ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

ME 519
Kinematic Analysis of Mechanisms

Fall 2020 *Distance Learning*

Dr. Ergin TÖNÜK

Department of Mechanical Engineering

Graduate Program of Biomedical Engineering

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Course Requirements

ME 208 Dynamics, 2-D Kinematics of Particle & Rigid Body is a **must**

ME 301 Theory of Machines I (or equivalent), Kinematics is a **must**

ME 431 Kinematic Synthesis of Mechanisms (or equivalent) is **strongly recommended**

> Burmester's theory [Ludwig Burmester (1840–1927)] for finitely separated position synthesis, circle point curve (K) and center point curve (M)]

Graduate level mathematics, complex number algebra, and geometry is **strongly recommended**

An interest to advanced *planar* kinematics in the graduate level is **recommended**

ME 519 KINEMATIC ANALYSIS OF MECHANISMS

Fall 2020 Course Policy (*Online Teaching*)

Course Instructor **E-mail**

Dr. Ergin TÖNÜK tonuk@metu.edu.tr

Course Grading (*Tentative*)

Midterm (25%), homework assignments (20%) term project (25%), final exam (30%).

References

Söylemez, Mechanisms, 5th Edition or earlier editions.

Sandor & Erdman Advanced Mechanism Design Analysis and Synthesis II, 1984.

Hall, Kinematics and Linkage Design, 1966.

Suh & Radcliff, Kinematics & Mechanism Design, 1978

Dijksman, Motion Geometry of Mechanisms, 1976

Hunt, Kinematic Geometry of Mechanisms, 1978

Bottema & Roth, Theoretical Kinematics, 1979

Rosenauer & Willis, Kinematics of Mechanisms, 1967 [or translation by Yüksel (1981)]

Müller, Kinematik Dersleri, 1963

Course Web Site

https://odtuclass.metu.edu.tr/

Examinations

The dates of all examinations will be arranged and announced by the Department

Make-up Examinations

Make-up examinations may be given to those with valid excuses approved by the Department. If you are eligible to take any of the make-up examinations, you must report to your course instructor within one week after the regular exam date. Expect a harder exam compared to the regular one.

Course Content (*tentative*)

1. Introduction & Review
2. Canonical Representation of Plane Motion
3. Curvature Theory: Infinitesimal Plane Motion
4. Cubic of Stationary Curvature

What to Expect

This is an advanced kinematics course. You will definitely need kinematics part of ME 301 Theory of Machines I. If you have already attended ME 431 Kinematic Synthesis of Mechanisms you would appreciate the analogy between finitely separated positions of ME 431 and infinitesimally separated positions of ME 519. Although the course title contains the word *analysis* we will use analysis methods to *synthesize* (i.e. design) planar mechanisms. There may be some analysis of spatial mechanisms like 3-D four-bar and slider-crank at the end if time permits.

Course Content (*Tentative*)

0. Introduction & Review[1]

1. Canonical Representation of Plane Motion[2]

2. Curvature Theory: Infinitesimal Plane Motion[2]

3. Cubic of Stationary Curvature[2]

Based on:
1 Söylemez Eres, Unpublished Lecture Notes on ME 431
2 Söylemez Eres, Unpublished Lecture Notes on ME 519

0. Introduction and Review
Mechanism

- It is a group of rigid bodies (*links*) connected to each other by rigid kinematic pairs (*joints*) to transmit force and motion.
- It is a *kinematic chain* where one of the links is fixed.
- A mechanical machine is defined as a combination of resistant bodies so arranged that by their means the mechanical forces of nature can be compelled to do work accompanied by certain determinate motion[1].

*Mechanisms are **the basic building blocks** of mechanical machines. A machine is designed for a specific task using appropriate mechanisms.*

1 Reulaeaux, Kinematics of Machinery, 1876

Kinematics of Mechanisms

a. Functional Synthesis: Determination of candidate mechanisms that can realize a set of given (or implied) functional requirements.

b. Type Determination: Investigation of known mechanisms for their *topological* characteristics.

c. Kinematic Analysis: Determination of kinematic characteristics (position, velocity and acceleration) of a *known* mechanism.

d. Kinematic Synthesis: Determination of mechanism parameters (mostly link lengths) to realize a given motion (position, velocity and/or acceleration) for a mechanism whose *topological characteristics are known*.

Four Methods of Dimensional Mechanism Synthesis

1. Multiple (Finitely Separated) Position Synthesis: Locate key geometric loci like revolute joints (on a circular path) or prismatic joints (on a straight path) using the kinematics of the required motion. Recall that you have a **finite** number of design parameters so you cannot (*mostly*) do the design for the entire (i.e. *infinitely many*) positions, a continuous path or function. This leads to Burmester's theory (ME 431).
2. Infinitesimally Separated Position/Order Synthesis: Order approximation of a mechanism existing at a point. For the real finite motion around the neighborhood of the design position, the motion is matched to the desired motion as much as possible. This leads to curvature theory (ME 519)
3. Optimization Synthesis: It involves minimizing or maximizing an objective function so that the desired motion is "*best*" matched. As a simple example, recall Chebyshev spacing for function synthesis and manually relocation of precision points in ME 431.
4. "*Best*" Match from a Catalogue or a Database: An extensive catalogue or database having many possible mechanisms is searched by a human expert, artificial intelligence, expert system, machine learning etc. for the "*best*" match. Some *ancient printed* catalogs are:
   - Hrones & Nelson, "Analysis of the four-bar linkage; its application to the synthesis of mechanisms", Technology Press of the Massachusetts Institute of Technology, and Wiley, New York, 1951, TJ183.H7.
   - Chironis, "Machine devices and instrumentation: mechanical, electromechanical, hydraulic, thermal, pneumatic, pyrotechnic, photoelectric and optical", New York, McGraw-Hill 1966, TJ213 C532.
   - (Sclater &) Chironis, "Mechanisms and mechanical devices sourcebook", McGraw-Hill 1965, (2001, 2007) TJ181.C4 (.S28 2001, 2007).
   - Artobolevskii, "Mechanisms in modern engineering design; a handbook for engineers, designers, and inventors", Mir Publishers, 1975-1980, TJ181.A7813 (*7 Volumes!*)

Degree of Freedom of Mechanisms

λ: Degree of freedom of the unconstrained bodies in the mechanism space

ℓ: Number of links of the mechanism (including fixed link)

j: Number of joints of the mechanism (*ternary, quarternary, etc. joints!*)

f_i: Degree of freedom of i^{th} joint

F: Degree of freedom of the mechanism

$$F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$$

Remember exceptions!

$F > 0$ mechanism requires F actuations for kinematically deterministic motion.

#of actuators < F: Under-actuation, motion is determined by forces (typical examples are car differential and safety stops).

$F = 0$ structure (immobile) *unless has special dimensions*.

$F < 0$ over-constraint (number of "*redundant*" constraints is |F|) and immobile unless has special dimensions (*also forces cannot be determined unless equations of equilibrium/motion are complemented by* $|F|$ *number of equations relating deformations of the links*).

Degree of Freedom of Mechanisms

Derivation:

In planar motion ℓ links with no joints has $F = 3(\ell - 1)$

k_1 joints (revolute & prismatic) constrain 2 freedoms

k_2 joints (cylinder in slot) constrain 1 freedom

$F = 3(\ell - 1) - 2k_1 - k_2$

Kutzbach formula!

Similarly in 3-D space:

$F = 6(\ell - 1) - 5k_1 - 4k_2 - 3k_3 - 2k_4 - k_5$

Replace 3 and 6 in the above equation with λ

Constraints imposed by i^{th} joint is $\lambda - f_i$

Constraints imposed by all joints $\sum_{i=1}^{j}(\lambda - f_i) = \lambda j - \sum_{i=1}^{j} f_i$

Then $F = \lambda(\ell - 1) - \left[\lambda j - \sum_{i=1}^{j} f_i\right]$

Simplification yields $F = \lambda(\ell - j - 1) + \sum_{i=1}^{j} f_i$

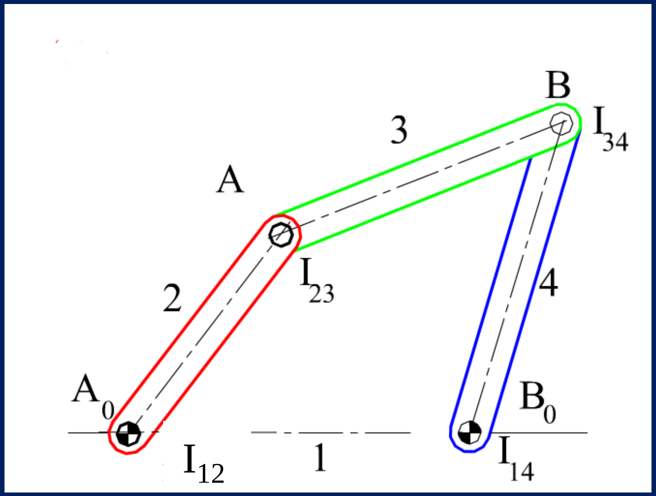
Instant Centers in Plane Motion

Aranhold-Kennedy Theorem: In planar motion the instant centers of any three links (whether they are connected by a joint or not) lay on a straight line.

The number of instant centers of a mechanism having ℓ links is $N = \frac{\ell(\ell-1)}{2}$

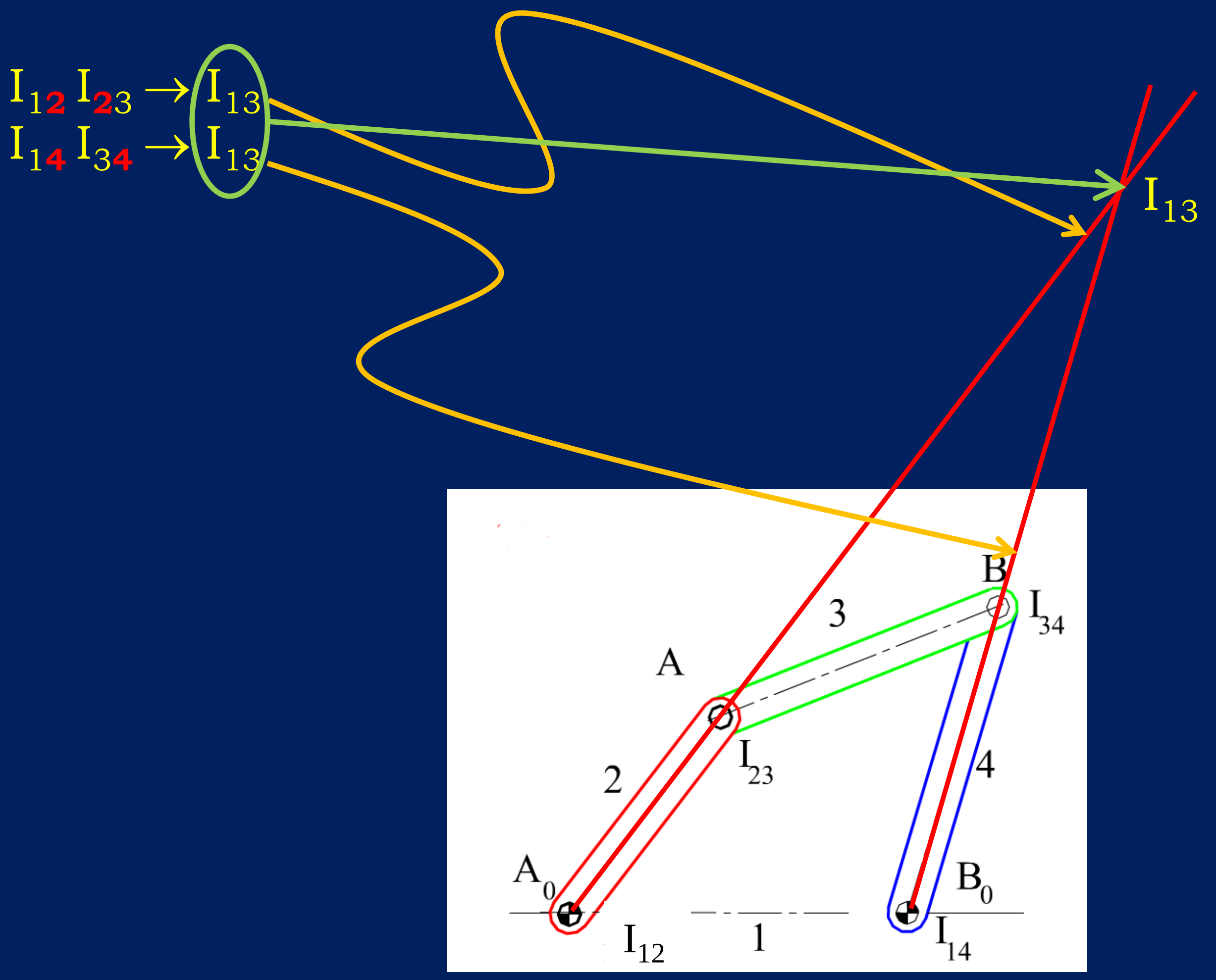
The instant center is denoted by I_{ij} and this point is momentarily coincident on links i and j (momentarily has zero relative velocity). If one of i or j is *1* then it is the absolute instant center with zero absolute velocity. Otherwise the relative velocity of this point with respect to link i (or j) on link j (or i) is momentarily zero. *Remember we may consider links as infinite planes.*

Instant Centers of Four-Bar



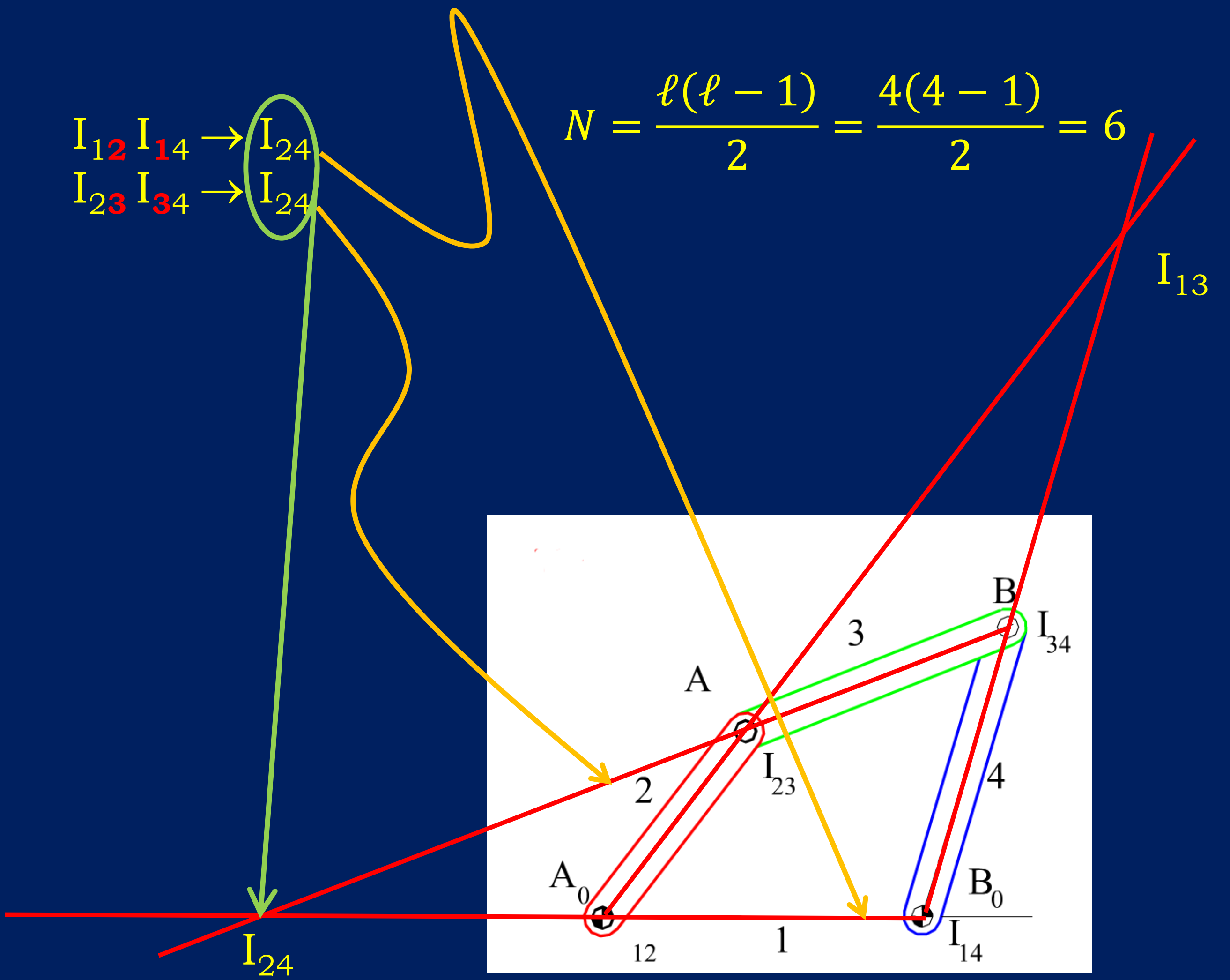
https://ocw.metu.edu.tr/pluginfile.php/1845/mod_resource/content/1/ch5/5-1.htm

Instant Centers of Four-Bar

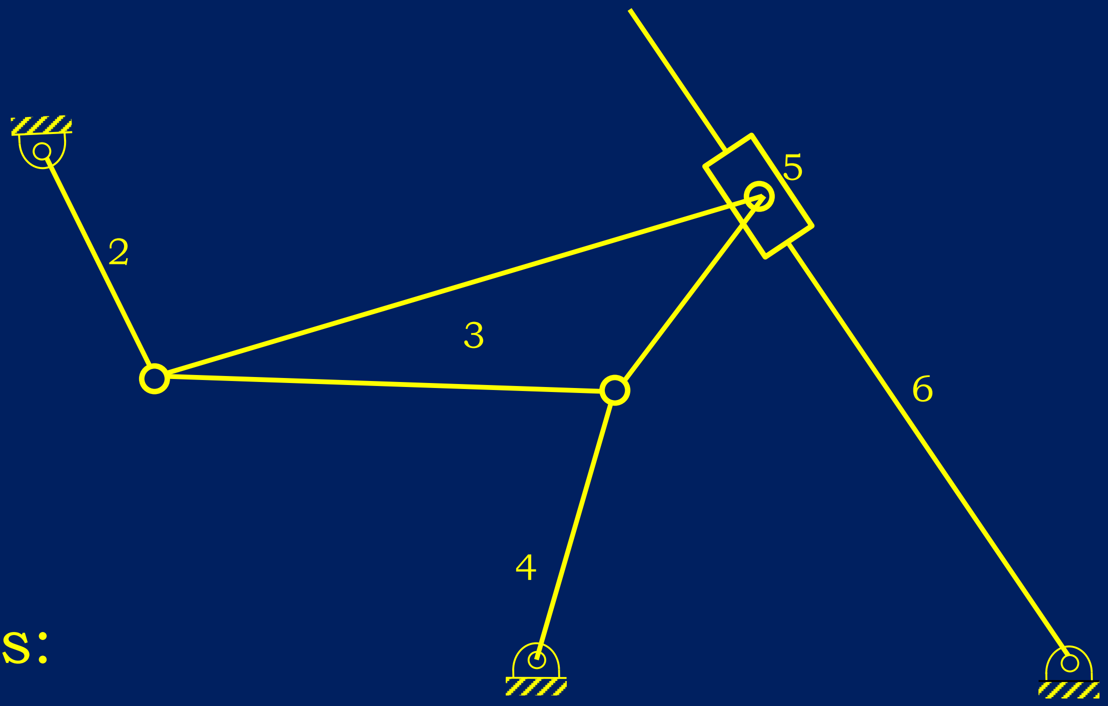


Instant Centers of Four-Bar

$$N = \frac{\ell(\ell-1)}{2} = \frac{4(4-1)}{2} = 6$$



Instant Centers of a Multi-Loop Mechanism

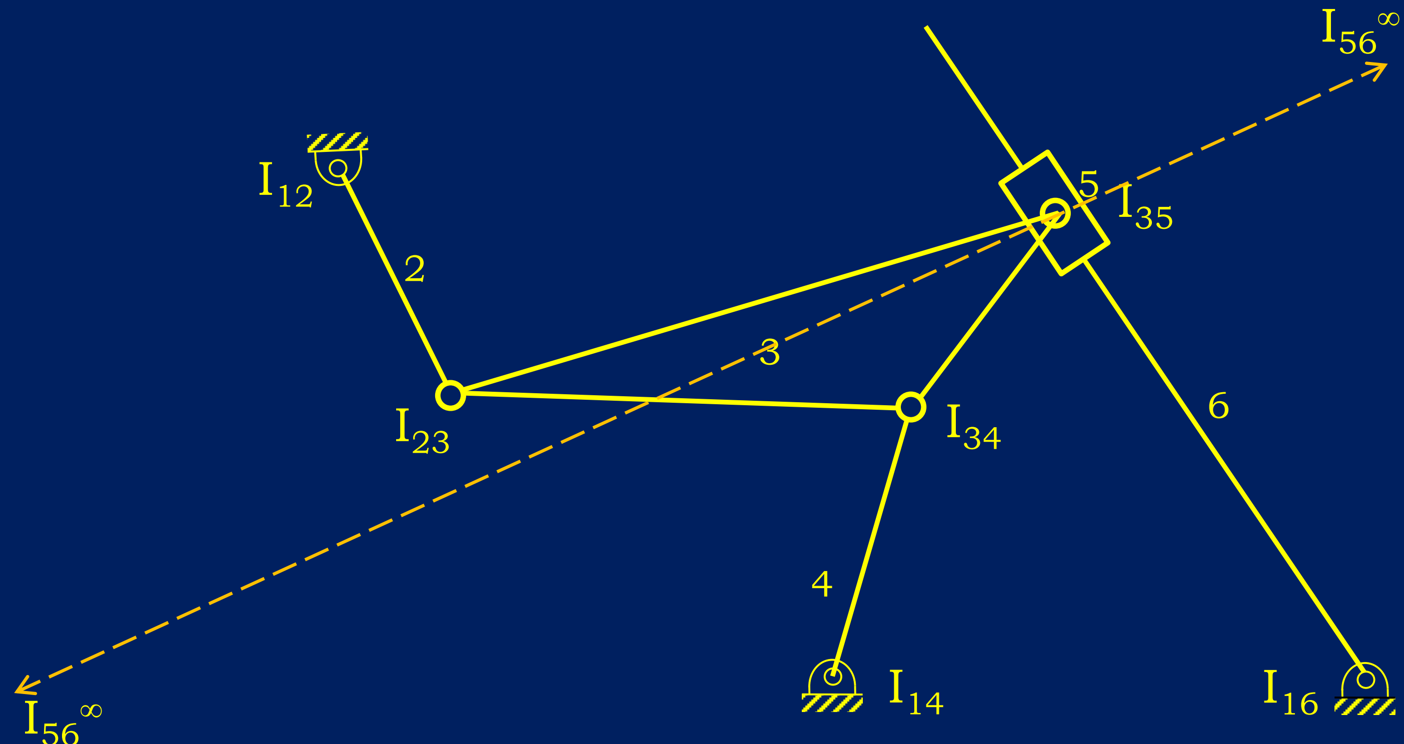


Number of instant centers:

$$N = \frac{\ell(\ell - 1)}{2} = \frac{6(6 - 1)}{2} = 15$$

Instant Centers of a Multi-Loop Mechanism

Obvious instant centers:



Instant Centers of a Multi-Loop Mechanism

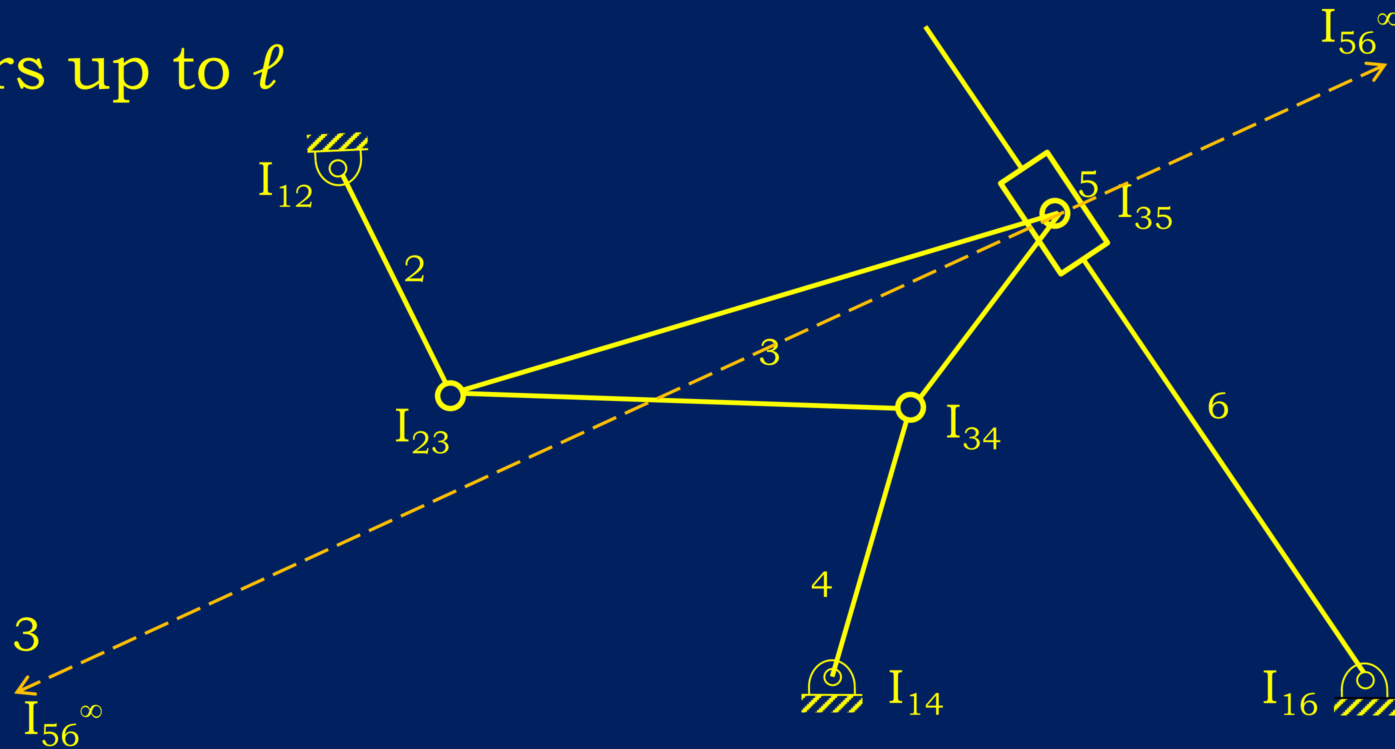
Determination of other instant centers:

