MATH 219

Fall 2020

Lecture 26

Lecture notes by Özgür Kişisel

Content: Fourier Series

Suggested Problems: (Boyce, Di Prima, 9th edition)

§10.2 : 1, 2, 4, 8, 9, 10, 17, 18, 21, 22, 27, 28

§10.3 : 1, 3, 5, 6

§10.4 : 7, 8, 11, 12, 17, 18, 19, 24, 26

In the previous lecture, we found a large family of solutions for the problem

$$u_t = \alpha^2 u_{xx}, \quad u(0,t) = u(L,t) = 0.$$

This family of solutions was

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

Here, the constants c_n can be arbitrary, except they should not grow extremely rapidly with n so as to make the series diverge for any value of t. However, since we have a term decaying like $exp(-n^2)$ in the formula, many plausible choices for c_n's work.

In order to finish the heat equation problem, we need to select the constants c_n so that for $0 < x < L$, the equality $f(x) = u(x,0)$ holds. If we put $t = 0$ in the formula above, this says that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

In plain language, we would like to express an arbitrarily given function $f(x)$ as an infinite linear combination of certain trigonometric functions. Although this looks like an ambitious statement, it turns out to be correct under mild assumptions on $f(x)$.

1 Periodic Functions

Recall that a real valued function $f(x)$ is called periodic with period T, if for all $x \in \mathbb{R}$ we have
$$f(x+T) = f(x).$$
We remark that if T is a period for $f(x)$, then so is any integer multiple nT of T. The primary examples of functions with period $T = 2L$ are
$$\sin\left(\frac{n\pi x}{L}\right), \quad \cos\left(\frac{n\pi x}{L}\right)$$
The question of interest for us is whether or not an arbitrarily given periodic function $f(x)$ can be expressed as an infinite linear combination of these functions? The answer is given by the following important theorem of Fourier:

Theorem 1.1 *(Fourier convergence theorem) Suppose that $f(x)$ is a piecewise continuous periodic function of period $T = 2L$. Then, there exist unique sets of constants $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that*
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$
for every value of x for which f is continuous. At the points of discontinuity, the right hand side of the equation converges to
$$\frac{f(x^-) + f(x^+)}{2}$$
where $f(x^-) = \lim_{z\to x^-} f(z)$ and $f(x^+) = \lim_{z\to x^+} f(z)$.

The proof of this theorem will be omitted. However, assuming that the theorem holds, we will see soon how we can find the coefficients a_n and b_n when $f(x)$ is given.

2 Orthogonality Relations

Definition 2.1 *Let $f(x)$ and $g(x)$ be two piecewise continuous functions, both periodic with the same period $T = 2L$. Their* ***integral inner product*** *is*
$$\langle f(x), g(x)\rangle = \int_{-L}^{L} f(x)g(x)dx.$$

By periodicity, the integral in the definition could be taken over any period (for instance, from 0 to $2L$).

The integral inner product defined above shares many common properties with the dot product of vectors. For instance,

- $\langle f, g\rangle = \langle g, f\rangle$,
- $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$,
- $\langle cf, g\rangle = c\,\langle f, g\rangle\,, \quad c \in \mathbb{R}$.

We would like to compute the inner products of functions of the form $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ first, where n takes integer values. For these computations, the following trigonometric identities will be useful:

$$\begin{aligned}\sin(a)\sin(b) &= \frac{1}{2}(\cos(a-b) - \cos(a+b)),\\ \cos(a)\cos(b) &= \frac{1}{2}(\cos(a-b) + \cos(a+b)),\\ \sin(a)\cos(b) &= \frac{1}{2}(\sin(a+b) + \sin(a-b)).\end{aligned}$$

Another useful observation is the following: If we integrate a sine or cosine function over one of its periods, then we get 0 by symmetry. Now:

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n-m)\pi x}{L}\right) dx - \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n+m)\pi x}{L}\right) dx$$

Here, $n, m \in \{1, 2, 3, \ldots\}$. The first integral is 0 if $n \neq m$ and L if $n = m$. The second integral is always 0. Therefore,

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right\rangle = \begin{cases} 0, & n \neq m,\\ L, & n = m.\end{cases}$$

