 \\ &\Leftrightarrow \quad kL = n\pi, \quad n \in \mathbb{Z} \\ &\Leftrightarrow \quad k = \frac{n\pi}{L}, \quad n \in \mathbb{Z} \\ &\Leftrightarrow \quad \lambda = \frac{n^2\pi^2}{L^2} \quad n \in \mathbb{Z}. \end{aligned}$$

For each such value of $\lambda$, the nontrivial solutions that we get are constant multiples of

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

It is enough to take $n \in \{1, 2, 3, \ldots\}$ since $n = 0$ gives a trivial solution and $-n$ gives the negative of the solution for $n$.

Next, let us solve the ODE for $T(t)$ for the special values of $\lambda = n^2\pi^2/L^2$ discovered above.

$$T'(t) + \frac{n^2\pi^2\alpha^2}{L^2} T(t) = 0.$$

Select a solution,

$$T_n(t) = e^{-n^2\pi^2\alpha^2 t/L^2}.$$

Set $u_n(x,t) = X_n(x)T_n(t)$. Now, we have one solution of the original problem in step 1 for each $n = 1, 2, 3, \ldots$

$$u_n(x,t) = e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

# 4 Superposition

In the previous section, we produced infinitely many solutions to the problem

$$u_t = \alpha^2 u_{xx}, \quad u(0,t) = u(L,t) = 0.$$

We can still produce more solutions by using the principle of superposition: If $u$ and $v$ satisfy all three conditions above, then it is easy to check that $c_1 u + c_2 v$ satisfies the same conditions for any choice of constants $c_1, c_2$. We can also try to use the same principle for an infinite linear combination of the form

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

Everything works out fine, except that there is a convergence issue. An infinite series may converge or diverge, depending on how fast the coefficients $c_n$ grow with $n$. But this is what happens for a general series. In our special case, the negative exponentials in the sum decay very rapidly to 0. So if the coefficients $c_n$ do not grow very quickly with $n$ (like $exp(n^2)$ or faster), then the sum converges and the principle of superposition holds. This gives us a very large family of solutions to the problem in step 1.