- Write numbers up to ℓ

1 2

6 3

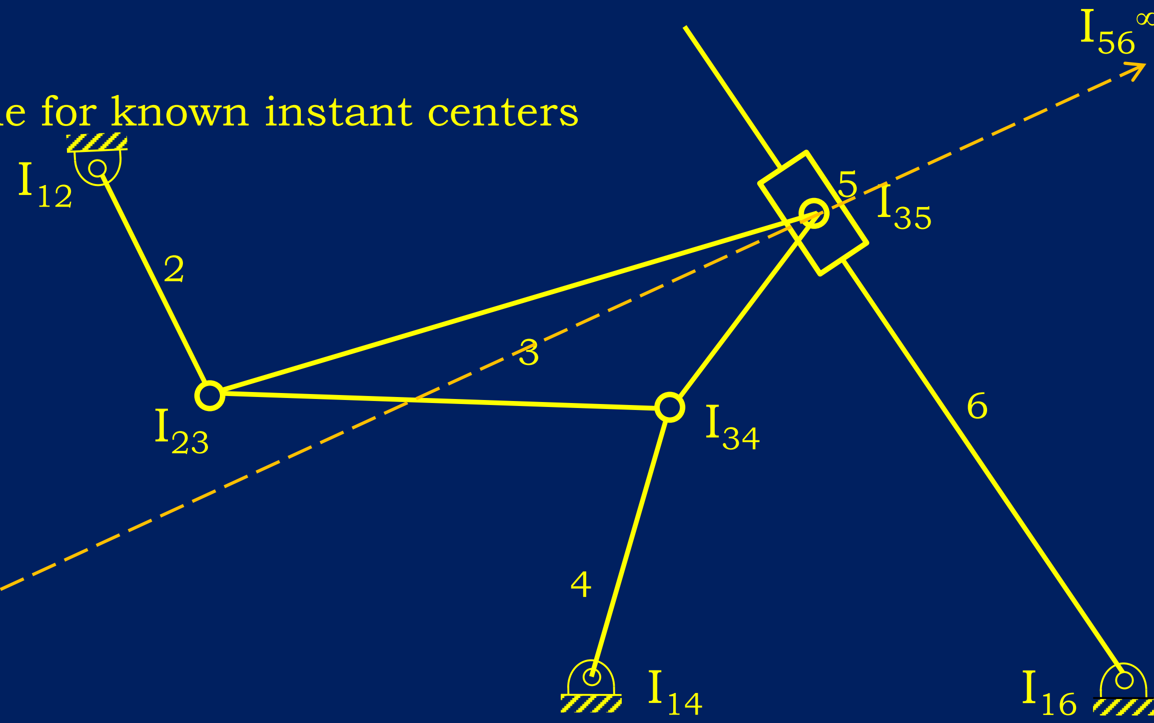
5 4

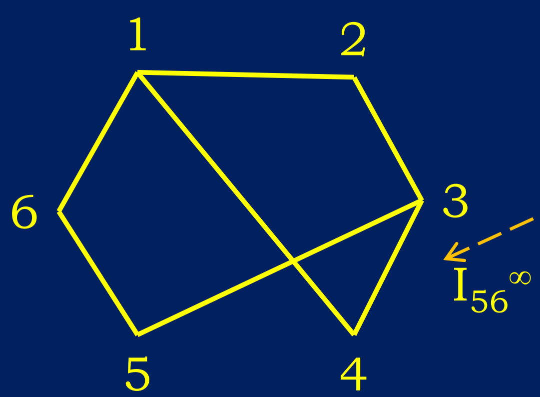


Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers

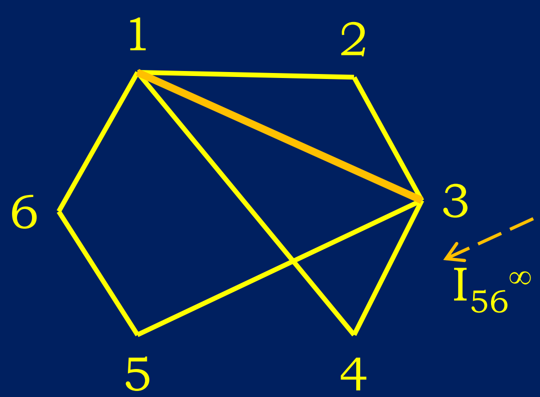


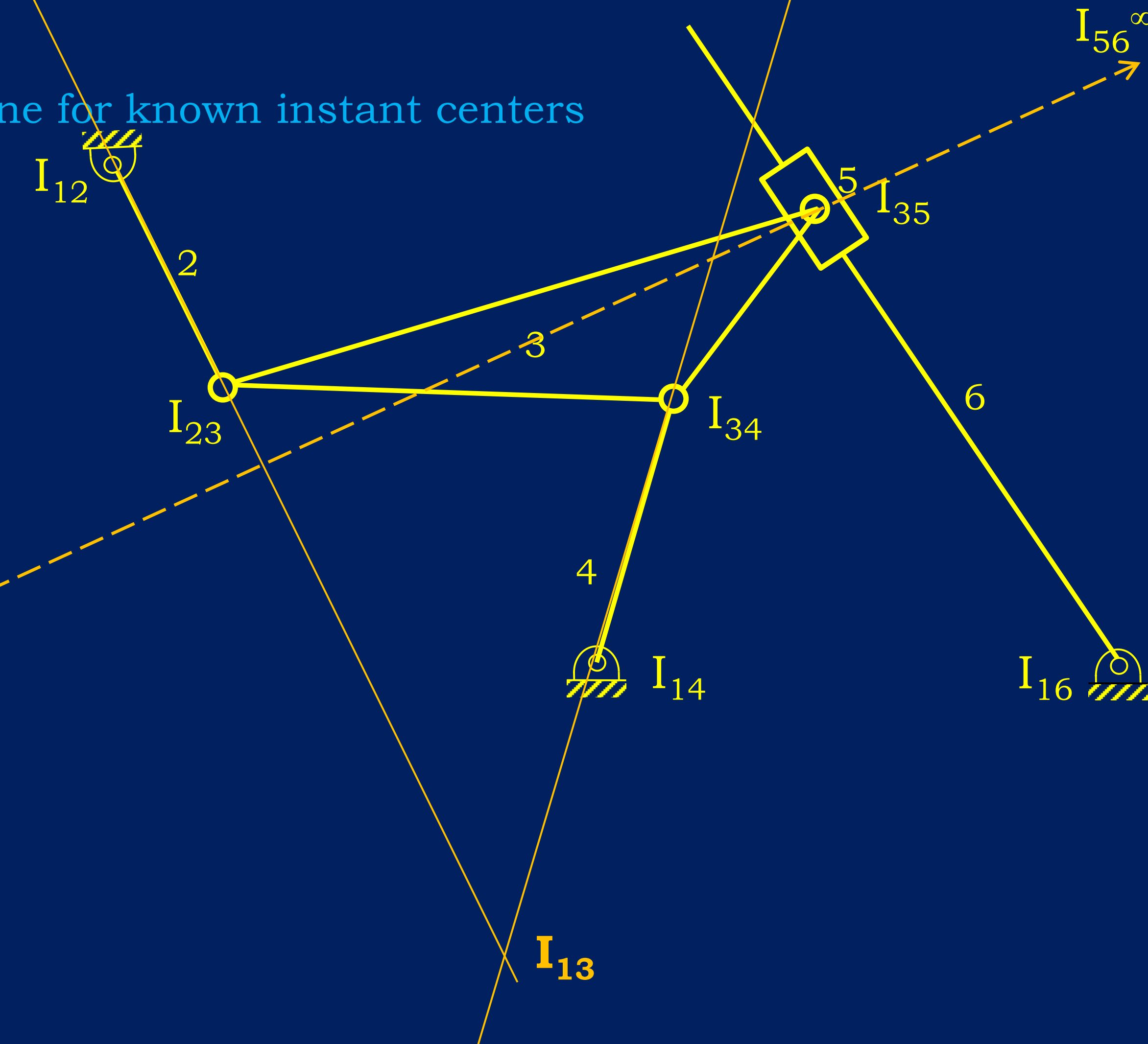


Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields $\mathbf{I_{13}}$

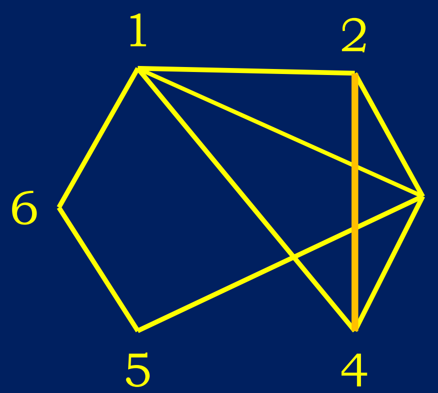


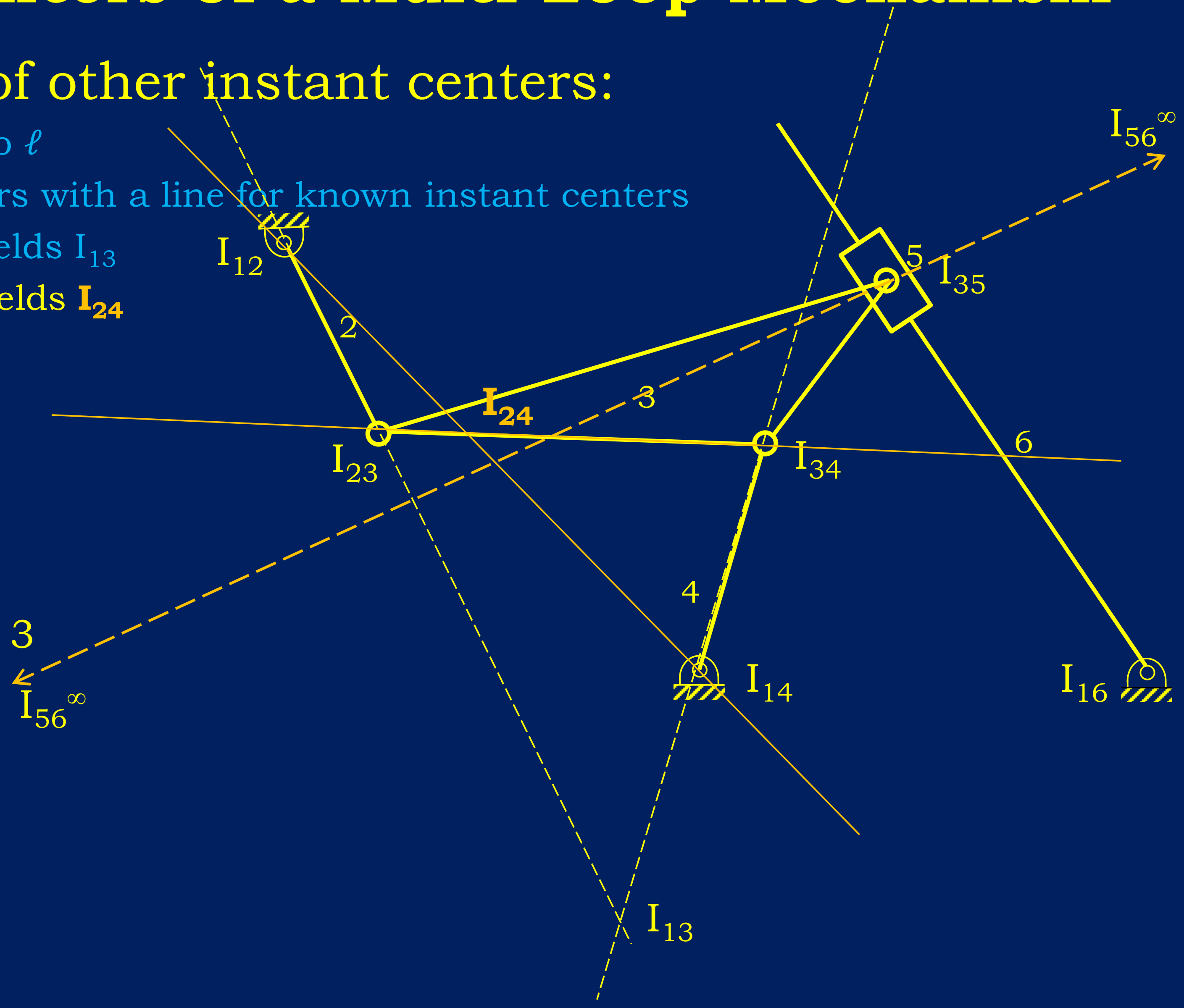


Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields I_{13}
- I_{23}, I_{34} and I_{12}, I_{14} yields $\mathbf{I_{24}}$

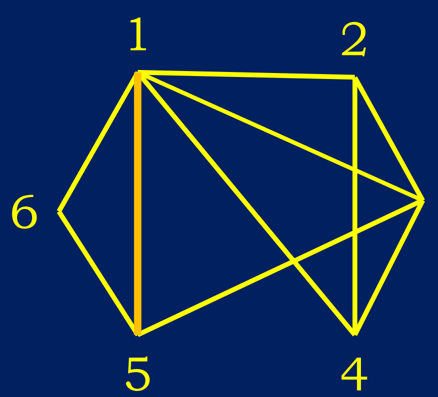


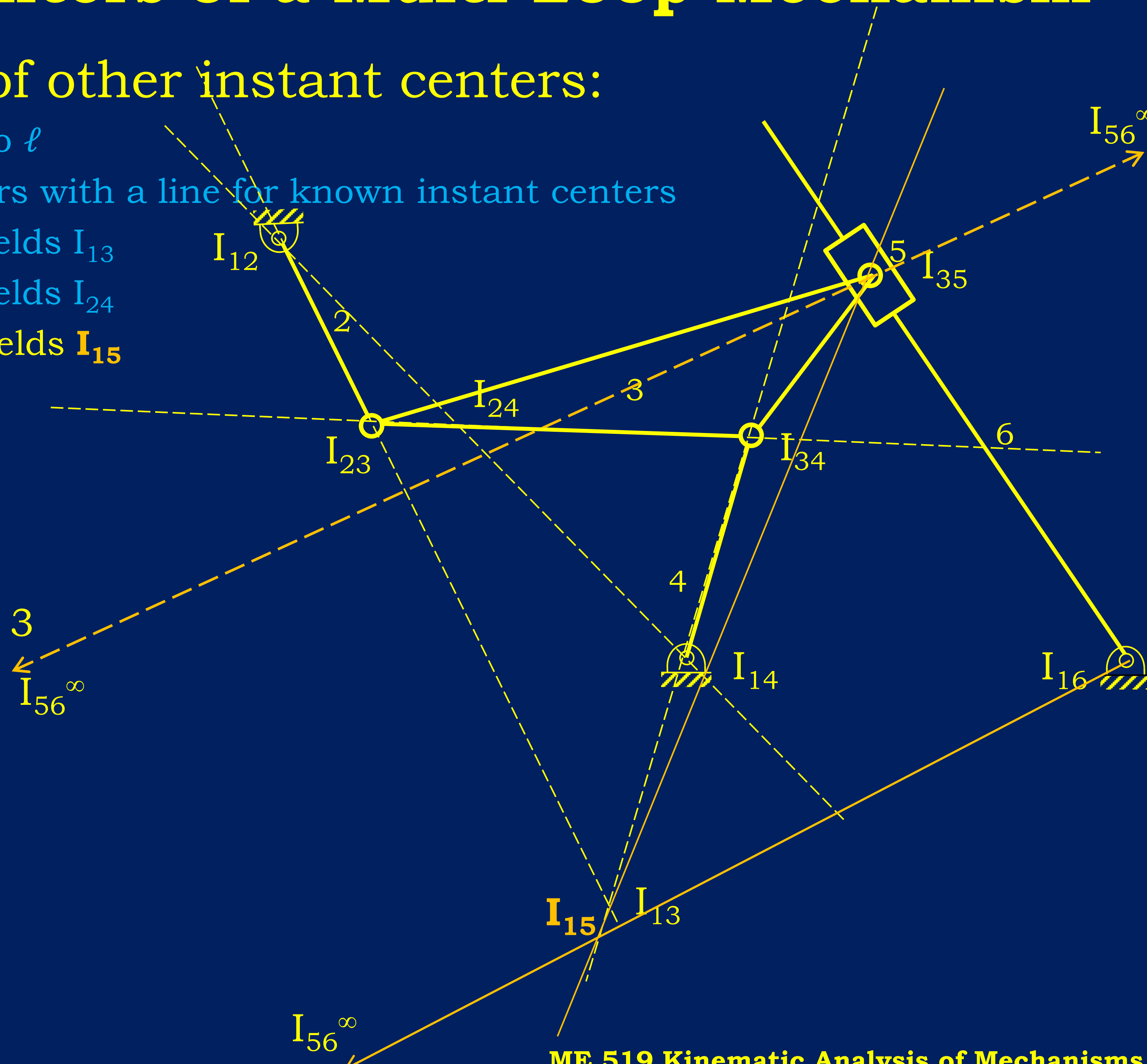


Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields I_{13}
- I_{23}, I_{34} and I_{12}, I_{14} yields I_{24}
- I_{13}, I_{35} and I_{16}, I_{56} yields $\mathbf{I_{15}}$

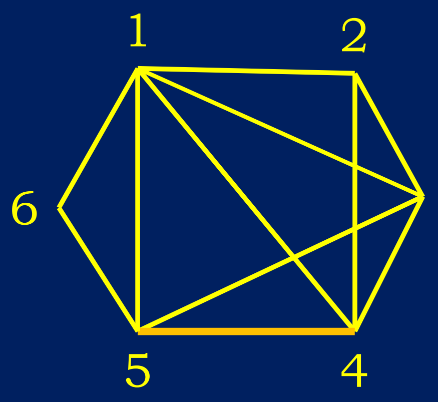


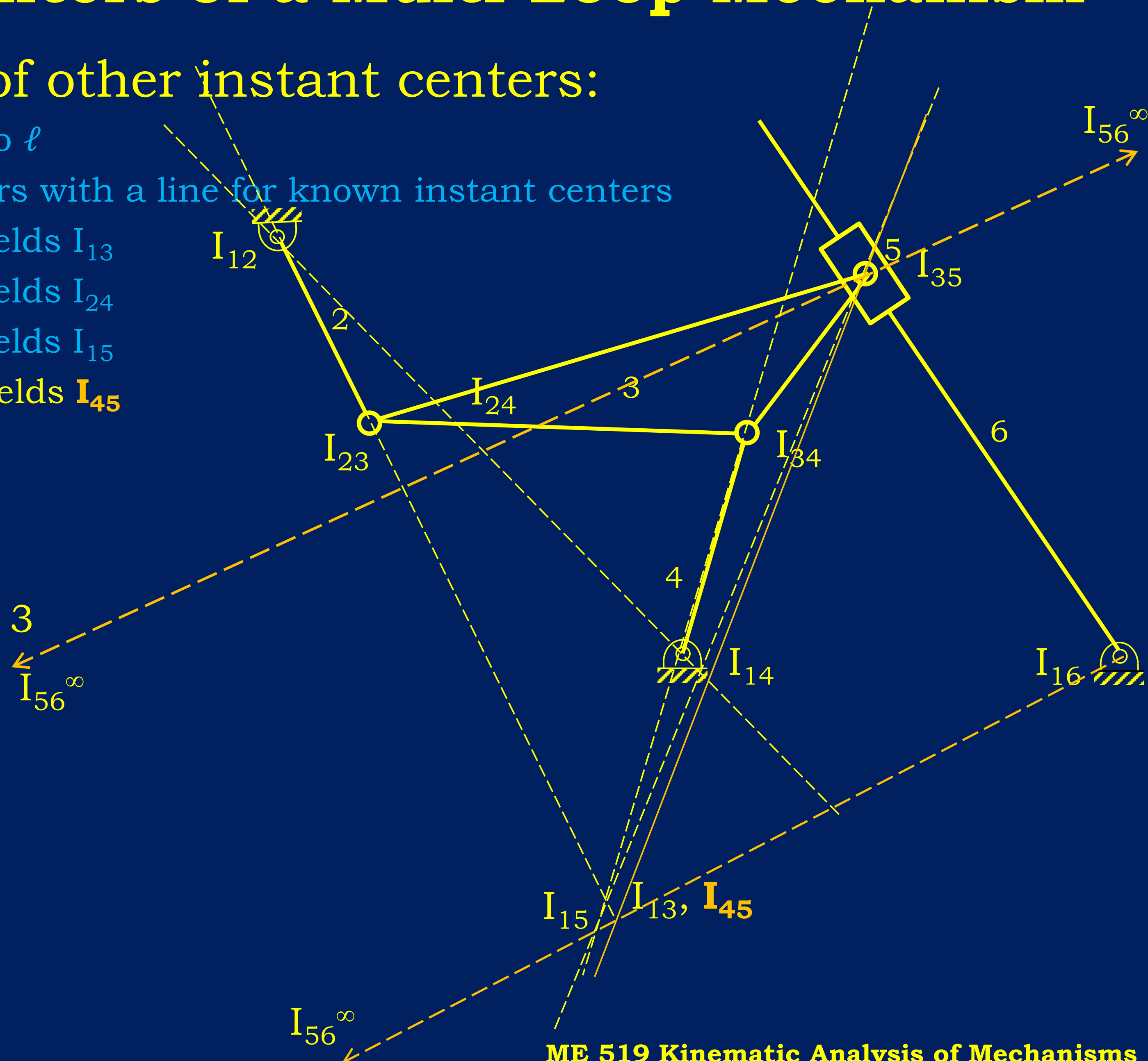


Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields I_{13}
- I_{23}, I_{34} and I_{12}, I_{14} yields I_{24}
- I_{13}, I_{35} and I_{16}, I_{56} yields I_{15}
- I_{13}, I_{35} and I_{16}, I_{56} yields $\mathbf{I_{45}}$

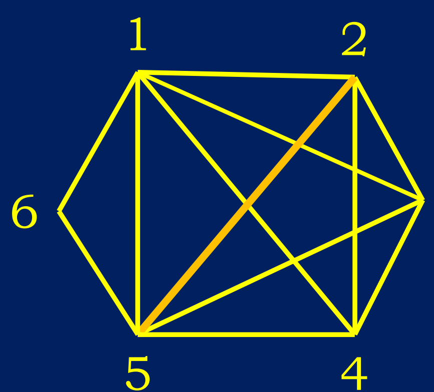


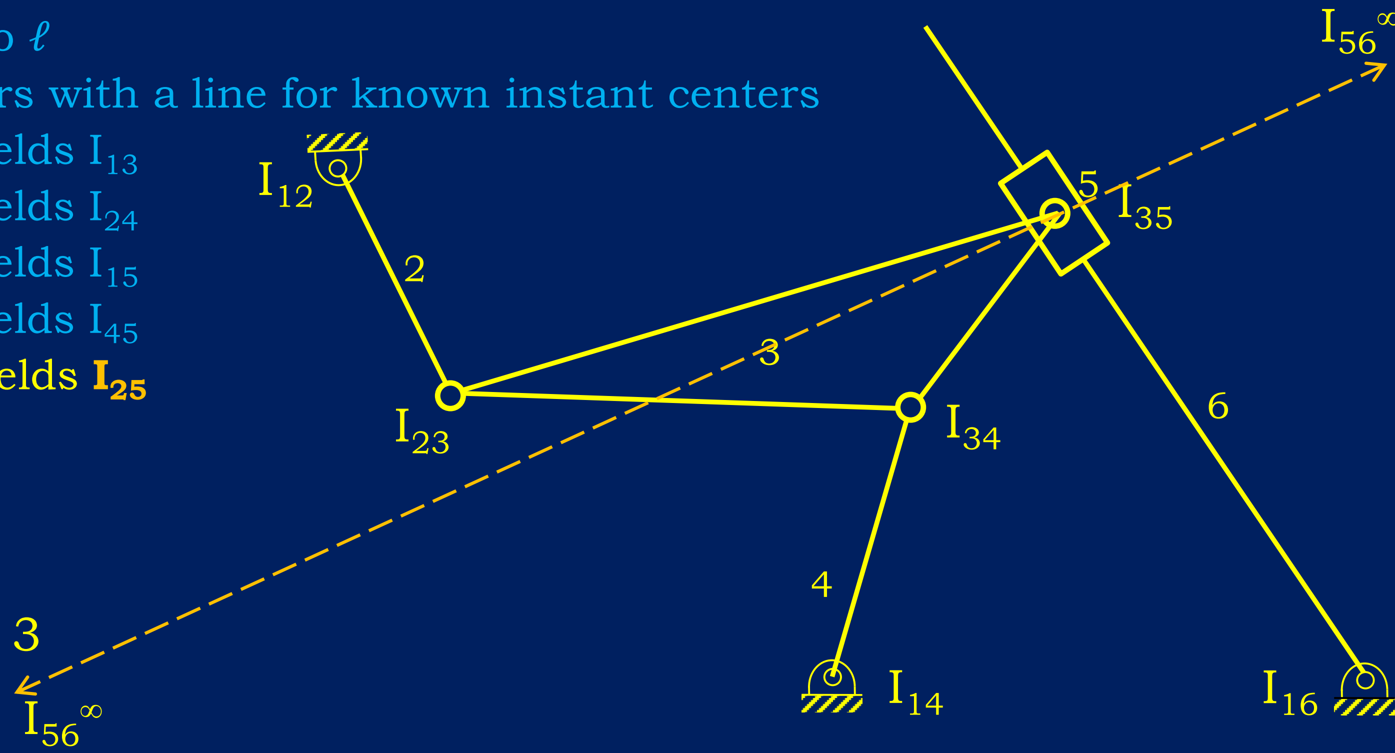


Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields I_{13}
- I_{23}, I_{34} and I_{12}, I_{14} yields I_{24}
- I_{13}, I_{35} and I_{16}, I_{56} yields I_{15}
- I_{14}, I_{45} and I_{16}, I_{56} yields I_{45}
- I_{12}, I_{15} and I_{23}, I_{35} yields $\mathbf{I_{25}}$



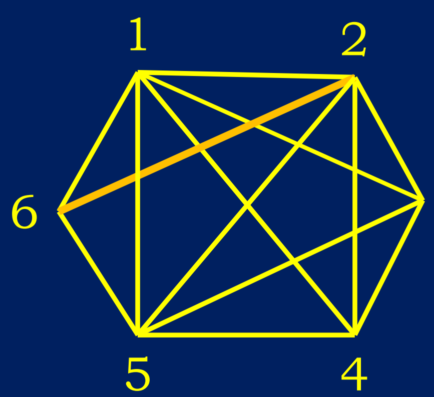


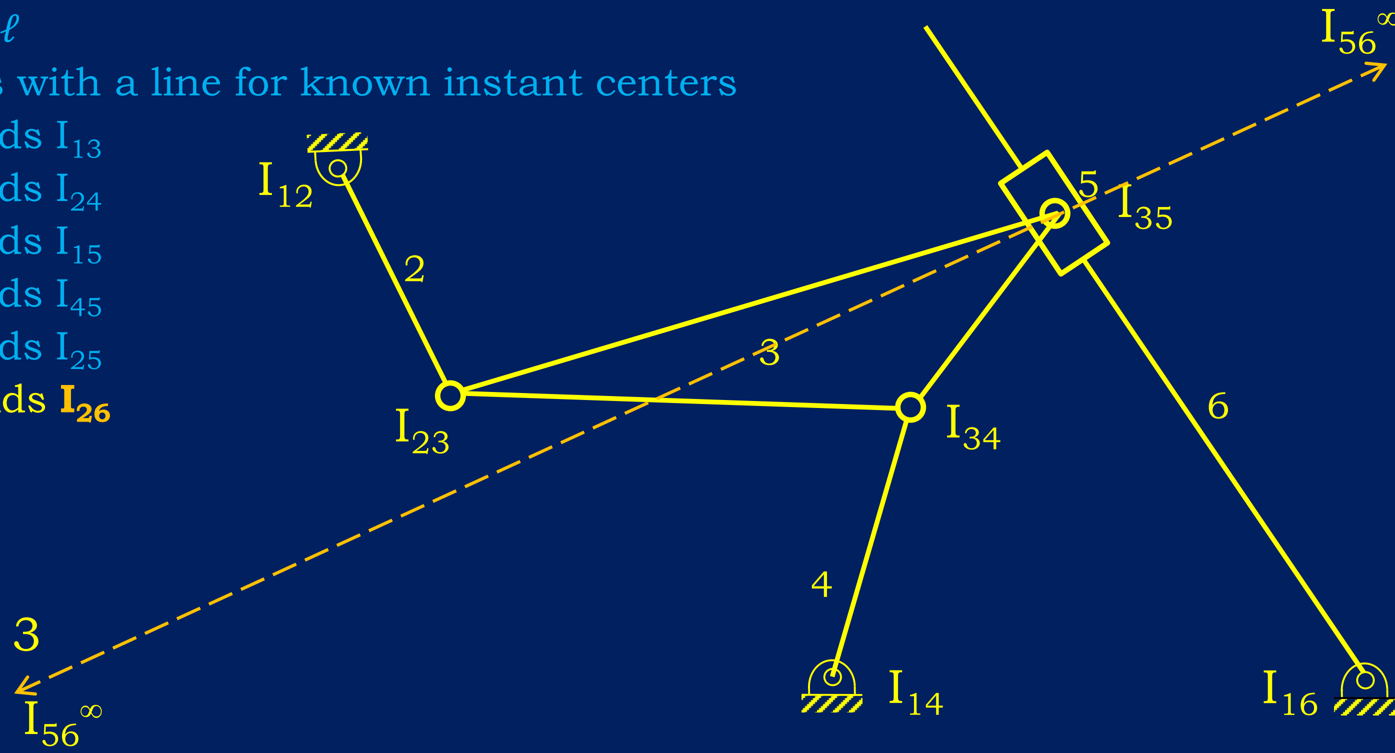
Continue the construction yourself!

Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields I_{13}
- I_{23}, I_{34} and I_{12}, I_{14} yields I_{24}
- I_{13}, I_{35} and I_{16}, I_{56} yields I_{15}
- I_{14}, I_{45} and I_{16}, I_{56} yields I_{45}
- I_{12}, I_{15} and I_{23}, I_{35} yields I_{25}
- I_{12}, I_{16} and I_{25}, I_{56} yields $\mathbf{I_{26}}$



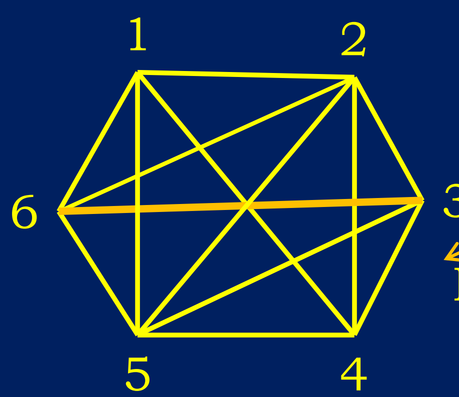


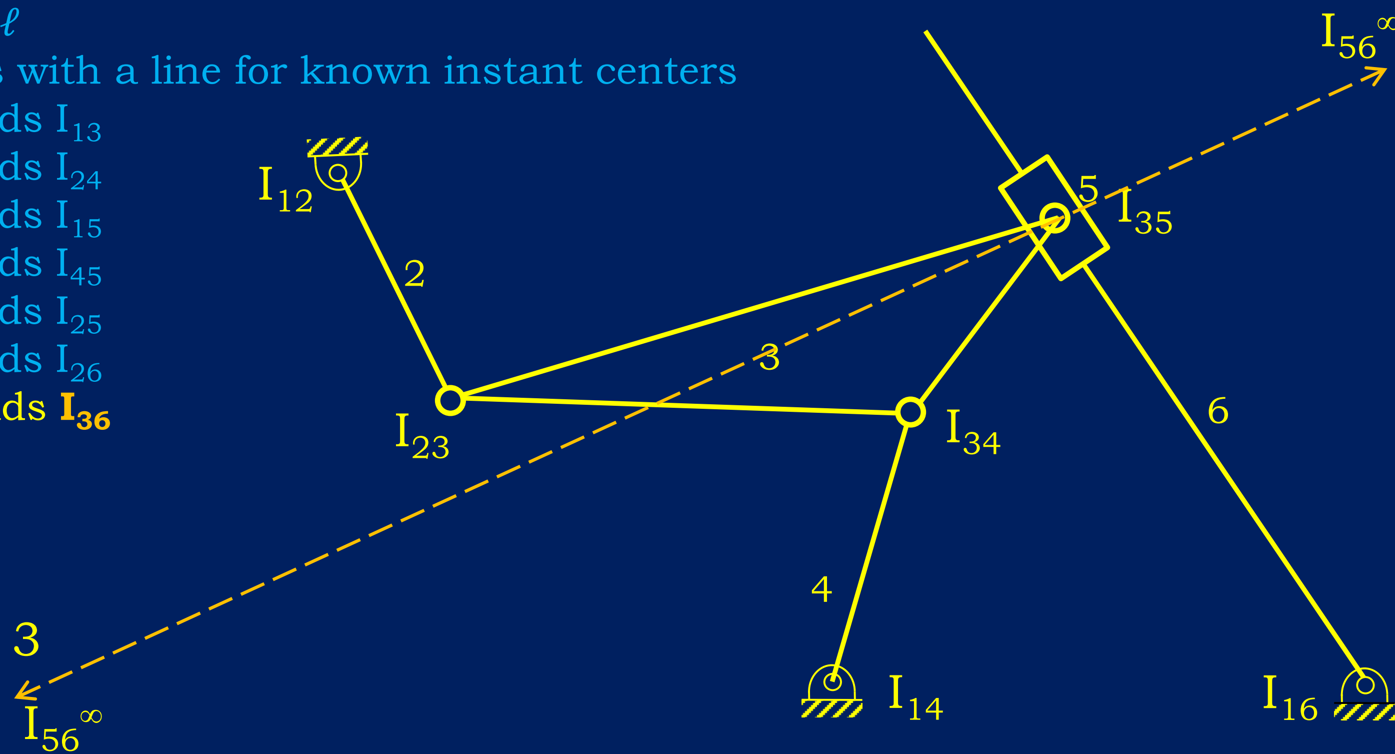
Continue the construction yourself!

Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields I_{13}
- I_{23}, I_{34} and I_{12}, I_{14} yields I_{24}
- I_{13}, I_{35} and I_{16}, I_{56} yields I_{15}
- I_{14}, I_{45} and I_{16}, I_{56} yields I_{45}
- I_{12}, I_{15} and I_{23}, I_{35} yields I_{25}
- I_{12}, I_{16} and I_{25}, I_{56} yields I_{26}
- I_{13}, I_{16} and I_{35}, I_{56} yields I_{36}



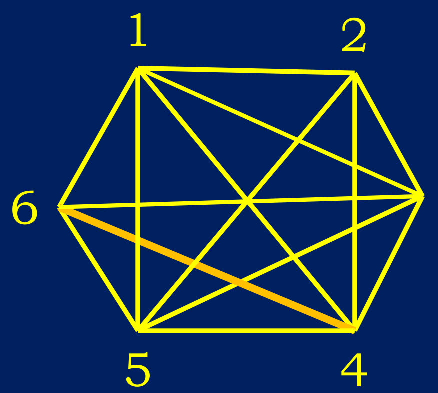


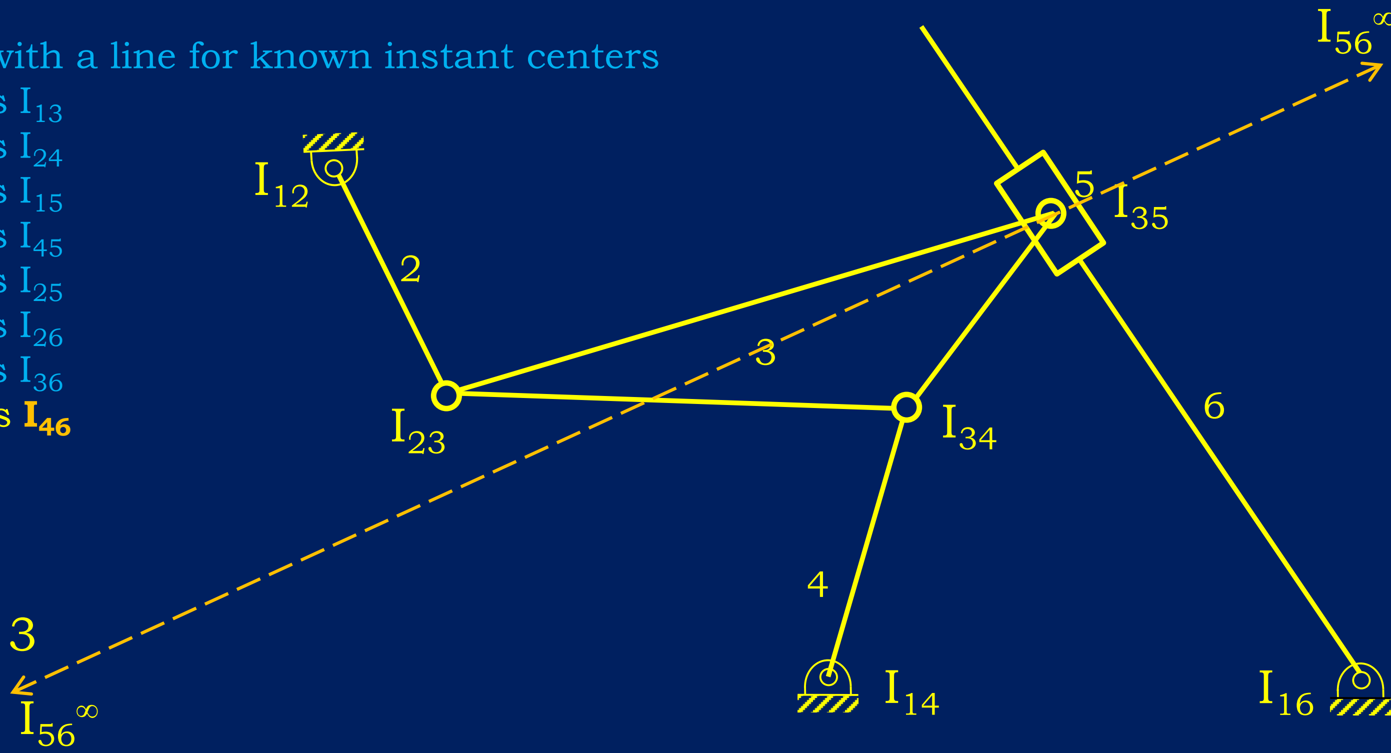
Continue the construction yourself!

Instant Centers of a Multi-Loop Mechanism

Determination of other instant centers:

- Write numbers up to ℓ
- Connect the numbers with a line for known instant centers
- I_{12}, I_{23} and I_{14}, I_{34} yields I_{13}
- I_{23}, I_{34} and I_{12}, I_{14} yields I_{24}
- I_{13}, I_{35} and I_{16}, I_{56} yields I_{15}
- I_{14}, I_{45} and I_{16}, I_{56} yields I_{45}
- I_{12}, I_{15} and I_{23}, I_{35} yields I_{25}
- I_{12}, I_{16} and I_{25}, I_{56} yields I_{26}
- I_{12}, I_{16} and I_{25}, I_{56} yields I_{36}
- I_{34}, I_{36} and I_{45}, I_{56} yields $\mathbf{I_{46}}$





Continue the construction yourself!

Very Brief & Over Simplified Summary of Burmester's Theory of ME 431 (1/7)

Two Positions: Two circle points can be selected freely on the moving plane. The center points can be anywhere on the perpendicular bisectors of the circle points.

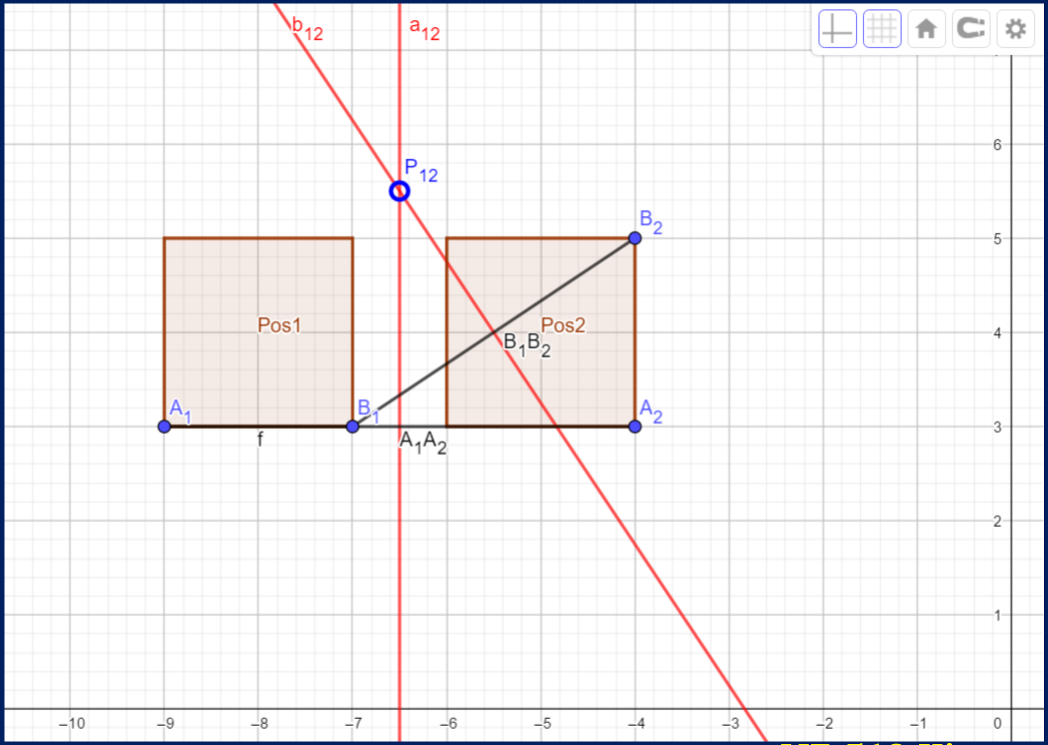
Three Positions: Two circle points can be selected freely on the moving plane. The center points are the centers of the circle points (unique for selected circle points!).

Four Positions: Two circle points *should be* select on Burmester's (K, circle point) curve on the moving plane. The center points are the corresponding points on Burmester's (M, center point) curve.

Five Positions: Use four positions at a time twice, say 1, 2, 3, 4 and 1, 2, 3, 5. Draw Burmester's K and M curves for both. The curves may intersect at most 3 or less points. Intersections of two K and two M curves are the circle and center point candidates respectively.

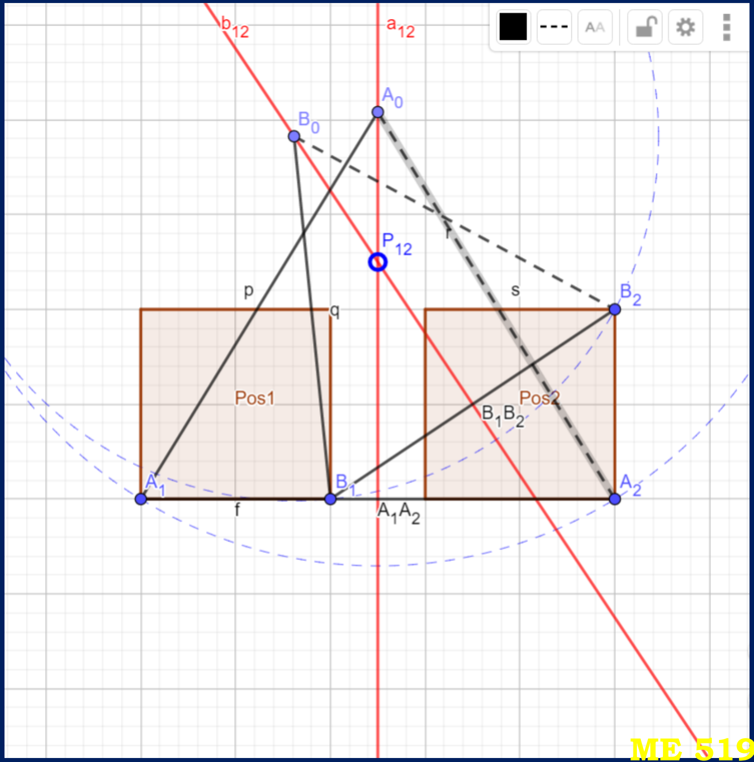
Very Brief & Over Simplified Summary of Burmester's Theory of ME 431 (2/7)

Two Positions: Two circle points can be selected *freely* on the moving plane (A_1 and B_1 in position 1, corresponding homologous points in Position 2 are A_2 and B_2). The center points can be anywhere on the perpendicular bisectors of the circle points.



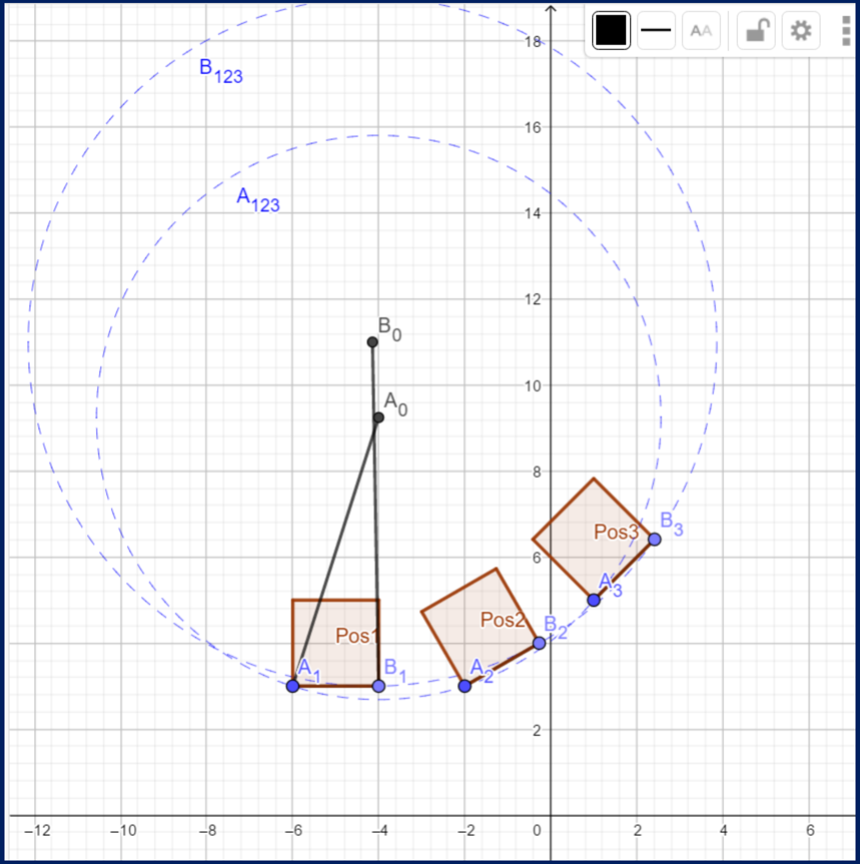
Very Brief & Over Simplified Summary of Burmester's Theory of ME 431 (3/7)

Two Positions: Two circle points can be selected *freely* on the moving plane (A_1 and B_1 in position 1, corresponding homologous points in Position 2 are A_2 and B_2). The center points (A_0 and B_0 respectively) can be anywhere on the perpendicular bisectors of the circle points a_{12} and b_{12} respectively.



Very Brief & Over Simplified Summary of Burmester's Theory of ME 431 (4/7)

Three Positions: Two circle points (A_1 and B_1) can be selected freely on the moving plane. The center points are the centers of the circles defined by three points (this time unique!).



Very Brief & Over Simplified Summary of Burmester's Theory of ME 431 (5/7)

Four Positions: Two circle points should be select on Burmester's (K, circle point) curve (because you ***cannot*** pass a circle through *arbitrarily selected* four points, these four points **must** define a circle!) on the moving plane. The center points are the corresponding points on Burmester's (M, center point) curve.

Very Brief & Over Simplified Summary of Burmester's Theory of ME 431 (6/7)

Five Positions: Use four positions at a time twice, say 1, 2, 3, 4 and 1, 2, 3, 5. Draw Burmester's K and M curves for both. The K_{1234} and K_{1235} curves and M_{1234} and M_{1235} curves may intersect at most 3 (or less points). Intersections of two K and two M curves are the circle and corresponding center point candidates respectively.

Please note that in four position synthesis you could trace the points on K curve (therefore had *infinitely* many solution candidates) however in five position synthesis you have just a *finite* number of solutions (at most 6 different four-bar mechanisms for a given motion, consequences in the next slide).

Very Brief & Over Simplified Summary of Burmester's Theory of ME 431 (7/7)

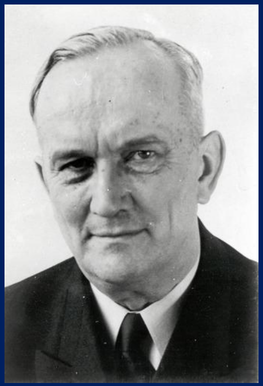
Some Problems of Position Synthesis

1. Although the mechanism exits in all design positions, all the positions may not be *in the same branch*.
2. The positions may not be followed in order when the mechanism is driven.
3. The transmission angle may not be favorable during the whole range of motion.
4. The link lengths may not be suitable for specific applications, joints may not be on desired areas.
5. During intermediate positions practical obstacles may not be avoided.
6. etc…

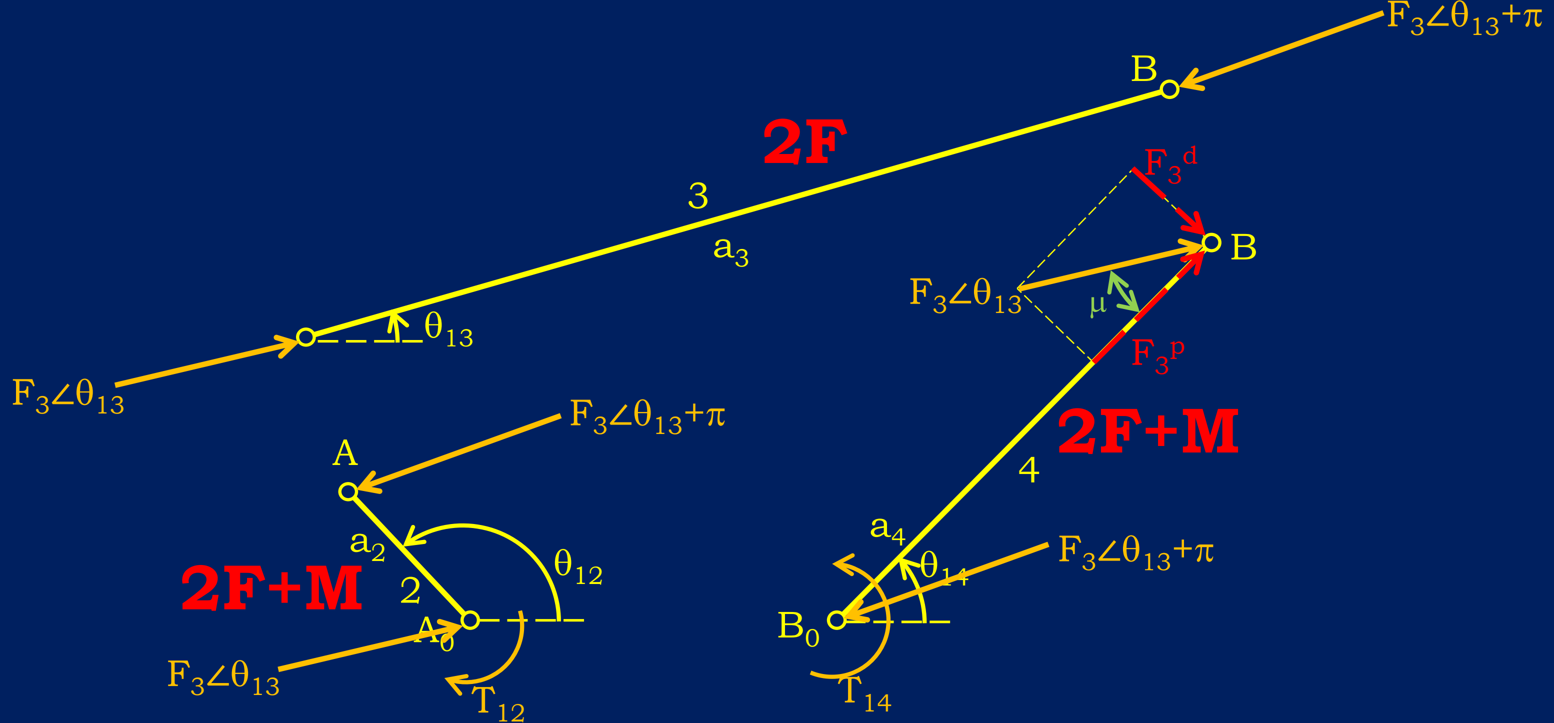
Transmission Angle:

Alt[1] defined the transmission angle as:

$$tan\mu = \frac{F_3{}^d}{F_3{}^p} \text{ or } sin\mu = \frac{F_3{}^d}{F_3}$$

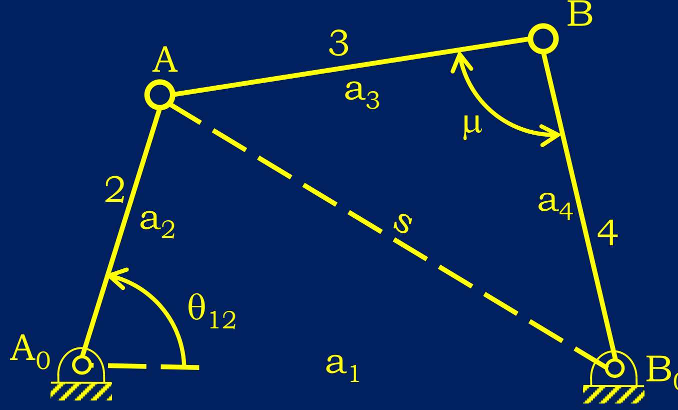


[1] Alt, Hermann (1889 - 1954). Der Übertragungswinkel und seine Bedeutung für das Konstruieren periodischer Getriebe (*The transmission angle and its importance for designing periodic mechanisms*). Werkstattstechnik 26 (1932) 61–64.