We make a similar computation for two cosine functions:

$$\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n-m)\pi x}{L}\right) dx + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n+m)\pi x}{L}\right) dx$$

This time, since $cos(0) \neq 0$, we take $n, m \in \{0, 1, 2, \ldots\}$. The first integral is 0 if $n \neq m$ and L if $n = m$. The second integral is L if $n = m = 0$ and 0 in all other cases. Therefore,

$$\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = \begin{cases} 0, & n \neq m, \\ L, & n = m \neq 0 \\ 2L, & n = m = 0. \end{cases}$$

Finally, we compute the inner product of a sine and a cosine function:

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$
$$= \frac{1}{2} \int_{-L}^{L} \sin\left(\frac{(n+m)\pi x}{L}\right) dx + \frac{1}{2} \int_{-L}^{L} \sin\left(\frac{(n-m)\pi x}{L}\right) dx$$

This time, both integrals are 0 regardless of the values of n and m. Therefore,

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = 0.$$

These equalities are called **orthogonality relations**. There is a geometric reason behind this: The inner product is similar to the dot product of vectors. Recall that the dot product of two vectors is 0 if and only if the vectors are orthogonal. So, the sine and cosine functions described above are orthogonal to one another in some abstract sense.

3 Fourier Inversion Formula

Suppose now that $f(x)$ is a piecewise continuous function with period $T = 2L$ and its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Fourier's convergence theorem guarantees that $f(x)$ is equal to the above series at all points of continuity. By definition of piecewise continuous functions, the points of discontinuity can at most be finitely many in every finite interval. Therefore, in an integral formula, we can assume that $f(x)$ is equal to its Fourier series, since differences in finitely many points will not change the value of the integral. So, in the integral formula we will simply insert

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Now, let us take the integral inner product of both sides of this equation with $\cos(m\pi x/L)$:

$$\begin{aligned}
\left\langle f(x), \cos\left(\frac{m\pi x}{L}\right)\right\rangle &= \left\langle \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle \\
&= \frac{a_0}{2}\left\langle 1, \cos\left(\frac{m\pi x}{L}\right)\right\rangle + \sum_{n=1}^{\infty}\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle \\
&+ \sum_{n=1}^{\infty} b_n \left\langle \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle \\
&= La_m.
\end{aligned}$$

(Using the distributive law for the infinite sum above requires some justification, which we omit). This gives us a formula for a_m:

$$\begin{aligned}
a_m &= \frac{1}{L}\left\langle f(x), \cos\left(\frac{m\pi x}{L}\right)\right\rangle \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) dx.
\end{aligned}$$

By a very similar argument, we get

$$\begin{aligned}
b_m &= \frac{1}{L}\left\langle f(x), \sin\left(\frac{m\pi x}{L}\right)\right\rangle \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right) dx.
\end{aligned}$$

This gives us a feasible way to compute all Fourier coefficients.

Example: Suppose that $f(x) = 1$ for $0 < x < 2$, $f(x) = -1$ for $-2 < x < 0$ and $f(x)$ is periodic with period 4. Let us find the Fourier coefficients of $f(x)$. We take

$L = 2$ (half of the period).

$$\begin{aligned}
a_m &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{m\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{0} -\cos\left(\frac{m\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2}\cos\left(\frac{m\pi x}{2}\right)dx \\
&= 0.
\end{aligned}$$

The last equality holds because *cos* is an even function. Similarly, we compute b_m's:

$$\begin{aligned}
b_m &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{m\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{0} -\sin\left(\frac{m\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2}\sin\left(\frac{m\pi x}{2}\right)dx \\
&= \int_{0}^{2}\sin\left(\frac{m\pi x}{2}\right)dx \\
&= -\frac{2}{m\pi}\cos\left(\frac{m\pi x}{2}\right)\Bigg|_{0}^{2} \\
&= -\frac{2}{m\pi}(\cos(m\pi) - 1) \\
&= \begin{cases} 4/m\pi, & m\,\text{odd}, \\ 0, & m\,\text{even} \end{cases}
\end{aligned}$$

Therefore, the Fourier series for $f(x)$ is

$$\frac{4}{\pi}\left(\sin(\pi x/2) + \frac{1}{3}\sin(3\pi x/2) + \frac{1}{5}\sin(5\pi x/2) + \dots\right)$$

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