Kinematic Analysis

1. *Graphical Solution of Loop Closure Equations*
2. ***Stepwise Solution of Loop Closure Equations***
3. *Analytic – Closed Form Solution*
4. *Numerical Solution*



Law of cosines:

$$s^2 = {a_1}^2 + {a_2}^2 - 2a_1a_1cos\theta_{12}$$

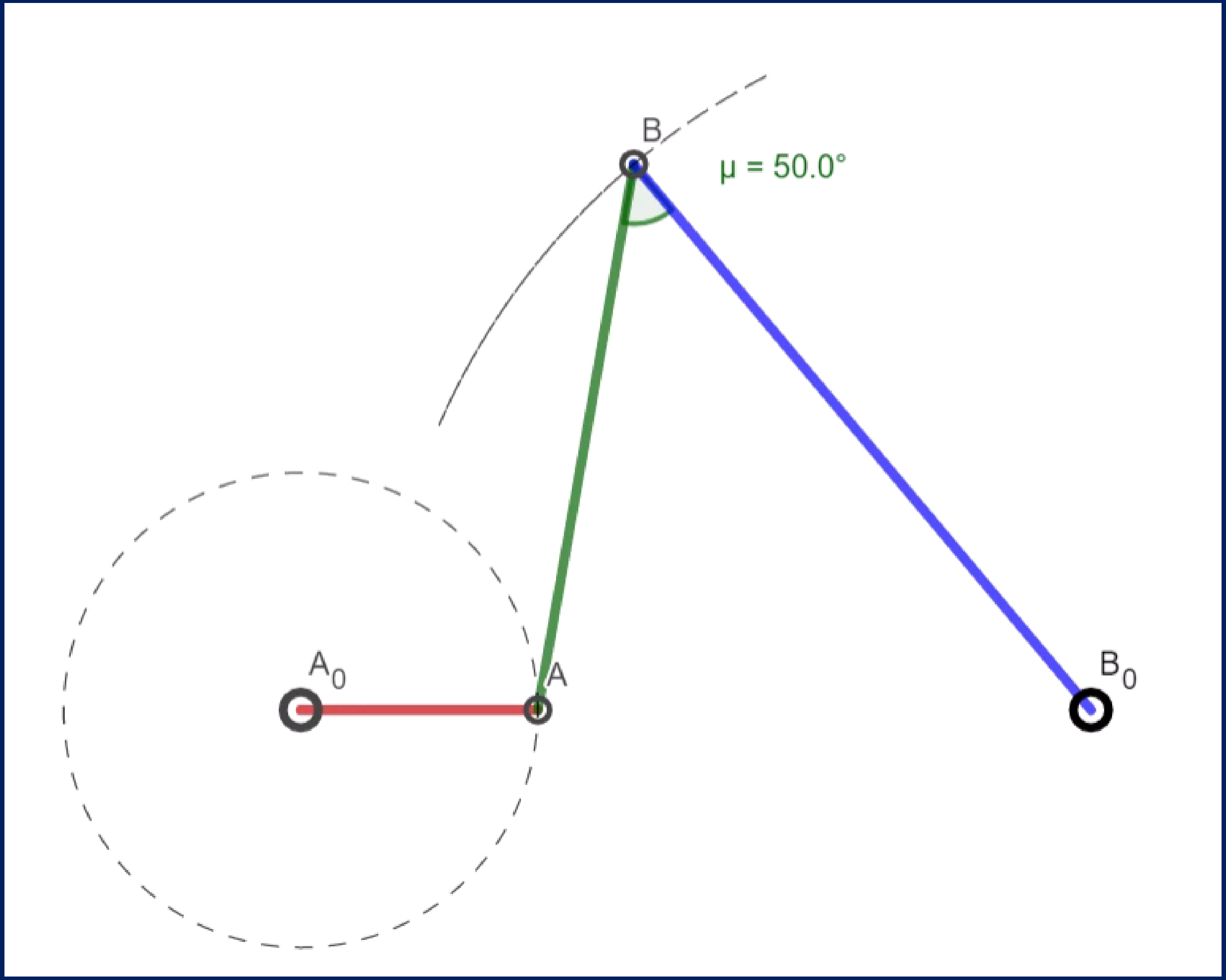
$$s^2 = {a_3}^2 + {a_4}^2 - 2a_3a_4cos\mu$$

$$cos\mu = \frac{{a_3}^2 + {a_4}^2 - {a_1}^2 - {a_2}^2 + 2a_1a_2cos\theta_{12}}{2a_3a_4}$$

The extremums of the transmission angle is

$$\frac{d\mu}{d\theta_{12}} = sin\theta_{12} = 0 \rightarrow \begin{cases} \theta_{12} = 0 \\ \theta_{12} = \pi \end{cases}$$

Transmission Angle:

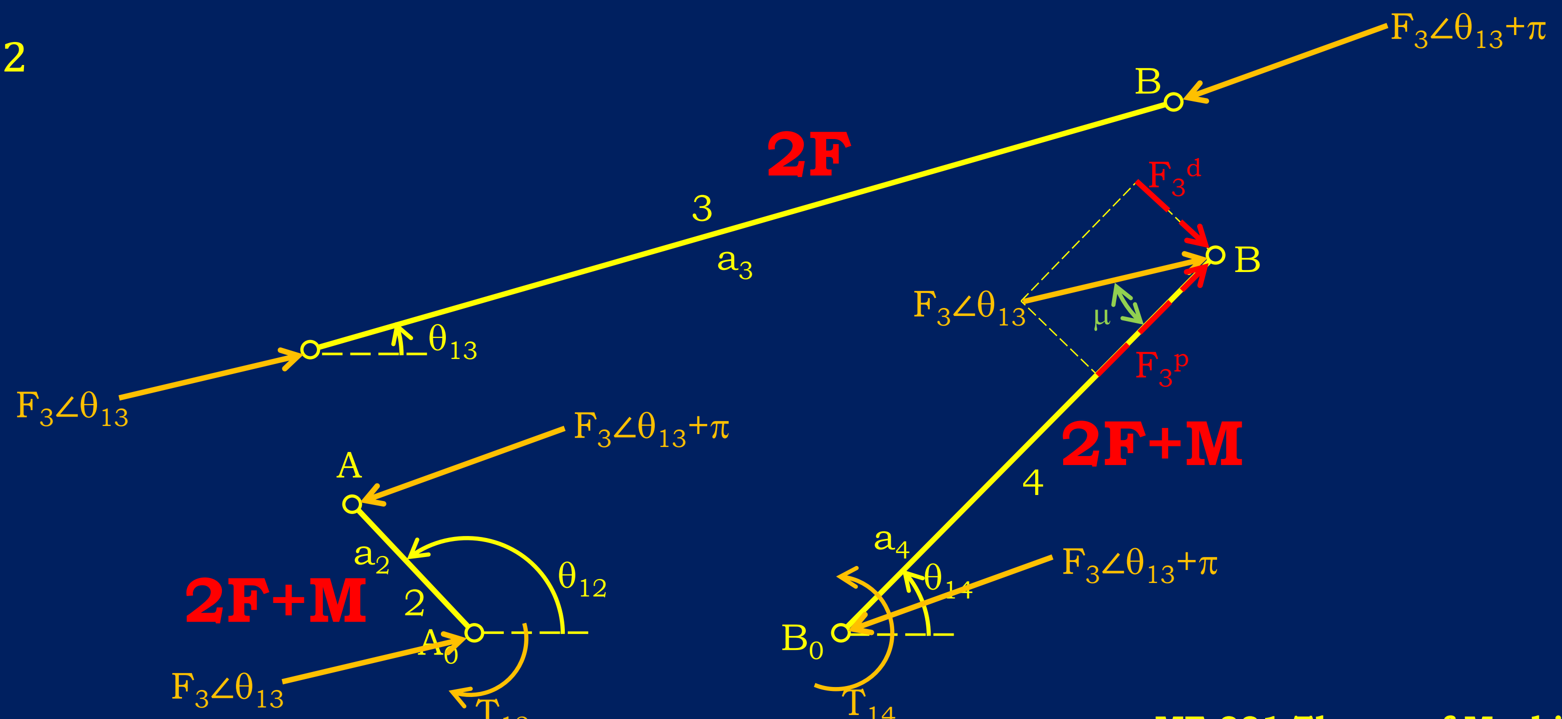


Mechanical Advantage:

Definition: The mechanical advantage of a mechanism is the instantaneous ratio of output torque (force) to input torque (force).

For a four bar mechanism where input is link 2 and output is link 4

$$MA = \frac{T_{14}}{T_{12}}$$



Mechanical Advantage:

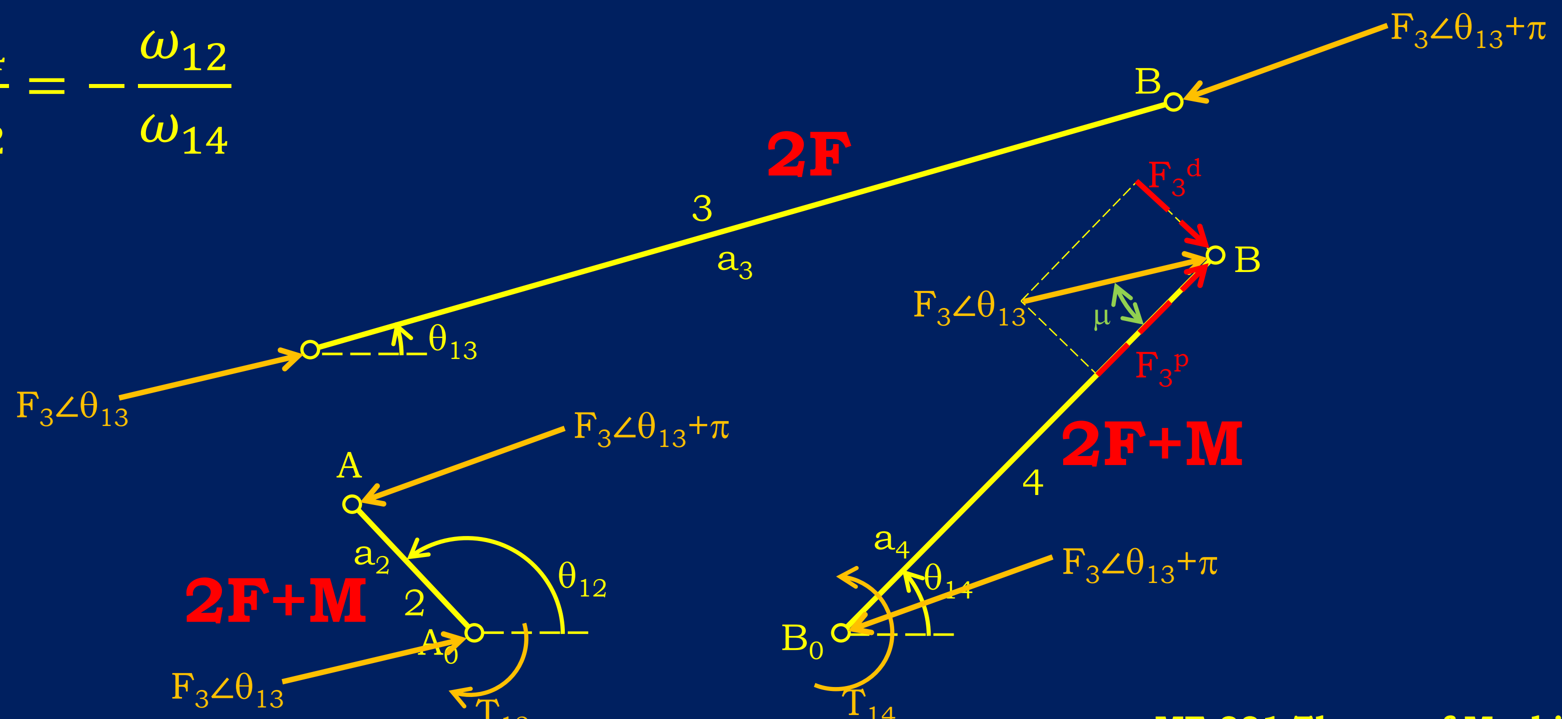
$$MA = \frac{T_{14}}{T_{12}}$$

Neglecting friction, kinetic and gravitational potential energy changes of the links (like quasi-static force analysis)

$$\mathbb{P}_{in} = \mathbb{P}_{out}$$

$$-T_{12}\omega_{12} = T_{14}\omega_{14}$$

$$MA = \frac{T_{14}}{T_{12}} = -\frac{\omega_{12}}{\omega_{14}}$$

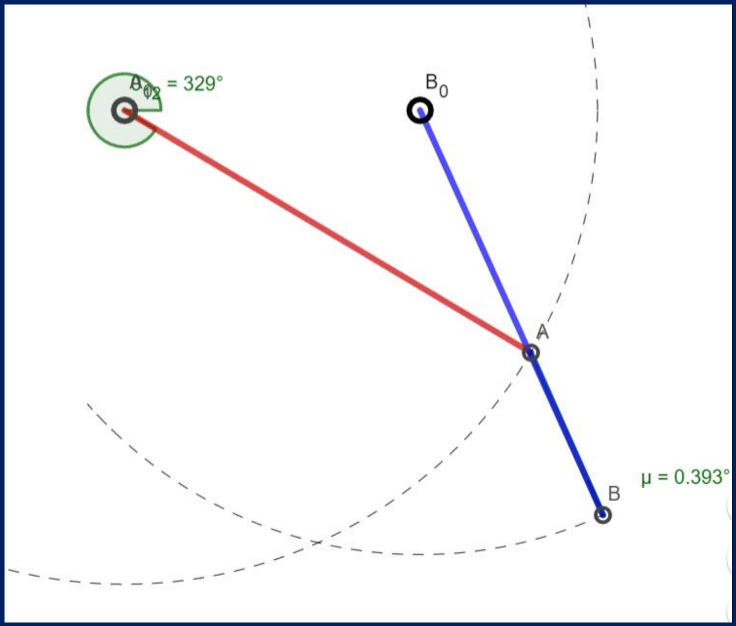


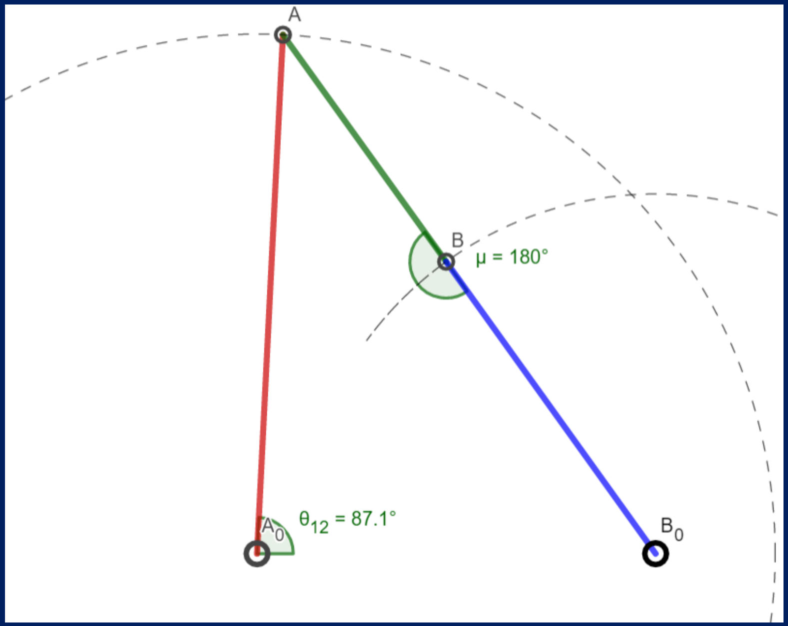
Mechanical Advantage:

$$MA = \frac{T_{14}}{T_{12}} = -\frac{\omega_{12}}{\omega_{14}} = -\frac{\dot{\theta}_{12}}{\dot{\theta}_{14}} = \frac{a_4 sin(\theta_{14} - \theta_{13})}{a_2 sin(\theta_{12} - \theta_{13})}$$

$sin(\theta_{12} - \theta_{13}) = 0, MA \rightarrow \infty$ Dead centers!

$sin(\theta_{14} - \theta_{13}) = 0, MA = 0, \mu = 0$ or $\mu = 180°$





Graphical Methods and GeoGebra

Geogebra is a free tool for mathematics, graphics, geometry etc.

Mechanism analysis and synthesis started with graphical methods, using drafting tools like ruler, compass, T-square etc. (i.e. geometry!)

With the evolution of digital computers the mathematics behind geometry was formulated as analytic methods.

With the evolution of parametric CAD software packages (e.g. SolidWorks, NX, Catia, etc.) the intuitive graphical methods became popular again.

Geogebra Classic which can be downloaded or used as a web application is a simple and intuitive tool to replace expensive CAD programs for mechanism analysis and synthesis.

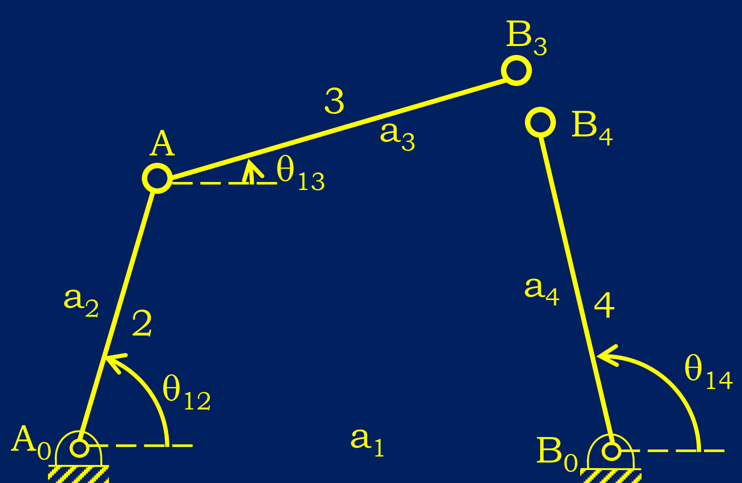
Graphical Methods and GeoGebra

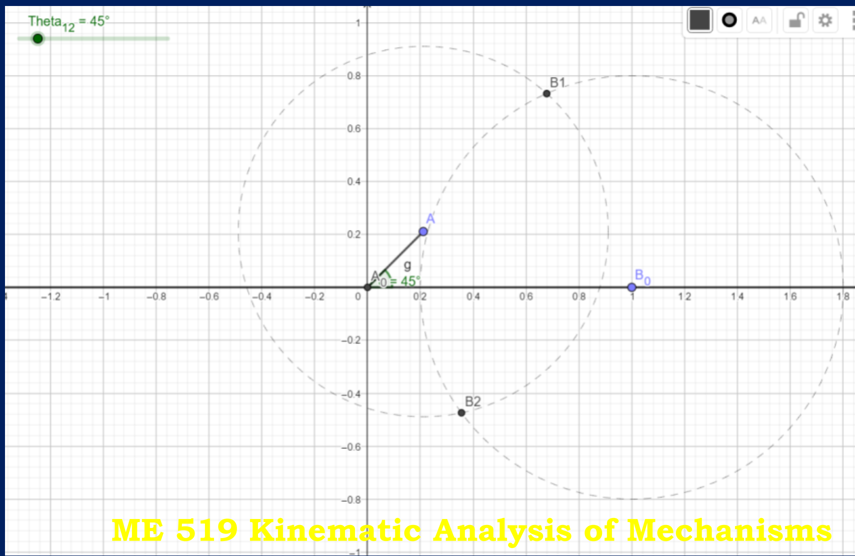
Position analysis of a crank-rocker four-bar utilizing graphical methods on Geogebra:

The loop closure equation for virtually disconnecting and re-connecting revolute joint B (i.e. B_3 and B_4 coincident) is:

$$a_2 e^{\theta_{12}} + a_3 e^{\theta_{13}} = a_1 + a_4 e^{\theta_{14}}$$

For a given θ_{12} A is fixed, B_3 traces a circle of radius a_3 centered at A and B_4 traces a circle of radius a_4 centered at B_0. The two intersection points of these two circles yield two locations of point B for the current position for two closures of the mechanism.





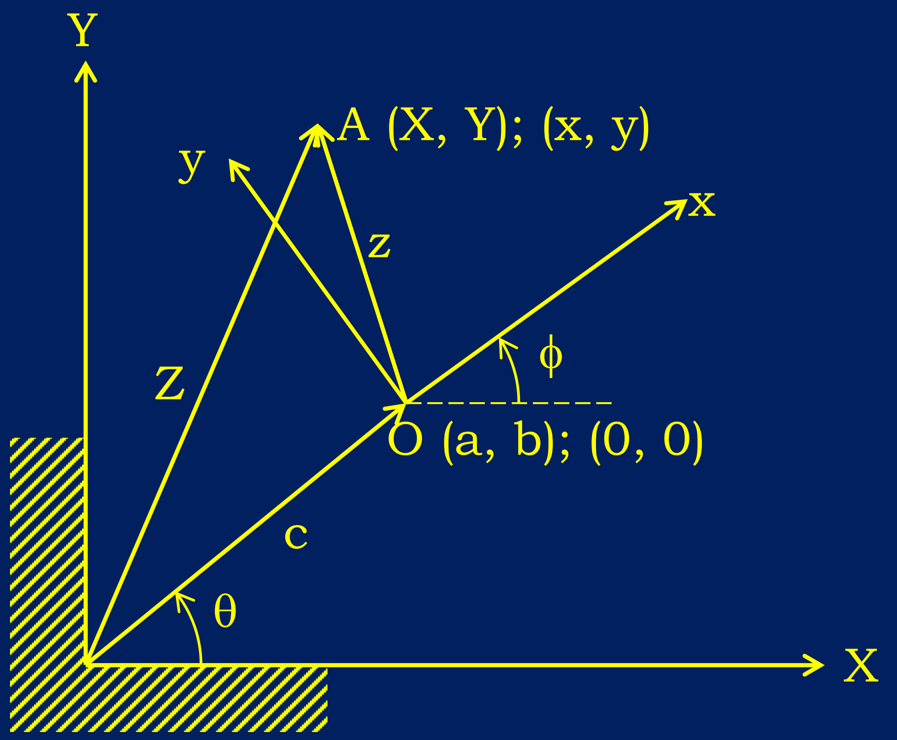
Graphical Methods and GeoGebra

Position analysis of a crank-rocker four-bar utilizing graphical methods on Geogebra:

Complete the two closures and run the mechanism by controlling given θ_{12} by the slider.

1. Canonical Representation of Plane Motion

Analysis of plane motion requires certain parameters. A typical selection may be *a*, *b* and ϕ (recall degree of freedom of a rigid body in plane motion is 3 therefore one needs three independent parameters to define the motion completely). Depending how the motion is defined, further, either two of these parameters may be defined as a function of the third parameter or all may be defined as a function of another independent parameter, most commonly time.



Selection of parameters to define the motion is totally arbitrary but by using *canonical representation* of plane motion one may define the plane motion in a ***unique way***.

1. Canonical Representation of Plane Motion

Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole[1].

Theorem 2: Every point on the moving plane rotates about instant center of zero velocity with a speed that is equal to the product of distance of the point to the instant center and the angular velocity of the plane. Recall from dynamics, $\omega = \frac{v_A}{r_{A/ICZV}} = \frac{v_B}{r_{B/ICZV}} = \frac{v_C}{r_{C/ICZV}} = \cdots = \frac{v_{\cdot}}{r_{\cdot/ICZV}}$ and $\vec{v}_A = \vec{\omega} \times \vec{r}_{A/ICZV}$

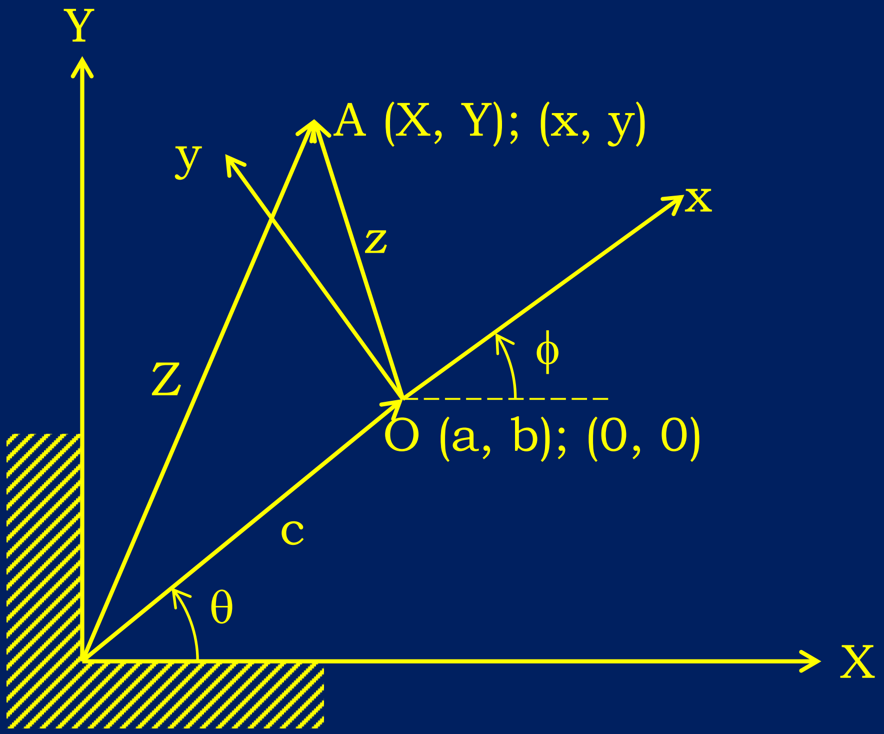
Theorem 3: The motion of the moving plane is pure rolling of *moving centrode* (locus of instant center on moving plane) on the *fixed centrode* (locus of instant center on fixed plane).

[1] *This can be extended to Mozzi–Chasles' theorem that the most general rigid body displacement can be produced by a translation along a line (called its screw axis or Mozzi axis) followed (or preceded) by a rotation about an axis parallel to that line in 3-D. Also recall Chasles' theorem for finitely separated two positions (ME 431) which boils down to the instant center of zero velocity at the limit when two positions are infinitesimally close!*

Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole.

Proof:

A is a *fixed* point in the moving plane x-y



$Z, z, c \ \in \mathbb{C}$

$X, Y, x, y, \theta, \phi \ \in \mathbb{R}$

$Z = X + iY$

$z = x + iy$

$c = a + ib$

$X = a + x\cos\phi - y\sin\phi$

$Y = b + x\sin\phi + y\cos\phi$

or

$Z = c + ze^{i\phi}$

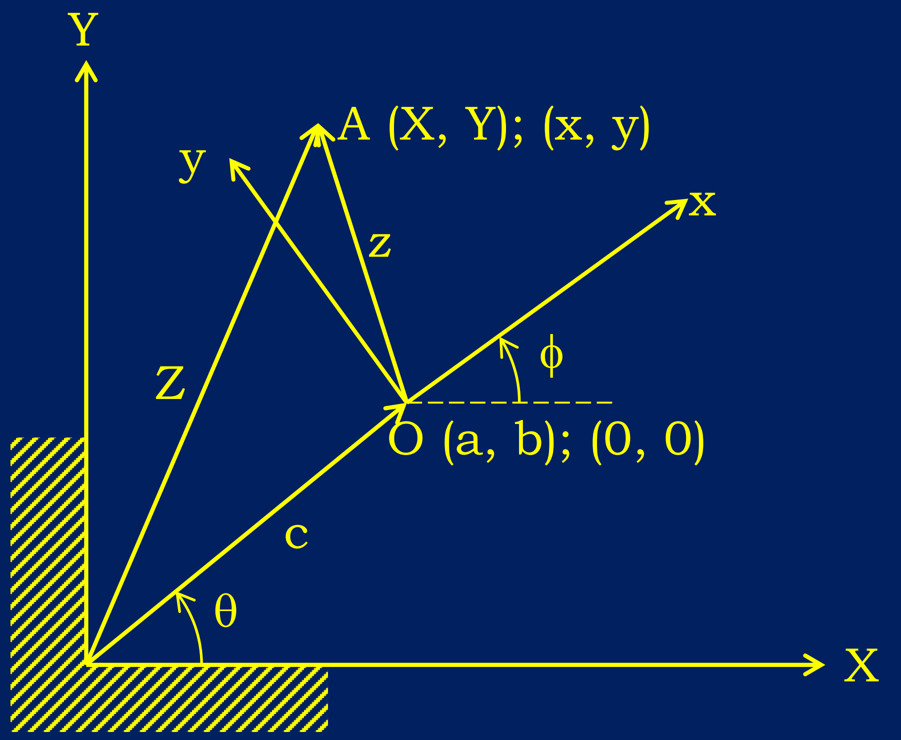
Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole.

Proof (cont'ed):

$$X = a + x\cos\phi - y\sin\phi$$

$$Y = b + x\sin\phi + y\cos\phi$$

$$Z = c + ze^{i\phi}$$



Taking time derivative:

$$\dot{X} = \dot{a} - \dot{\phi}x\sin\phi - \dot{\phi}y\cos\phi$$

$$\dot{Y} = \dot{b} + \dot{\phi}x\cos\phi - \dot{\phi}y\sin\phi$$

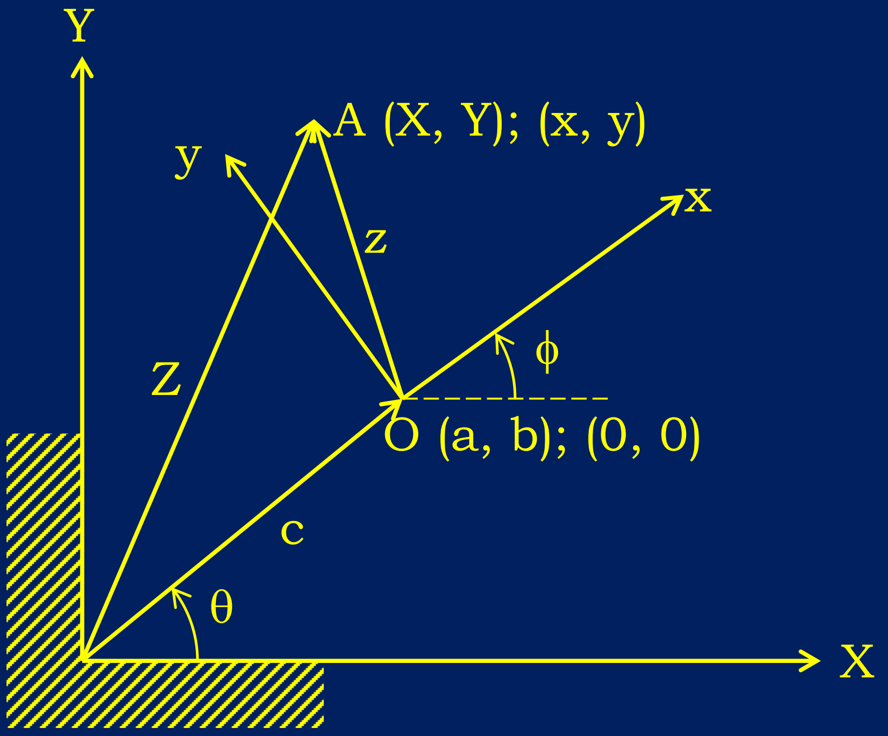
$$\dot{Z} = \dot{c} + i\dot{\phi}ze^{i\phi}$$

For $\dot{\phi} \neq 0$

$$(\quad\dot{}\quad) = \frac{d(\quad)}{dt} = \frac{d(\quad)}{d\phi}\frac{d\phi}{dt} = \dot{\phi}\,\frac{d(\quad)}{d\phi}$$

$$\frac{d}{d\phi} = (\quad)'$$

Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole.



Proof (cont'ed):

$$\frac{dZ}{d\phi}\frac{d\phi}{dt} = \frac{dc}{d\phi}\frac{d\phi}{dt} + i\frac{d\phi}{dt}ze^{i\phi}$$

$$\frac{dZ}{d\phi} = \frac{dc}{d\phi} + ize^{i\phi}$$

$$Z' = c' + ize^{i\phi}$$

$$c' = a' + ib'$$

In Cartesian coordinates

$$\left.\begin{aligned} X' &= a' - x\sin\phi - y\cos\phi \\ Y' &= b' + x\cos\phi - y\sin\phi \end{aligned}\right\} Eq.1$$

Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole (**P**).

Proof (cont'd):

Instant center has zero velocity

$Z' = 0, X' = 0$ and $Y' = 0$

Location of instant center

- on moving plane $p(x_P, y_P)$
- on fixed plane $P(X_P, Y_P)$

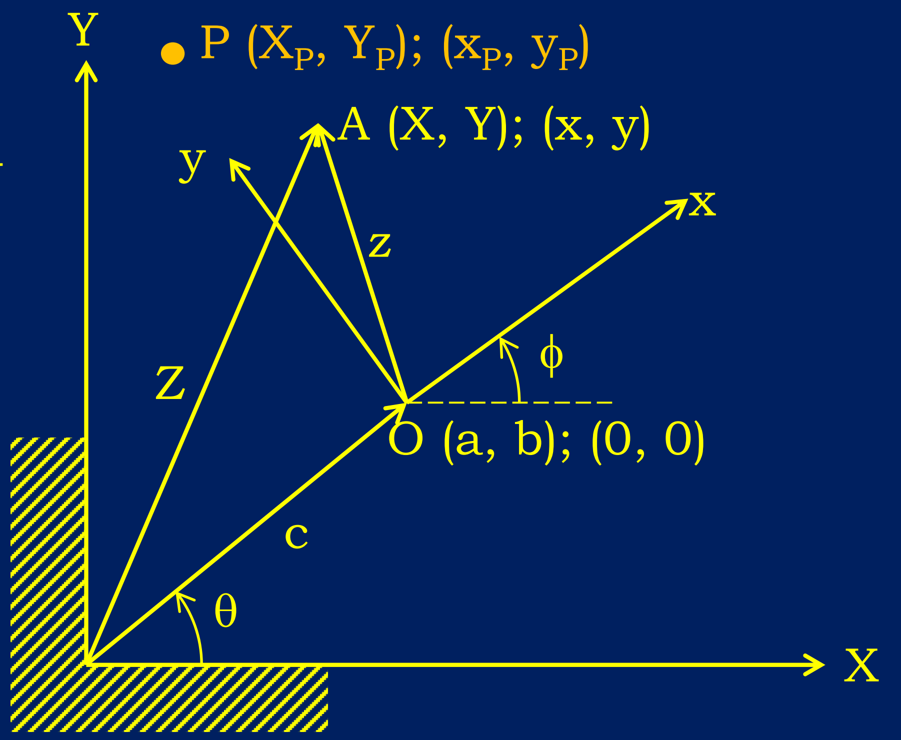
From Eq. 1

$x_P sin\phi + y_P cos\phi = a'$

$x_P cos\phi - y_P sin\phi = -b'$

Solution yields

$$\left.\begin{aligned} x_P &= a' sin\phi - b' cos\phi \\ y_P &= a' cos\phi - b' sin\phi \\ z_P &= ic' e^{i\phi} \end{aligned}\right\} Eq.\ 2$$



Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole.

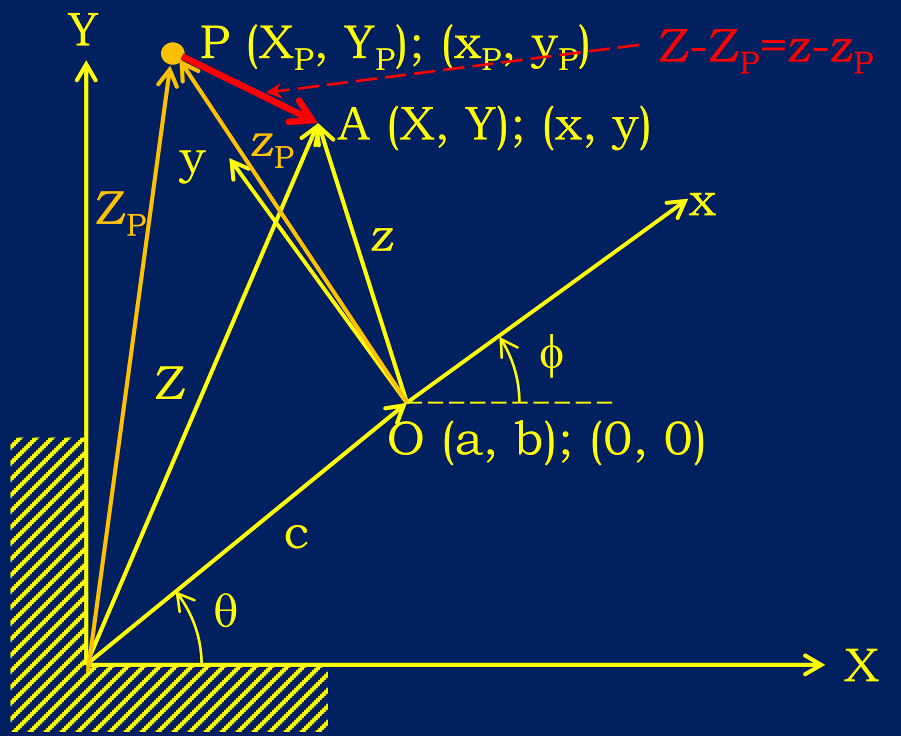
Proof (cont'ed):

In fixed plane

$$X_P = a - b'$$
$$Y_P = b + a'$$

$$Z_P = c + z_P e^{i\phi}, z_P = ic' e^{-i\phi}$$

$$Z_P = c + ic'$$



$$\left.\begin{aligned} X_P &= a - b' \\ Y_P &= b + a' \\ Z_P &= c + ic' \end{aligned}\right\} Eq.\ 3$$

This shows instant center of zero velocity/pole exists!

Canonical Representation of Plane Motion

Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole[1].

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Theorem 3: The motion of the moving plane is pure rolling of moving centrode (locus of instant center on moving plane) on the fixed centrode (locus of instant center on fixed plane).

[1] *This can be extended to Mozzi–Chasles' theorem that the most general rigid body displacement can be produced by a translation along a line (called its screw axis or Mozzi axis) followed (or preceded) by a rotation about an axis parallel to that line in 3-D.*

Theorem 2: Every point on the moving plane rotates about instant center of zero velocity with a speed that is equal to the product of distance of the point to the instant center and the angular velocity of the plane.

Proof:

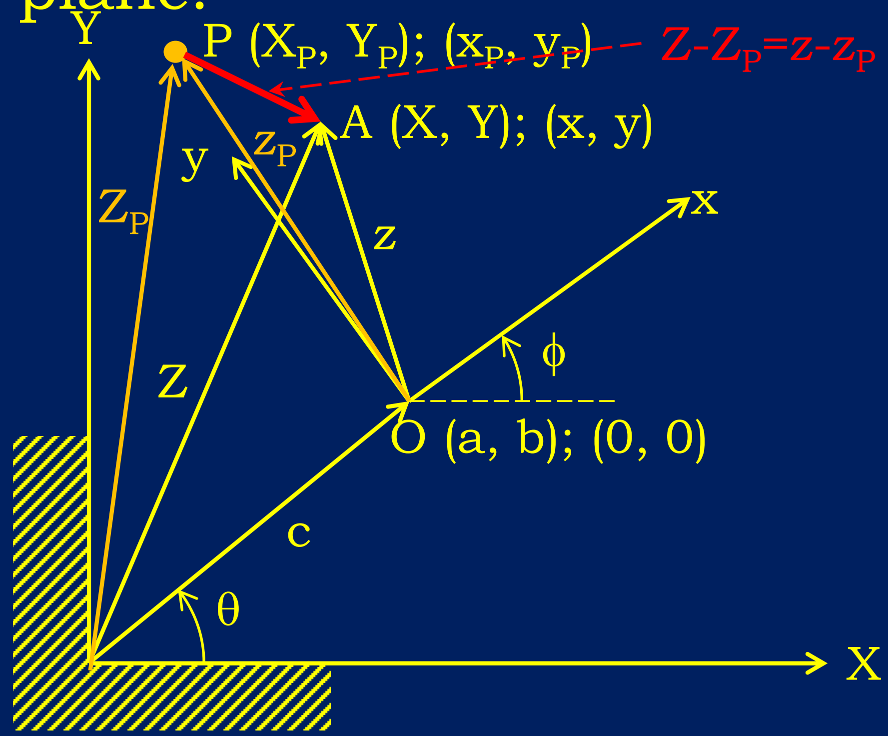
$$Z = c + ze^{i\phi}$$

$$Z - Z_P = (z - z_P)e^{i\phi}$$

Taking time derivative:

$$\dot{Z} = (z - z_P)e^{i\phi}i\dot{\phi}$$

$$\vec{v}_P = \vec{\omega} \times \vec{r}_{P/ICZV}$$



Considering the trajectory of P

- on moving plane (Eq. 2) the *moving centrode*
- on fixed plane (Eq. 3) the *fixed centrode*

are obtained (leads to *Theorem 3*).

Canonical Representation of Plane Motion

Theorem 1: In every plane motion there exists a point which has zero velocity at the instant considered. This point is called the instant center of zero velocity/rotation pole[1].

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Theorem 3: The motion of the moving plane is pure rolling of moving centrode (locus of instant center on moving plane, Eq. 2) on the fixed centrode (locus of instant center on fixed plane, Eq. 3).

Proof: Rolling without slipping requires

$$\frac{dS_P}{d\phi} = \frac{ds_P}{d\phi}$$

$$\left(\frac{ds_P}{d\phi}\right)^2 = \left(\frac{dx_P}{d\phi}\right)^2 + \left(\frac{dy_P}{d\phi}\right)^2$$

Recall [Eq. 2]

$$x_P = a'sin\phi - b'cos\phi, y_P = a'cos\phi - b'sin\phi$$

$$\frac{dx_P}{d\phi} = x_P{}' = a''sin\phi + a'cos\phi - b''cos\phi + b'sin\phi$$

$$\frac{dy_P}{d\phi} = y_P{}' = a''cos\phi - a'sin\phi + b''sin\phi + b'cos\phi$$

$$\left(\frac{ds_P}{d\phi}\right)^2 = (a'' + b')^2 + (a' - b'')^2$$

Theorem 3: The motion of the moving plane is pure rolling of moving centrode (locus of instant center on moving plane) on the fixed centrode (locus of instant center on fixed plane).

Proof (cont'ed):

$$\left(\frac{dS_P}{d\phi}\right)^2 = \left(\frac{dX_P}{d\phi}\right)^2 + \left(\frac{dY_P}{d\phi}\right)^2$$

Recall [Eq. 3]

$$X_P = a - b'$$

$$Y_P = b - a'$$

$$\frac{dX_P}{d\phi} = X_P{}' = a' - b''$$

$$\frac{dY_P}{d\phi} = Y_P{}' = b' - a''$$

$$\left(\frac{dS_P}{d\phi}\right)^2 = (a'' + b')^2 + (a' - b'')^2 = \left(\frac{ds_P}{d\phi}\right)^2$$

Theorem 3: The motion of the moving plane is pure rolling of moving centrode (locus of instant center on moving plane) on the fixed centrode (locus of instant center on fixed plane).

Proof (cont'ed): Further,

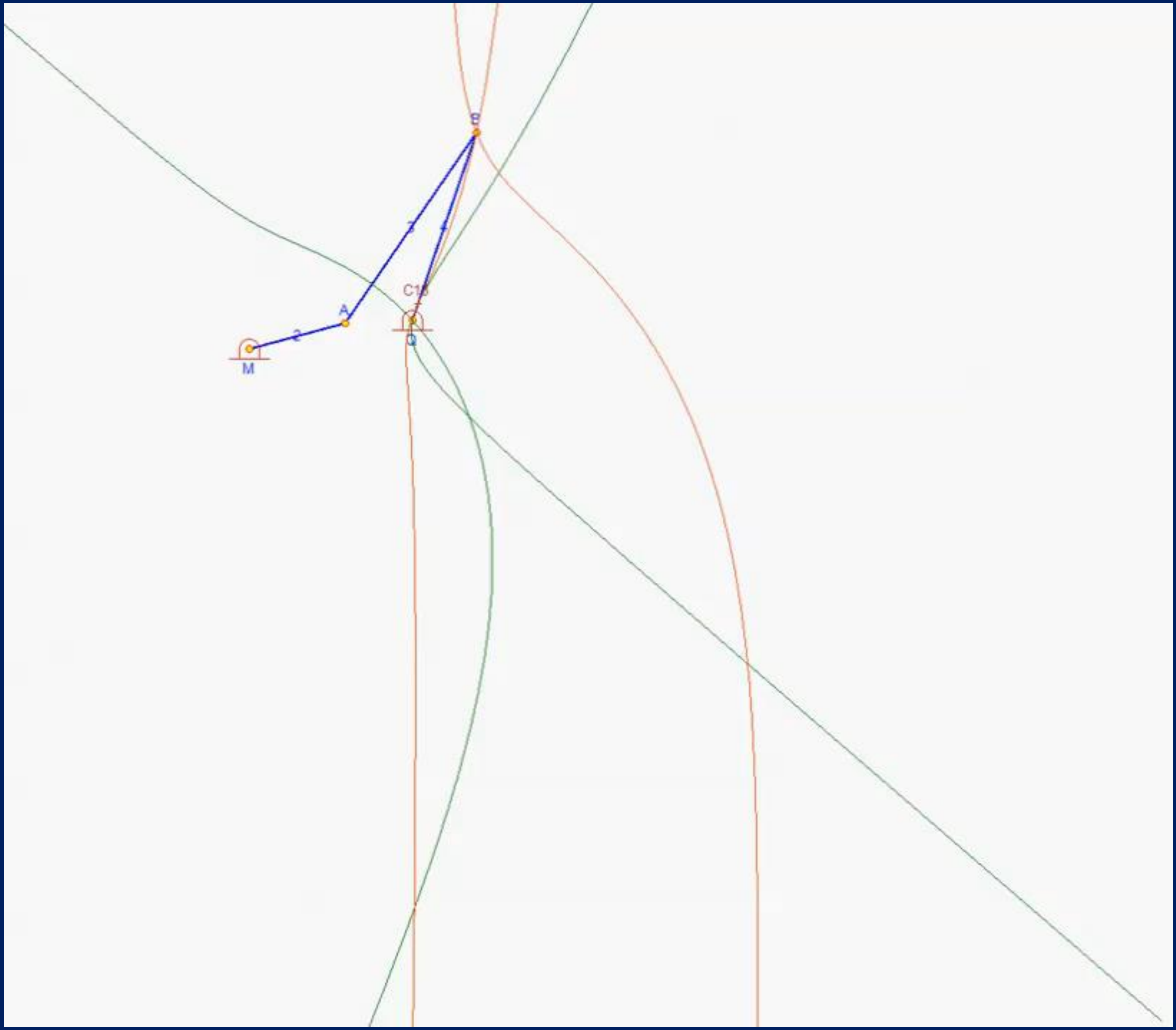
$$\frac{dy_P/d\phi}{dx_P/d\phi} = \frac{b' + a''}{a' - b''}$$

$$\frac{dY_P/d\phi}{dX_P/d\phi} = \frac{b' + a''}{a' - b''}$$

Therefore moving and fixed centrodes share the same tangent at the contact point which is the pole/instant center of zero velocity!

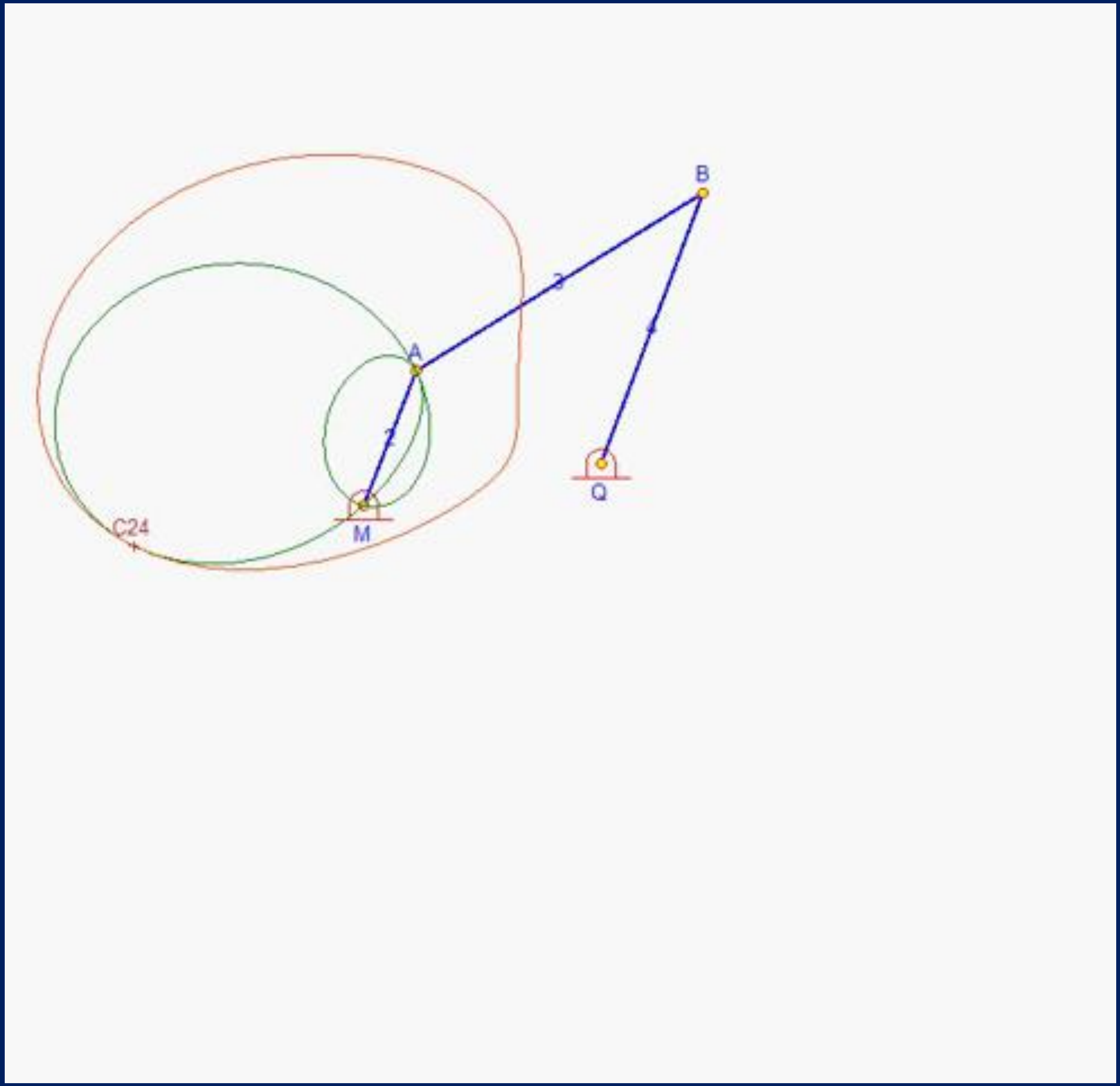
Q. E. D.

Motion of Instant Center of Four-Bar I_{13}



https://www.youtube.com/watch?v=5fEIhVH1doU

Motion of Instant Center of Four-Bar I_{24}



https://www.youtube.com/watch?v=ayyafcEBsqc

Centrodes of Some Common Motions

Case 1: Revolute Joint/Fixed Axis Rotation:

$|c| = r = const.$

$\theta = \phi + const.$

$Z_P = z_P = 0$

Centrodes reduce to the axis of rotation.

Case 2: Prismatic Joint/Rectilinear Translation:

$\phi = const., \dot{\phi} = 0$

Pole at infinity in a direction perpendicular to translation axis.

Case 3: Cardanic Motion:

$|c| = r_0 = const.$

$\theta = -\phi$

Recall [Eq. 2]

$z_P = ic'e^{i\phi}$

$c = re^{i\theta} = a + ib = r_0 e^{-i\phi}$

$c' = -ir_0 e^{-i\phi}$

Centrodes of Some Common Motions

Case 3: Cardanic Motion (cont'd):

$z_P = ic'e^{i\phi}$

$c = re^{i\theta} = a + ib = r_0 e^{-i\phi}$

$c' = -ir_0 e^{-i\phi}$

$z_P = i(-ir_0 e^{-i\phi})e^{i\phi} = r_0 e^{-2i\phi}$

Recall [Eq. 3]

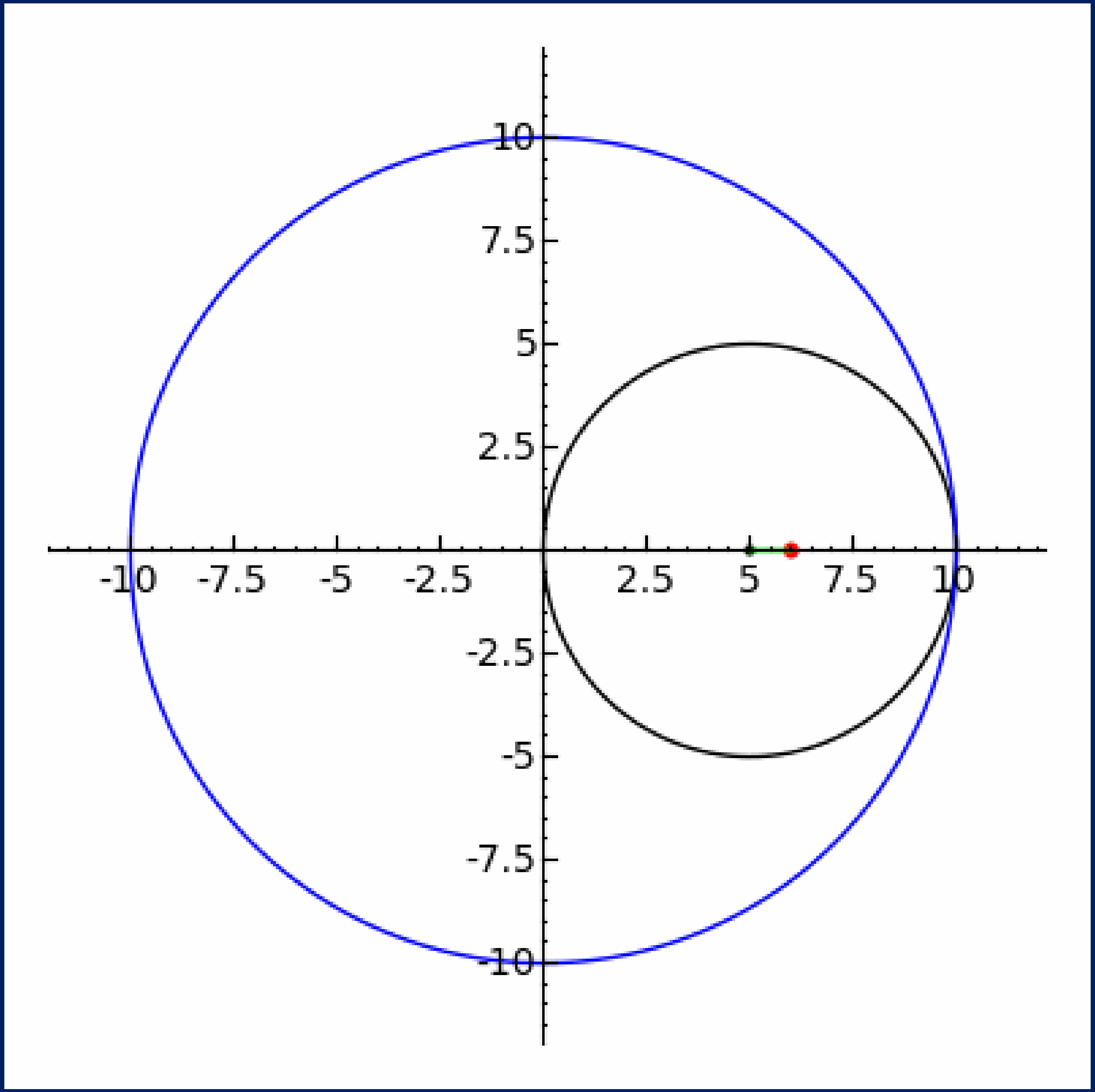
$Z_P = c + ic' = r_0 e^{-i\phi} + r_0 e^{-i\phi} = 2r_0 e^{-i\phi}$

} Two circles of radii r_0 and $2r_0$

This is like the motion of a planet gear of radius r_0 inside a fixed ring gear of radius $2r_0$ or a cylinder of radius r_0 rolling without slipping inside a fixed hollow cylinder of radius $2r_0$.

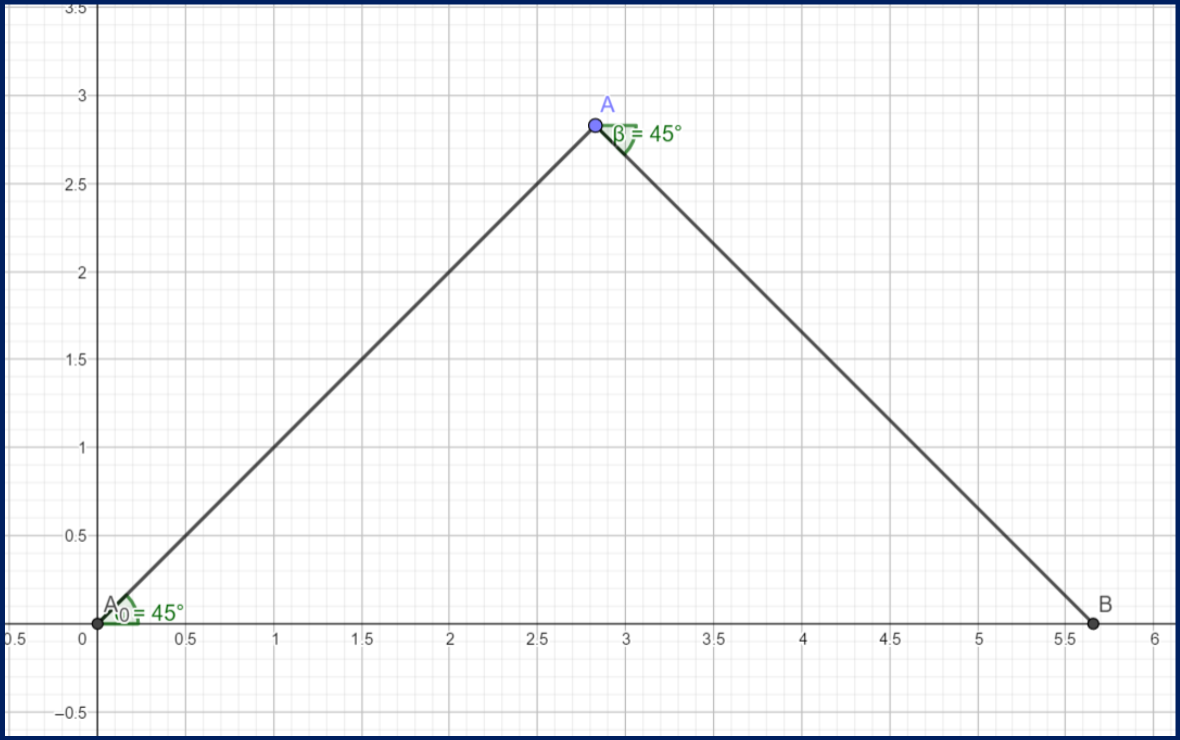
The pitch circles of the gears or the cylinders are known as *Cardan circles*.

Cardan Circles



In-Line Slider-Crank (Equal Crank and Coupler Lengths)

Crank angle, θ, coupler angle, $\phi = -\theta$



Double Slider

Consider the coupler (floating link) of the double slider. Please note that the two slider axes need not to be perpendicular to each other however they are not allowed to be parallel.

$$|AB| = p + q = c$$

$$X = p\,cos\theta + h\,sin\theta$$

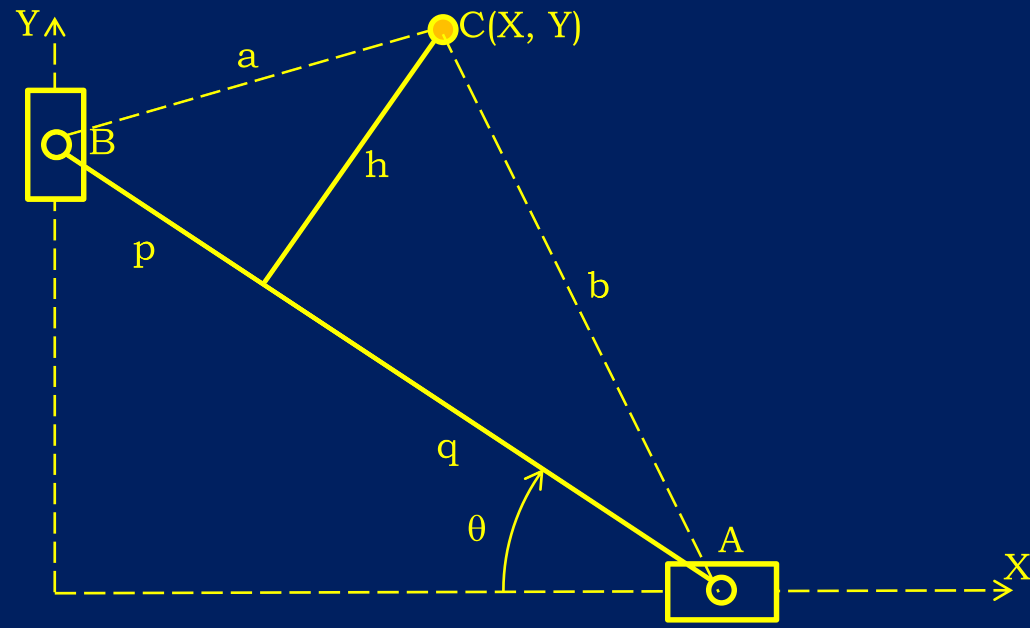
$$Y = q\,sin\theta + h\,cos\theta$$

$$cos\theta = \frac{qX - hY}{pq - h^2}$$

$$sin\theta = \frac{pY - hX}{pq - h^2}$$

$$sin^2\theta + cos^2\theta = 1$$

$$(pY - hX)^2 + (qX - hY)^2 = (pq - h^2)^2$$



Double Slider

$$(pY - hX)^2 + (qX - hY)^2 = (pq - h^2)^2$$

$$(h^2 + q^2)X^2 + (h^2 + p^2)Y^2 - 2h(p + q)XY = (pq - h^2)^2$$

$$(h^2 + q^2) = b^2$$

$$(h^2 + p^2) = a^2$$

$$(p + q) = c$$

$$b^2X^2 + a^2Y^2 - 2hcXY = (pq - h^2)^2$$

This is the equation of an ellipse!

For C on |AB|

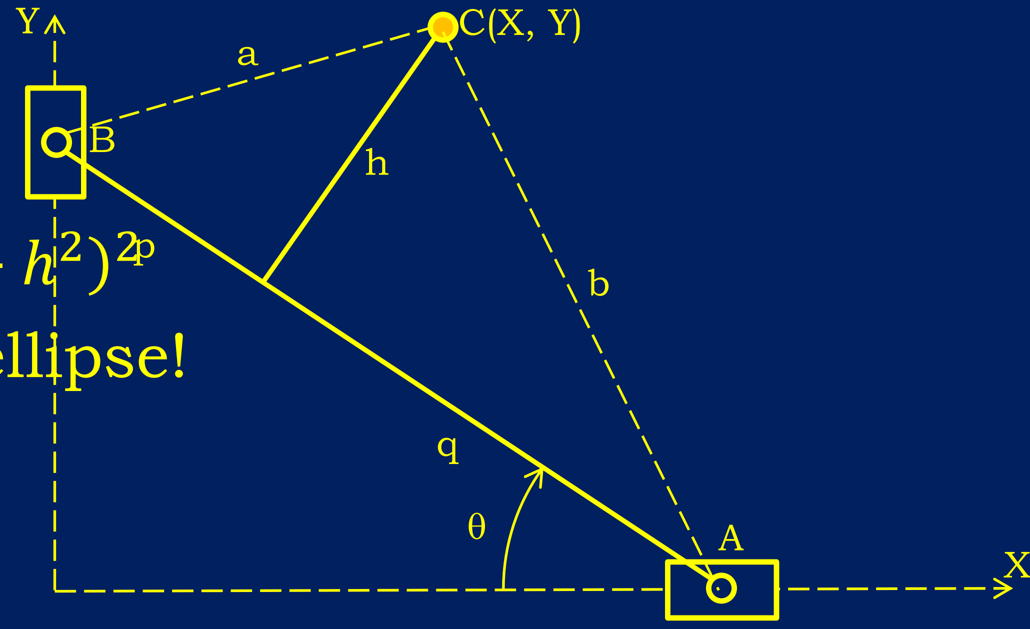
h = 0, p = a, q = b

$$b^2X^2 + a^2Y^2 = a^2 + b^2$$

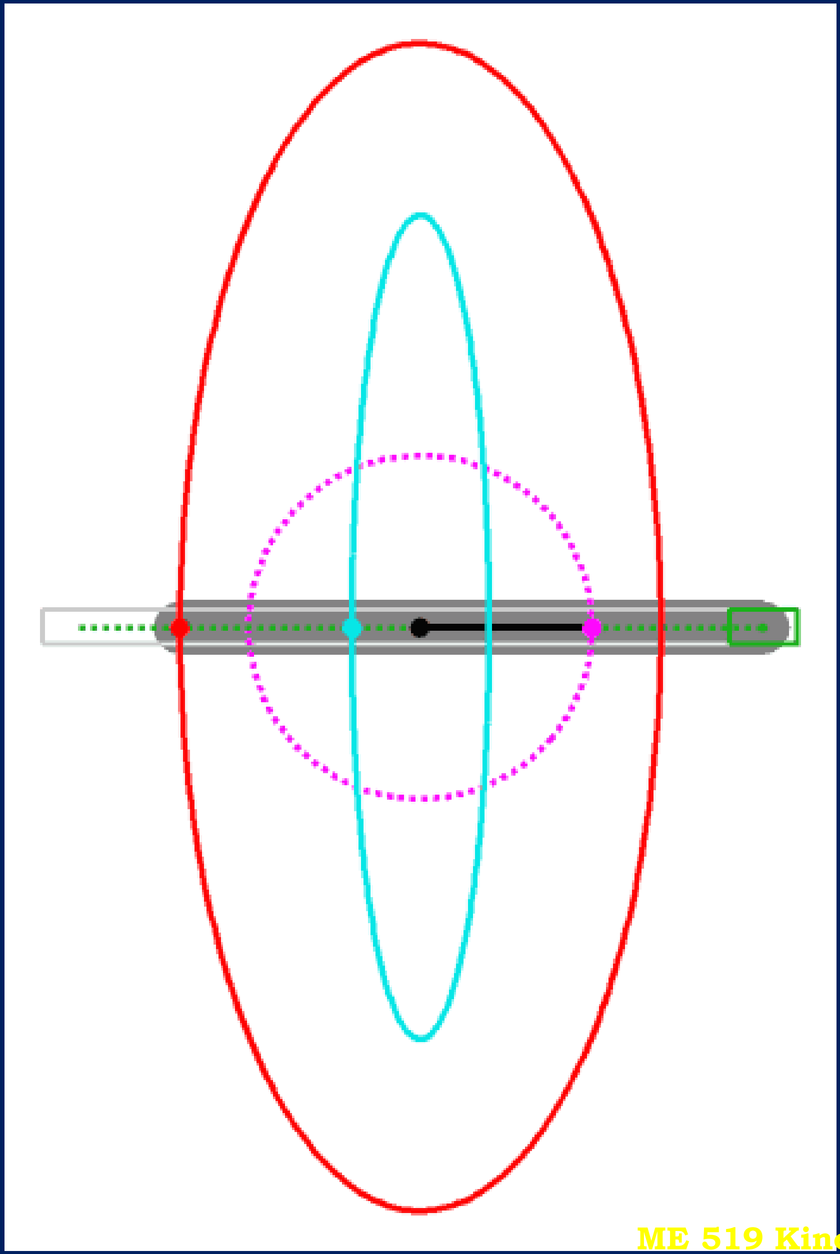
or

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

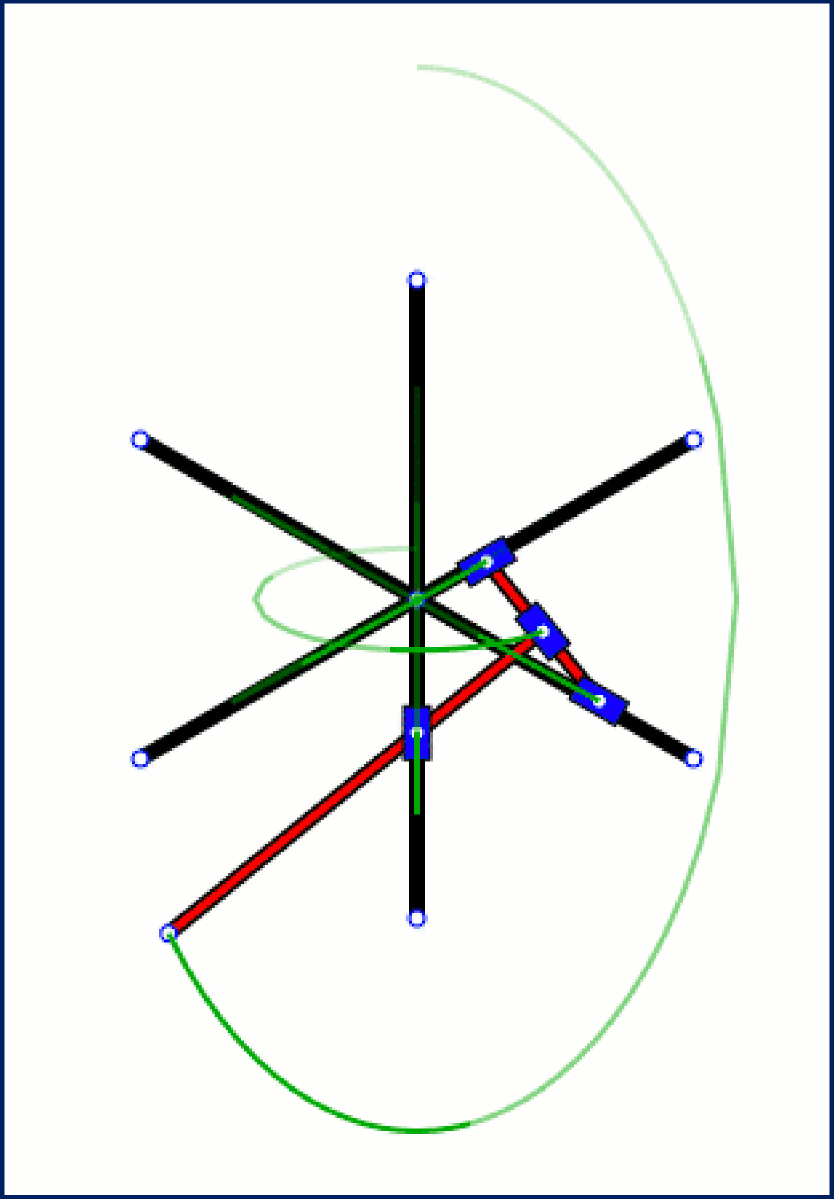
For a = b it would be a circle.



Cardan Ellipses



Cardan Ellipses



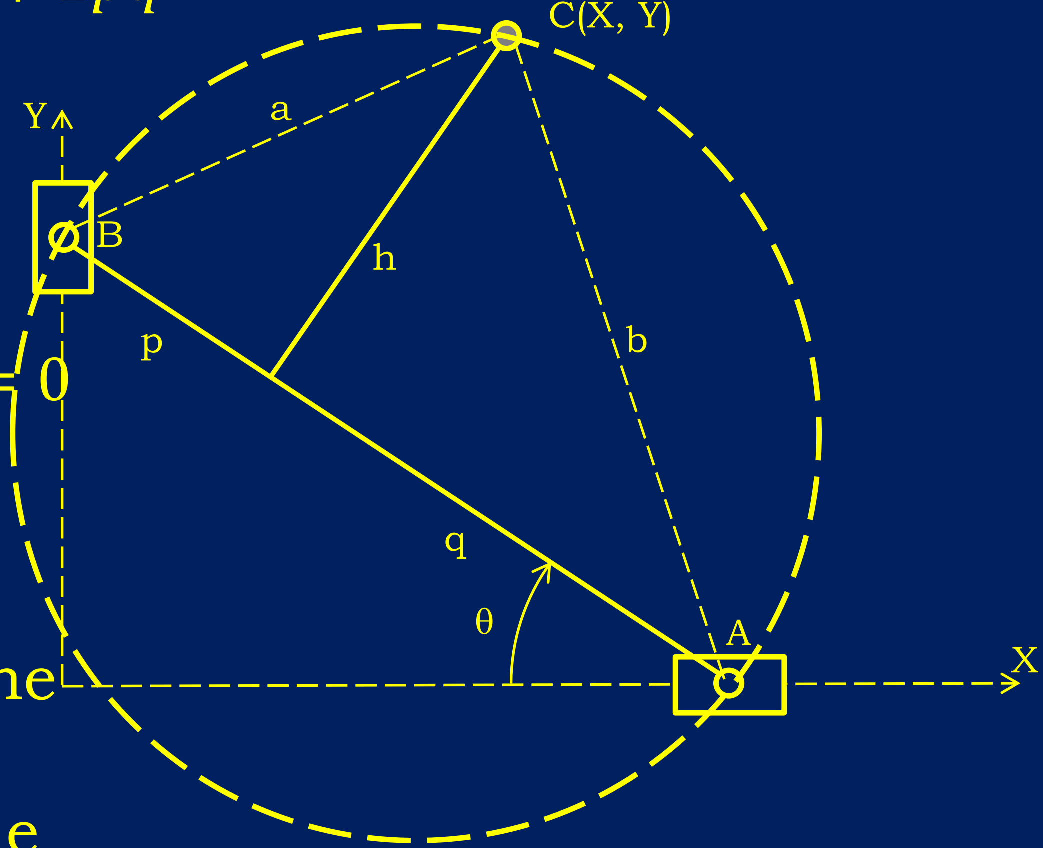
Double Slider

If C is selected on a circle whose diameter is |AB| then

$$a^2 + b^2 = (p+q)^2 = p^2 + q^2 + 2pq$$
$$a^2 = h^2 + p^2$$
$$b^2 = h^2 + q^2$$

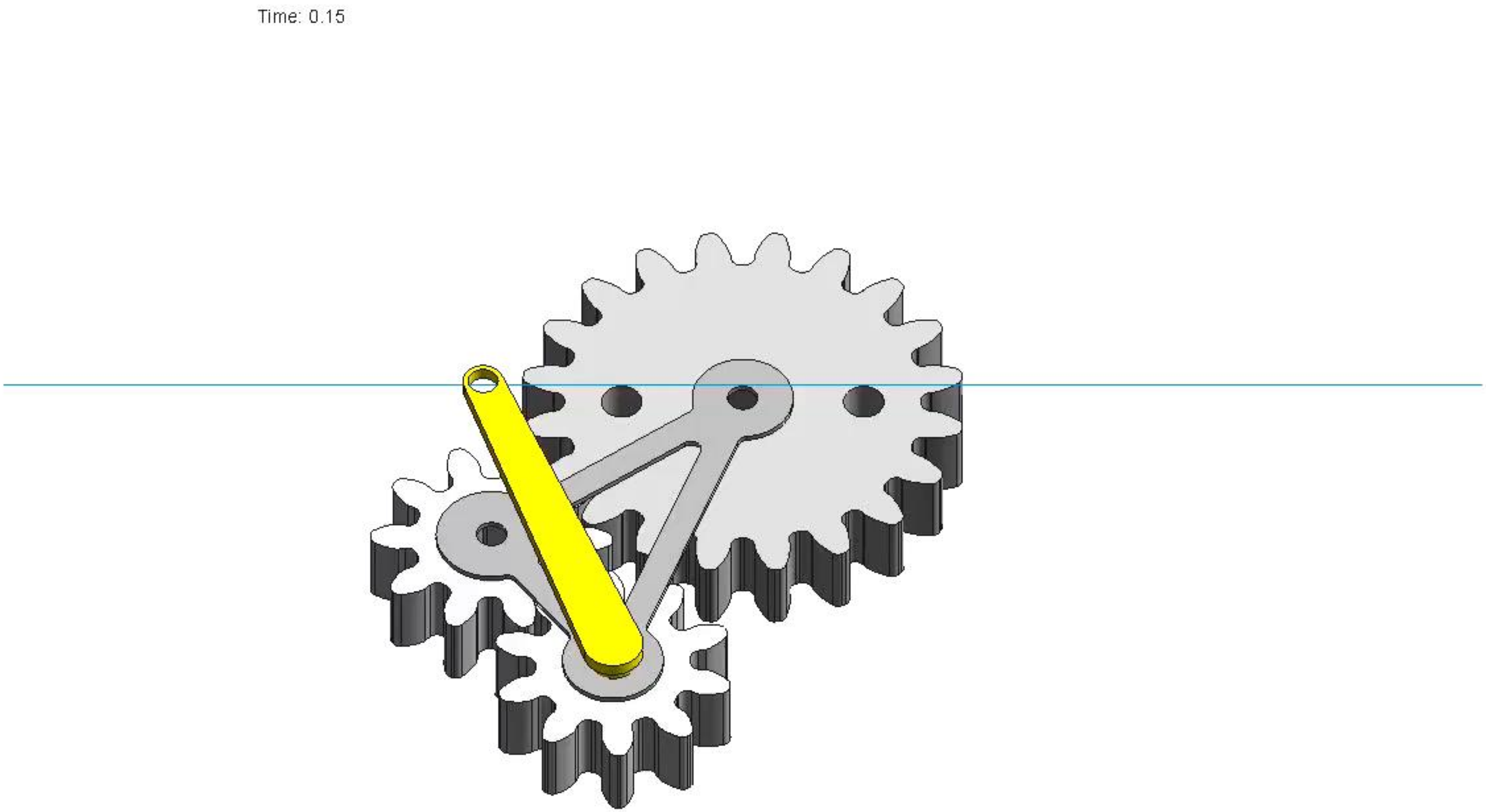
Substitution yields

$$h^2 = pq$$
$$b^2X^2 + a^2Y^2 - 2h(p+q)XY = 0$$
$$(aY - bX)^2 = 0$$
$$Y = \frac{b}{a}X$$



This is an exact straight line motion mechanism whose coupler point curve is a line with a slope b/a passing through origin!

Cardan Motion

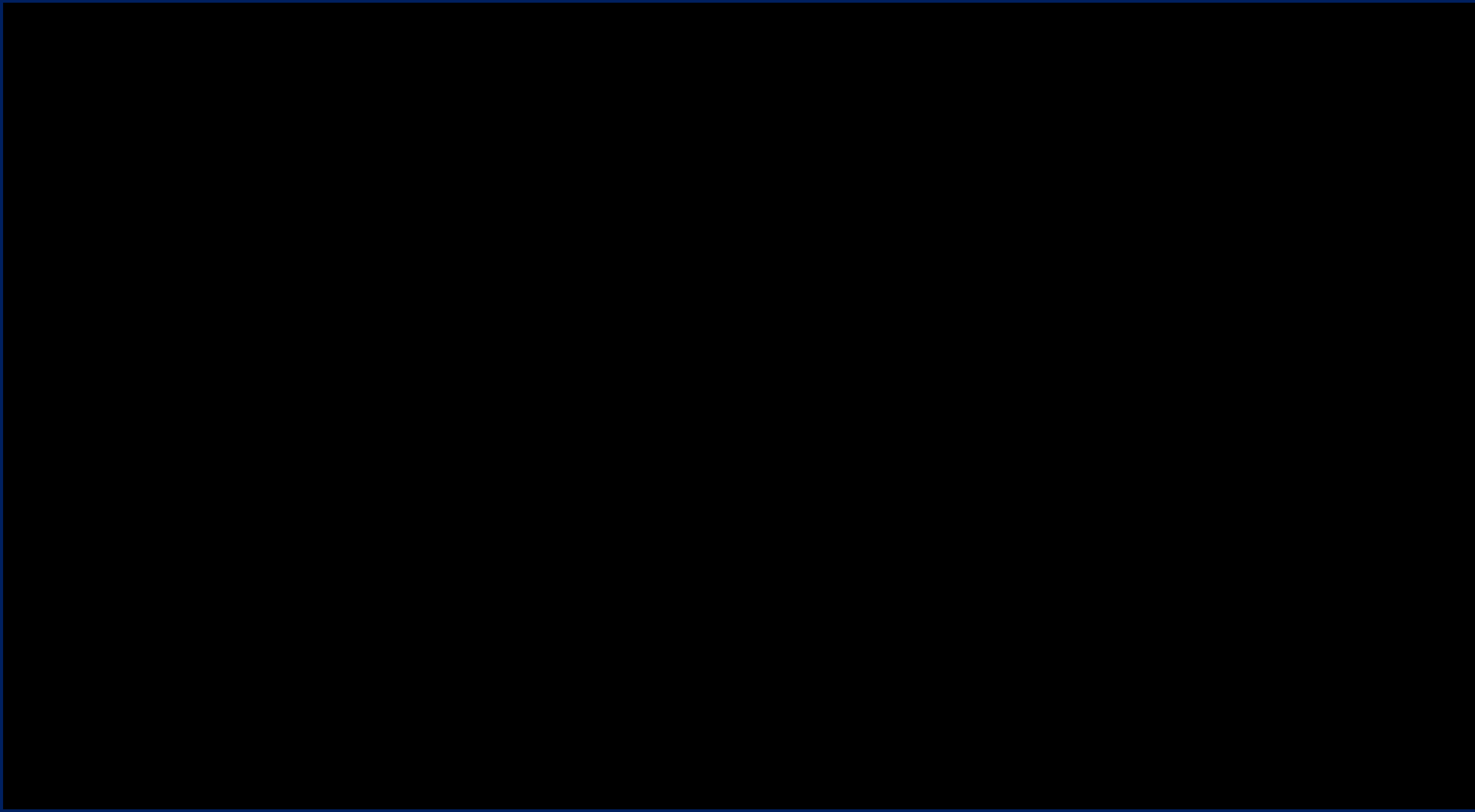


Cardan Motion

The rod is connected to the external gear on its pitch circle with a revolute joint. Since the radius of the pitch circle of the internal gear is double of that of the external gear, the revolute joint on the pitch circle of the external gear will draw a straight line along the diameter of the pitch circle of the internal gear.

http://140.116.71.92/cmd/model/page/model/ntu/_L09.htm

Sample Uses of Cardanic Motion



Centrodes of Some Common Motions

Case 4: Cycloidal Motion (r = const, θ = kϕ, k ∈ ℜ):

$$-\frac{T_2}{T_1} = \frac{\omega_{11} - \omega_{13}}{\omega_{12} - \omega_{13}}$$

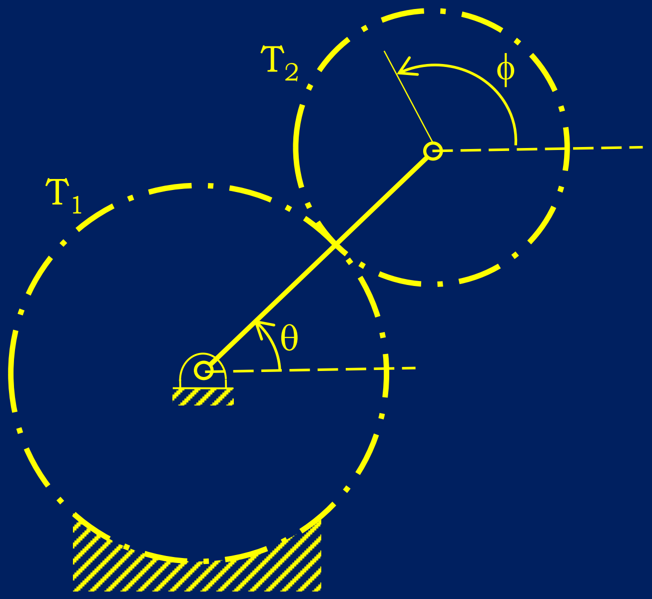
$$\frac{T_2}{T_1} = \frac{\theta}{\phi - \theta}$$

$$\frac{T_2}{T_1}(\phi - \theta) = \theta$$

$$\frac{T_2}{T_1}\phi - \frac{T_2}{T_1}\theta = \theta$$

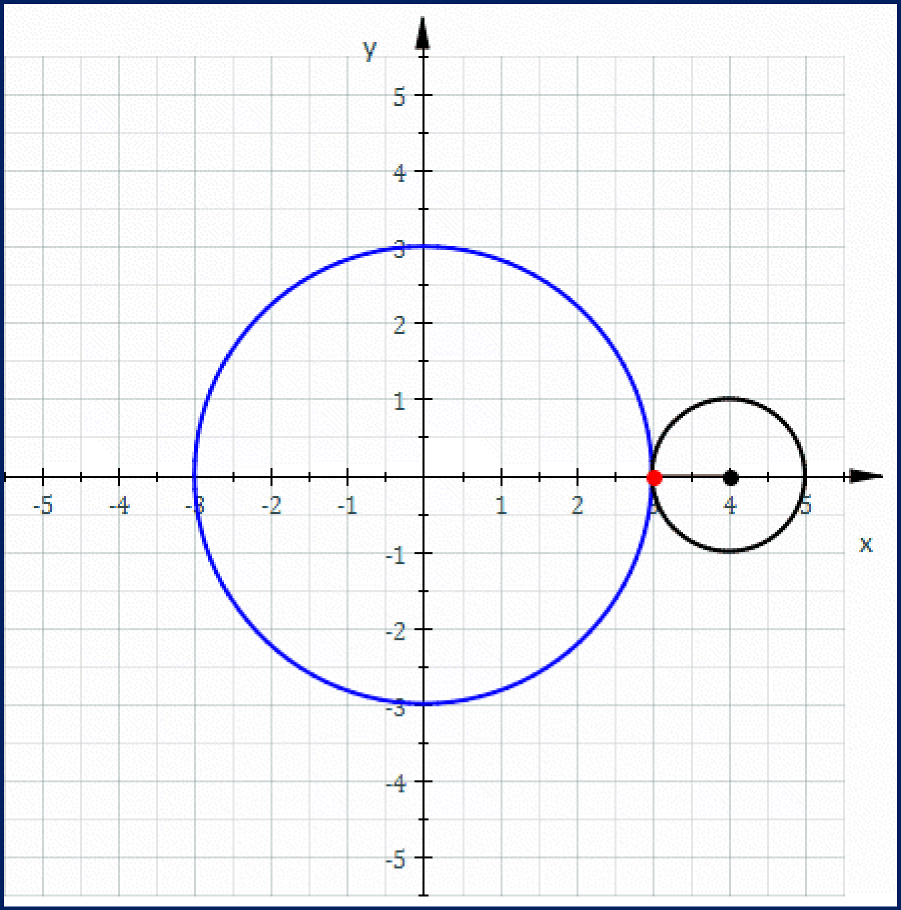
$$\frac{T_2}{T_1}\phi = \left(1 + \frac{T_2}{T_1}\right)\theta$$

$$\theta = \frac{{}^{T_2}/_{T_1}}{1 + {}^{T_2}/_{T_1}}\phi = k\phi, k = \frac{{}^{T_2}/_{T_1}}{1 + {}^{T_2}/_{T_1}}$$

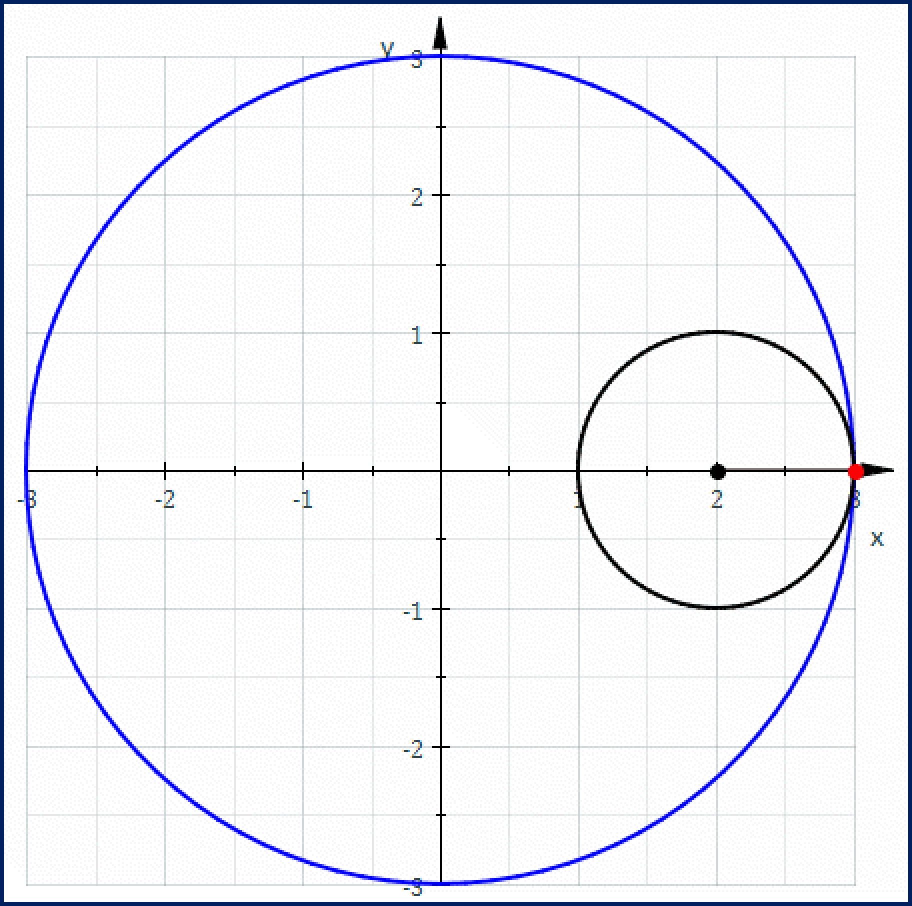


Centrodes of Some Common Motions

Case 4: Cycloidal Motion (r = const, $\theta = k\phi$, $k \in \Re$):



Epicycloid



Hypocycloid

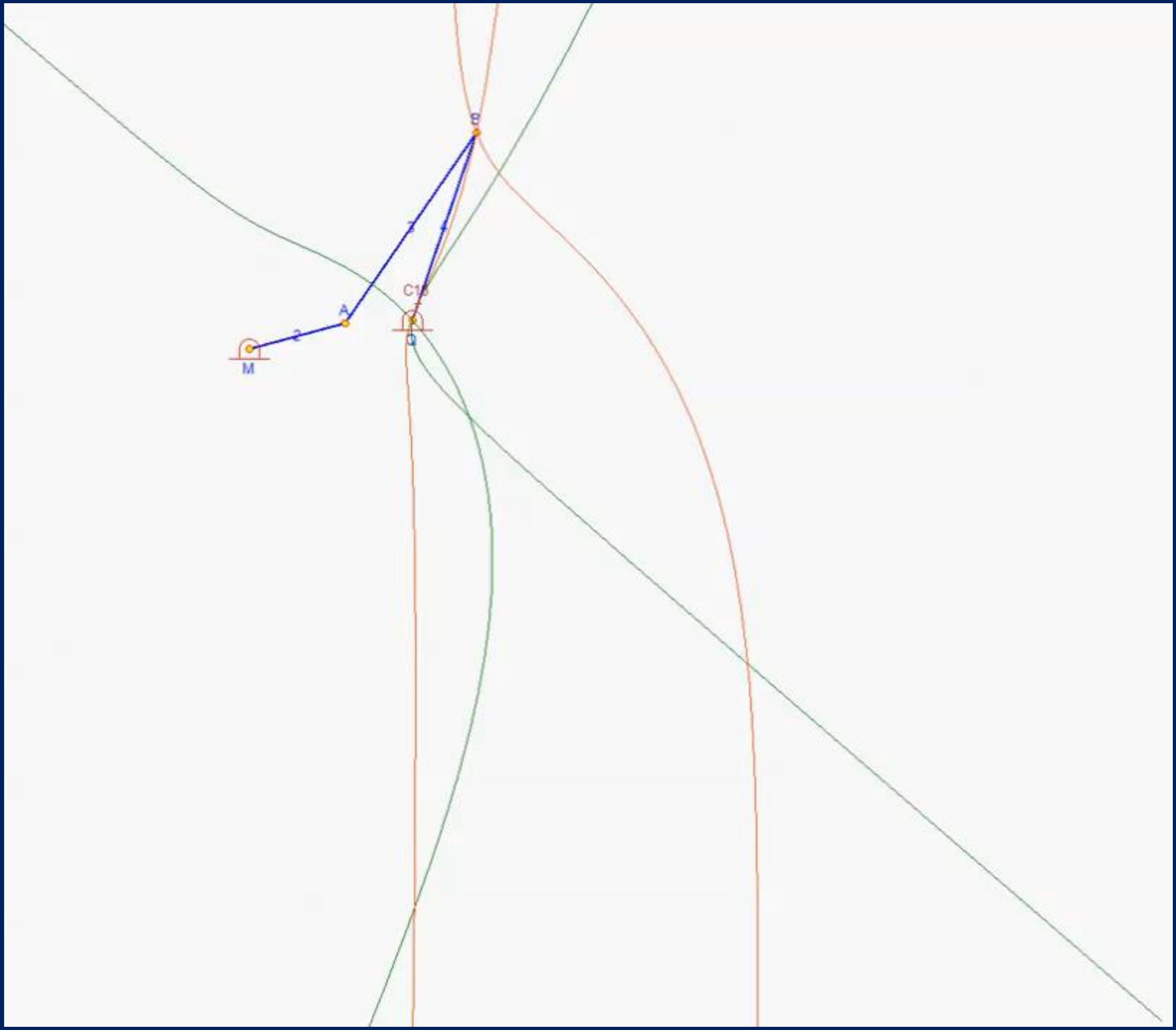
Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism:

Centrodes, in general, are not simple curves

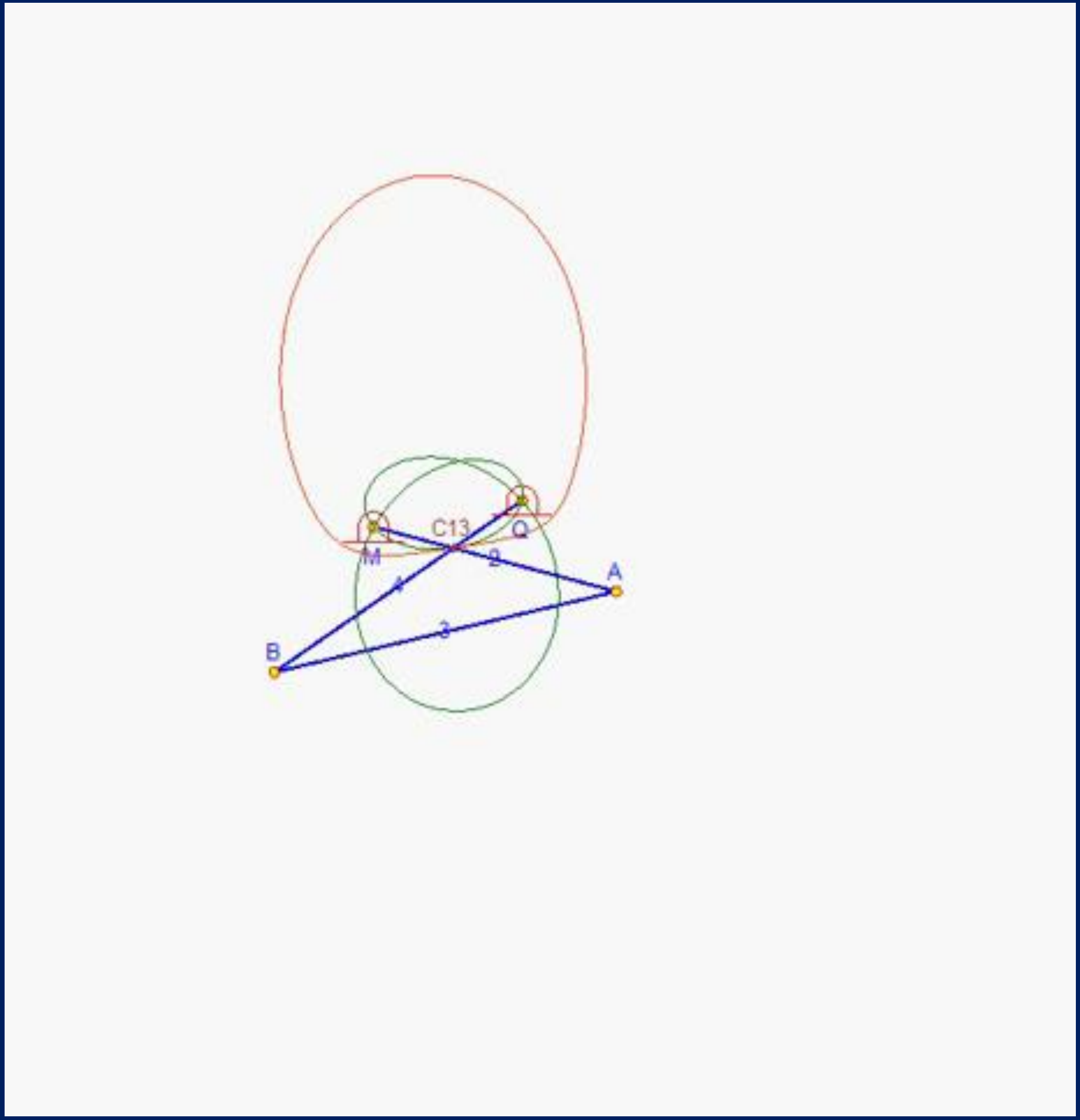
- For a crank-rocker the centrodes tend to infinity
- For drag-link (double crank) centrodes are closed curves

Centrodes of Crank-Rocker



https://www.youtube.com/watch?v=5fEIhVH1doU

Centrodes of Double-Crank



https://www.youtube.com/watch?v=GEFZE33Vabc

Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism (cont'ed)**:**

The loop closure equation in complex form:

$$a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} = a_1 + a_4e^{i\theta_{14}}$$

Eliminate θ_{14} using loop closure equation and its complex conjugate:

$$a_2e^{i\theta_{12}} + a_3\,e^{i\theta_{13}} - a_1 = a_4e^{i\theta_{14}}$$

$$a_2e^{-i\theta_{12}} + a_3e^{-i\theta_{13}} - a_1 = a_4e^{-i\theta_{14}}$$

$$\left(a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} - a_1\right)\left(a_2e^{-i\theta_{12}} + a_3e^{-i\theta_{13}} - a_1\right) = a_4{}^2$$

Simplification yields

$$a_1{}^2 + a_2{}^2 + a_3{}^2 - a_4{}^2 + a_2a_3\left(e^{i(\theta_{12}-\theta_{13})} + e^{-i(\theta_{12}-\theta_{13})}\right)$$
$$- a_1a_2\left(e^{i\theta_{12}} + e^{-i\theta_{12}}\right) - a_1a_3\left(e^{i\theta_{13}} + e^{-i\theta_{13}}\right) = 0$$

Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism (cont'ed)**:**

$$a_1{}^2 + a_2{}^2 + a_3{}^2 - a_4{}^2 + a_2a_3\left(e^{i(\theta_{12}-\theta_{13})} + e^{-i(\theta_{12}-\theta_{13})}\right)$$
$$- a_1a_2\left(e^{i\theta_{12}} + e^{-i\theta_{12}}\right) - a_1a_3\left(e^{i\theta_{13}} + e^{-i\theta_{13}}\right) = 0$$

Recall Euler's identity

$$cos\theta + isin\theta = e^{i\theta}, cos\theta - isin\theta = e^{-i\theta}$$

Sum of the two yields

$$2cos\theta = e^{i\theta} + e^{-i\theta}$$
$$a_1{}^2 + a_2{}^2 + a_3{}^2 - a_4{}^2 + 2a_2a_3cos(\theta_{12} - \theta_{13})$$
$$-2a_1a_2cos\theta_{12} - 2a_1a_3cos\theta_{13} = 0$$

Utilizing previous notation, $\theta = \theta_{12}$ and $\phi = \theta_{13}$

$$a_1{}^2 + a_2{}^2 + a_3{}^2 - a_4{}^2 + 2a_2a_3cos(\theta - \phi)$$
$$-2a_1a_2cos\theta - 2a_1a_3cos\phi = 0$$
$$f(\theta,\phi) = a_1{}^2 + a_2{}^2 + a_3{}^2 - a_4{}^2 + 2a_2a_3cos(\theta - \phi) - 2a_1a_2cos\theta - 2a_1a_3cos\phi = 0$$

Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism (cont'ed)**:**

$f(\theta,\phi) = {a_1}^2 + {a_2}^2 + {a_3}^2 - {a_4}^2 + 2a_2a_3cos(\theta - \phi) - 2a_1a_2cos\theta - 2a_1a_3cos\phi = 0$

$c = a_2e^{i\theta}$

$z_p = ic'e^{-i\phi}$

$$c' = c\frac{\partial f/\partial\phi}{\partial f/\partial\theta}$$

$$z_p = ic\frac{\partial f/\partial\phi}{\partial f/\partial\theta}e^{-i\phi}$$

$$Z_p = c + ic' = c\left(1 + \frac{\partial f/\partial\phi}{\partial f/\partial\theta}\right)$$

One could use another set of variables as:

$\{r, \theta, \phi\}, r = |c|, \theta = arg(c), c = e^{i\theta}$

Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism (cont'ed):

Example: Anti-parallel equal crank four-bar (a > b)

$|A_0A| = |B_0B| = a$

$|A_0B_0| = |AB| = b$

$I_{13} = P$

For any position of the mechanism

$|A_0P| + |PA| = a$

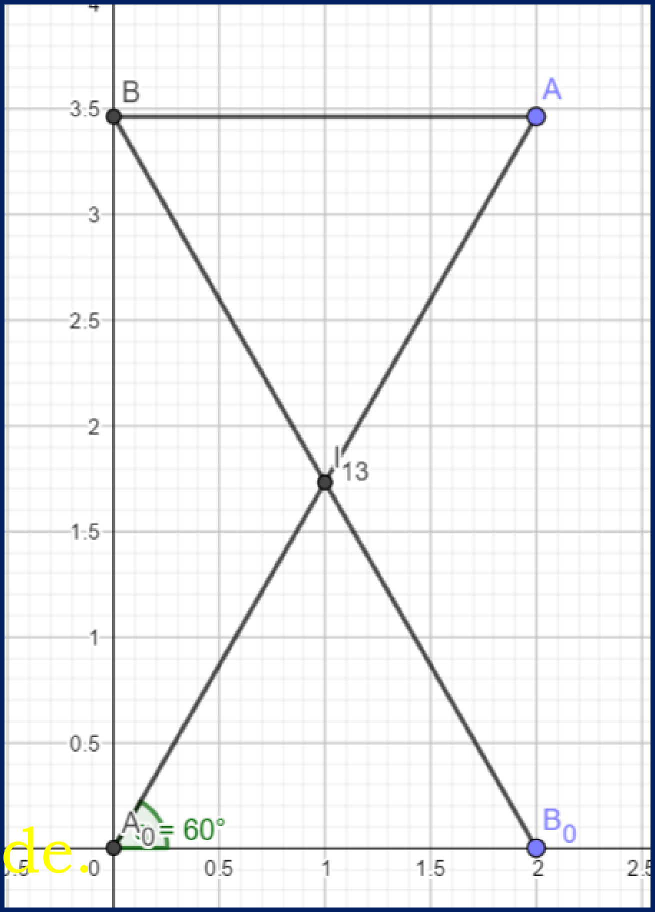
$|B_0P| + |PB| = a$

$\Delta PBA = \Delta PB_0A_0$

$|A_0P| + |B_0P| = const.$

Fixed centrode is an ellipse.

Invert the motion, fixed centrode of inverted motion is the moving centrode.

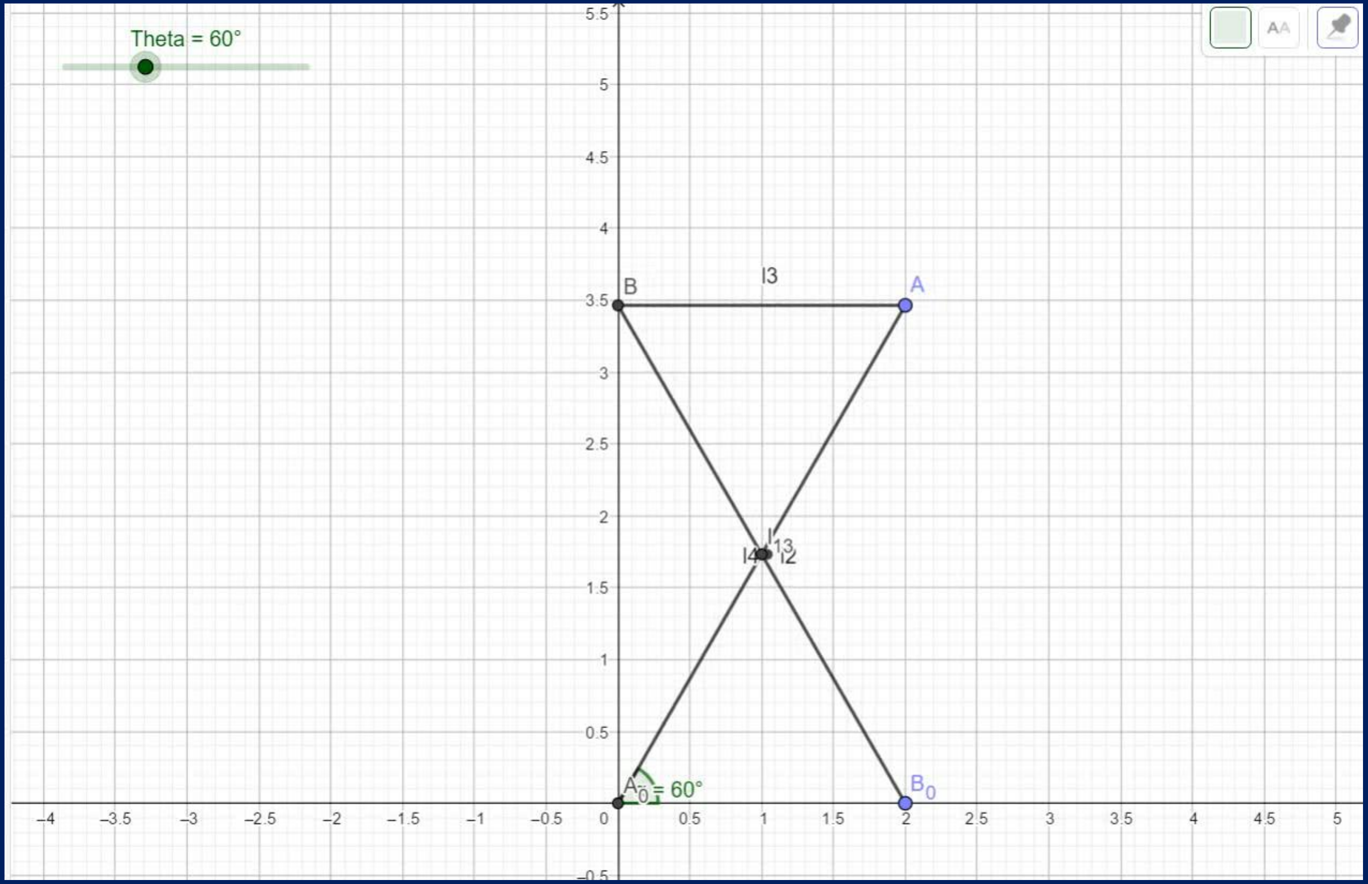


Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism (cont'ed)**:**

Example: Anti-parallel equal crank four-bar (a > b)

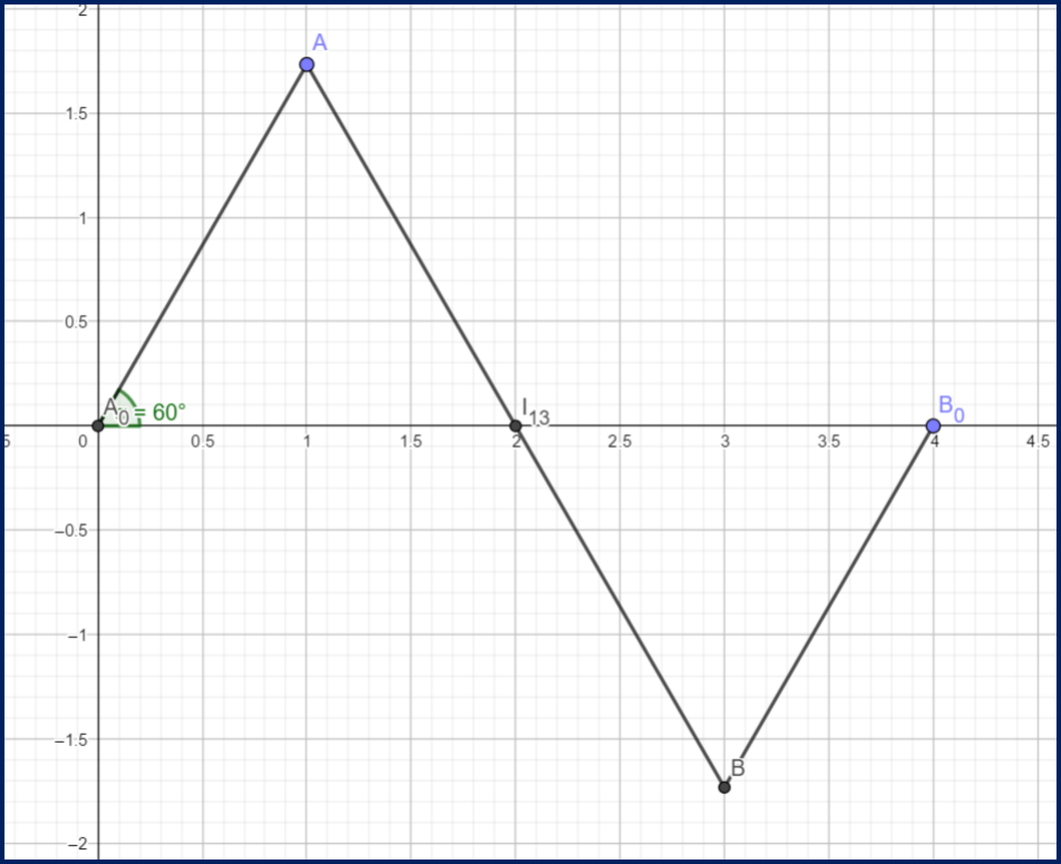
Fixed centrode is an ellipse.



Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism (cont'ed)**:**

Example: Anti-parallel equal crank four-bar (a < b)



Centrodes of Some Common Motions

Case 5: Coupler Motion of a Four Bar Mechanism (cont'ed)**:**

Example: Anti-parallel equal crank four-bar (a < b)

Canonical Representation of Plane Motion

Theorem 4: For the inverted motion (i.e. fixed plane is moving and moving plane is fixed) the moving and fixed centrodes change their roles. The angular velocity of the moving plane is negative angular velocity of the original motion.

Recall [See]

$Z_P = c + z_P e^{i\phi}$

$z_P e^{i\phi} = -c + Z_P$

$z_P = -ce^{-i\phi} + Z_P e^{-i\phi}$

Let

$u = -ce^{-i\phi}$ and $\psi = -\phi$

$z_P = u + Z_P e^{i\Psi}$

Moving centrode of inverted motion: $Z_P = iue^{-i\Psi}$

Fixed centrode of inverted motion: $z_P = u + iu'$

Determination of Centrodes

Graphical Method

Fixed Centrode:

1. Select two appropriate points on the moving plane
2. Move the mechanism in small increments using graphical position analysis
3. Mark location of I_{1i} on the fixed link
4. Go to 2 till you trace the necessary portion of the centrode
5. Connect the points by a smooth curve

Moving Centrode:

Same as fixed centrode but replace step 3 as:

3. Mark location of I_{1i} on the i^{th} link

This can be achieved on any parametric CAD software easily and accurately!

Determination of Centrodes

Graphical Method

Assignment: Think of a practical method of plotting moving centrode on Geogebra.

Determination of Centrodes
Analytic Method

Recall canonical reference frame:

$X = a + x\cos\phi - y\sin\phi$

$Y = b + x\sin\phi + y\cos\phi$

$Z = c + ze^{i\phi}$ where $c = a + ib, z = x + iy$

Location of pole on moving plane:

$x_P = a'\sin\phi - b'\cos\phi$

$y_P = a'\cos\phi + b'\sin\phi$

$z_P = ic'e^{i\phi}$ where $' = \dfrac{d}{d\phi}$

Location of pole on fixed plane:

$X_P = a - b'$

$Y_P = b + a'$

$Z_P = c + ic'$

Determination of Centrodes

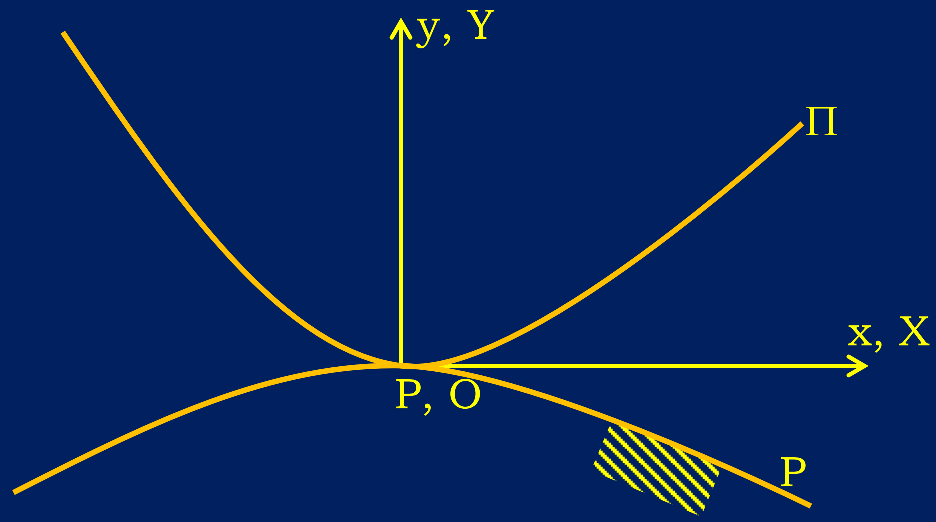
Analytic Method-Canonical Reference Frame

Select your reference frames such that:

1. Two reference frames are coincident at the instant considered, i.e. $c = 0, \phi = 0$ but $\dot{\phi} \neq 0$
2. Take P as the origin of both frames
3. Take X and x axis coincident with the path tangent

$$\frac{dX_p}{d\phi} = a' - b'' \; and \; \frac{dY_p}{d\phi} = b' + a''$$

$$\frac{dY_p}{dX_p} = \frac{dY_p/d\phi}{dX_p/d\phi} = \frac{b' + a''}{a' - b''} = 0$$



$$a'' = 0$$
$$b'' \neq 0$$

Determination of Centrodes

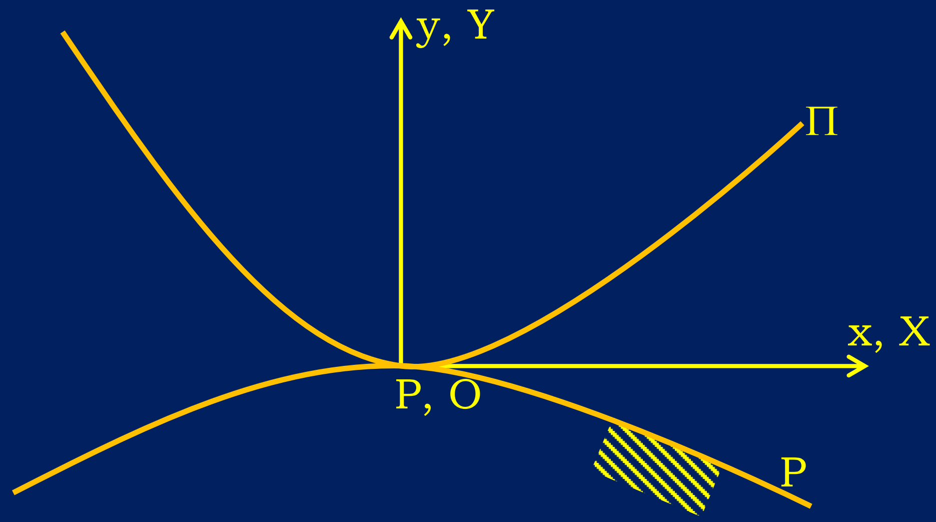
Analytic Method-Canonical Reference Frame

$$a'' = 0$$
$$b'' \neq 0$$

$$\frac{b' + a''}{a' - b''} = 0 \Rightarrow b' + a'' = 0 \vee a' + b'' \neq 0$$

$$x_P = a' sin0 - b' cos0 = 0 \Rightarrow b' = 0 \therefore a'' = 0$$

$$y_P = a' cos0 + b' sin0 = 0 \Rightarrow a' = 0 \therefore b'' \neq 0$$



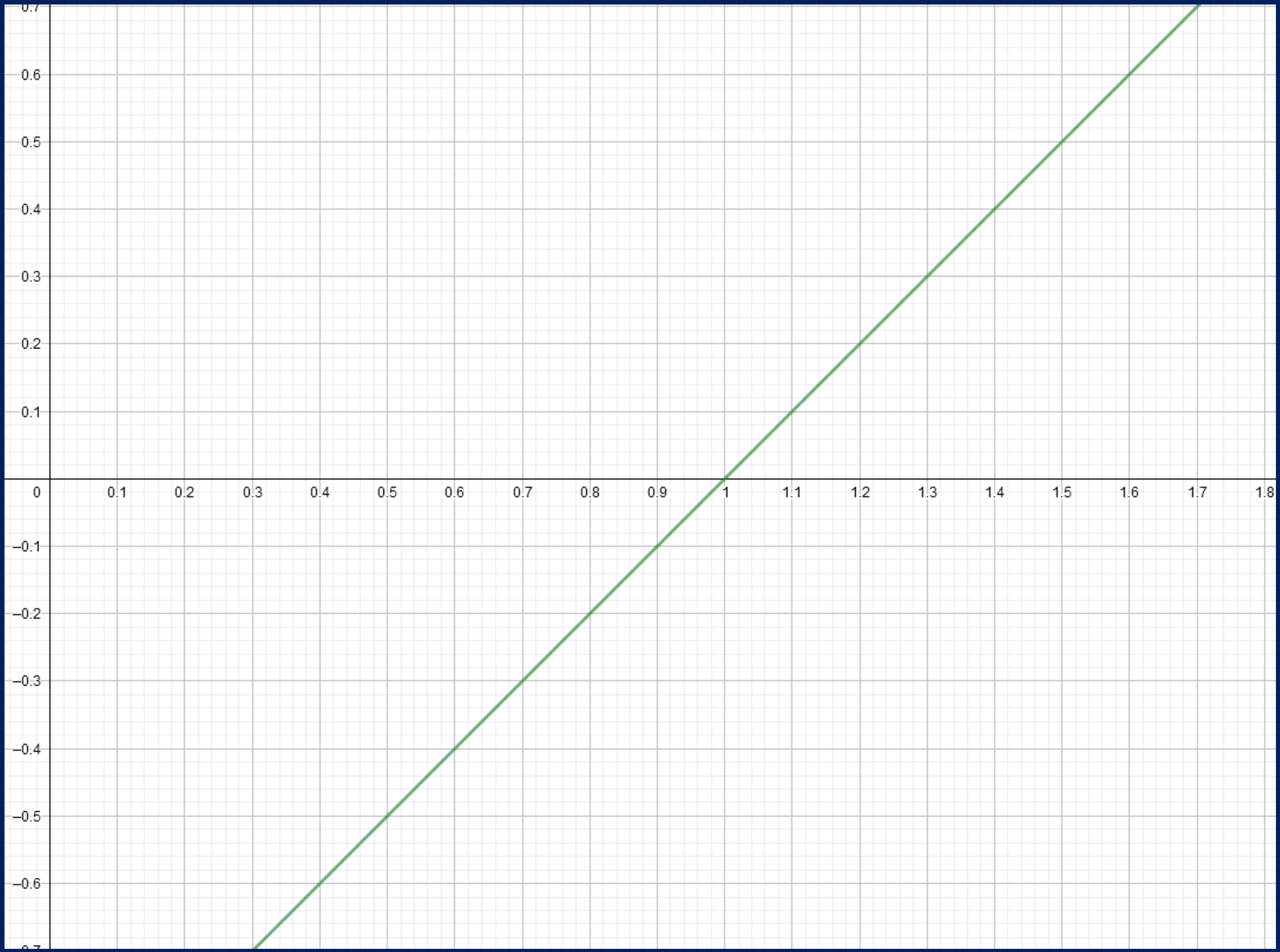
Determination of Centrodes
Analytic Method-Overview

- *Two infinitesimally separated positions:* One position and first rate of change of this position (i.e. velocity) changes at this position.
- *Three infinitesimally separated positions:* One position, first rate of change of this position and how the tangent (i.e. curvature) changes at this position.
- *Four infinitesimally separated positions:* One position, first rate of change of this position, how the tangent (i.e. curvature) and rate of curvature changes at this position.

Recall Burmester's theory for finitely separated positions!

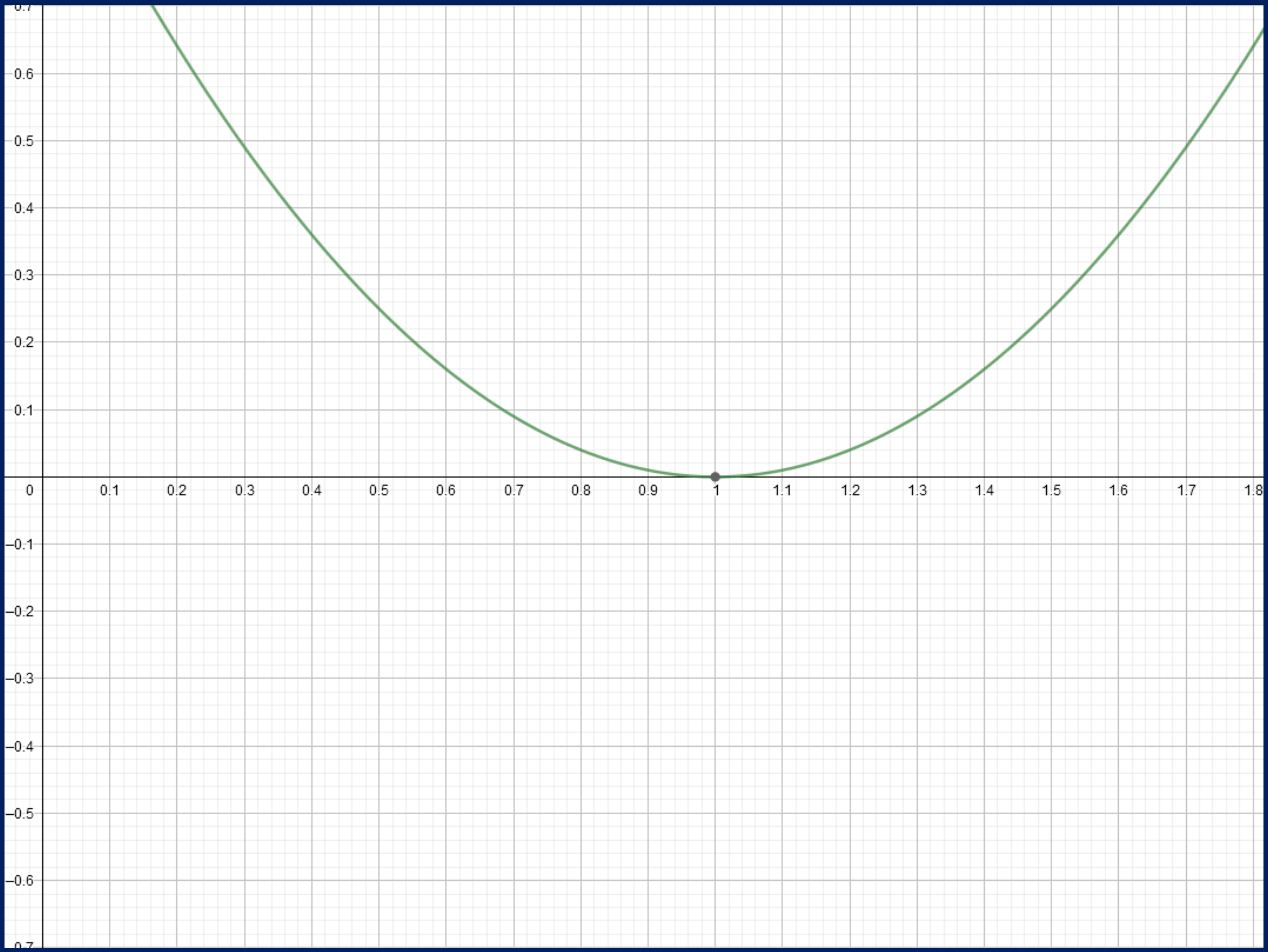
Contact of Two Curves

One Point Contact



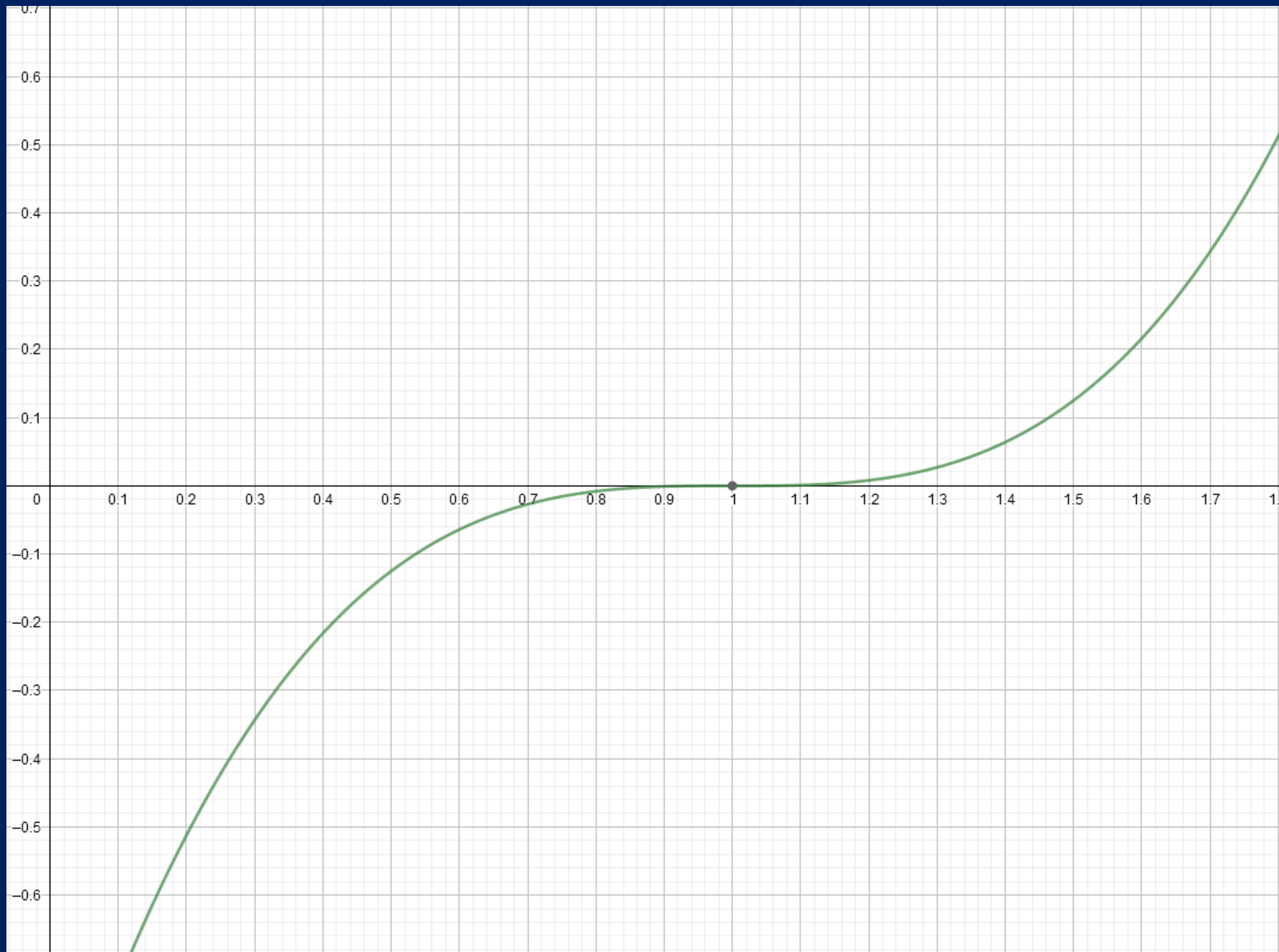
Contact of Two Curves

Two Point Contact (Same Tangent/Slope)



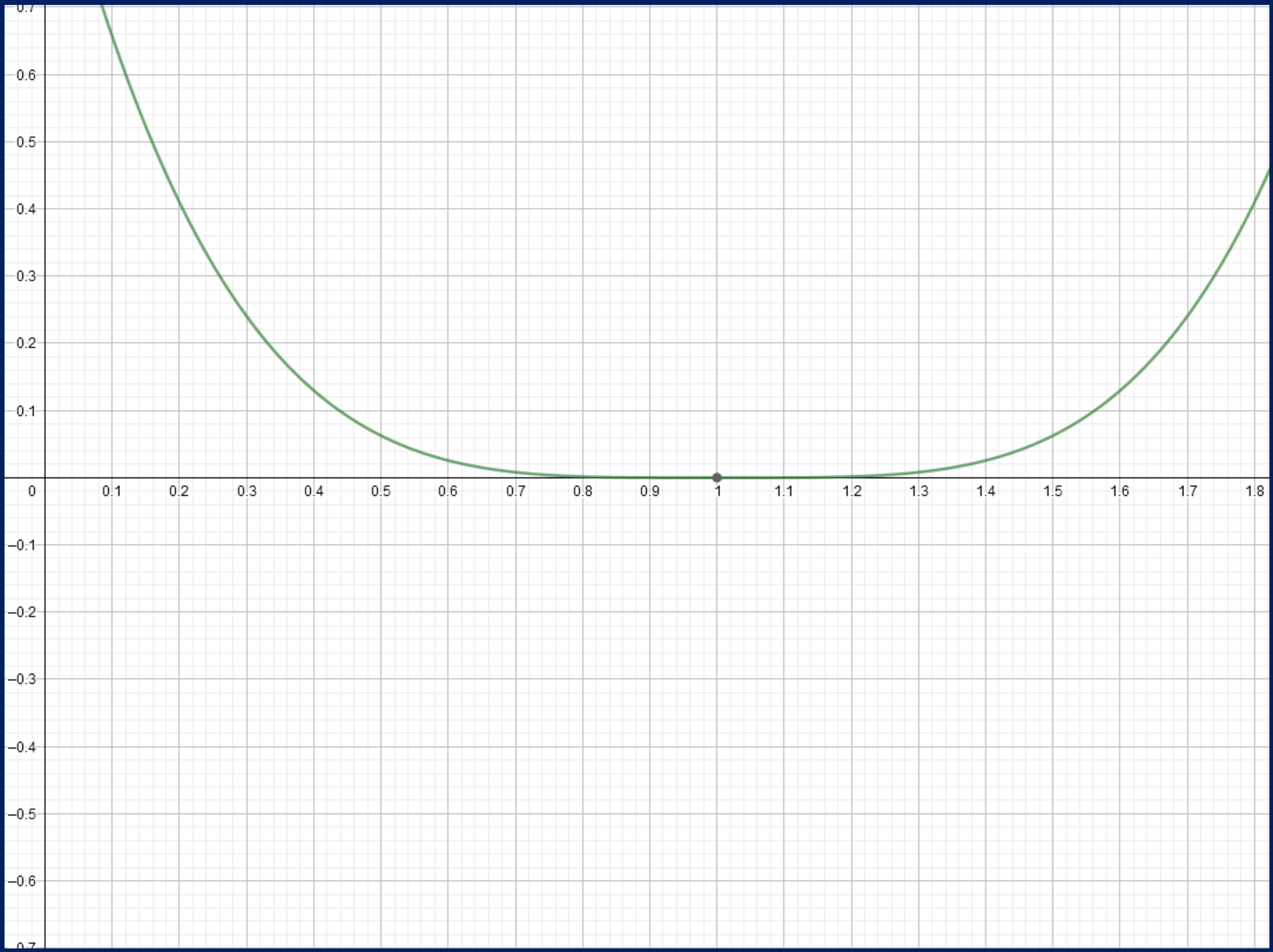
Contact of Two Curves

Three Point Contact (Same Radius of Curvature)



Contact of Two Curves

Four Point Contact (Same Rate of Change of Radius of Curvature)



Two Infinitesimally Separated Positions

The Euler-Savary equation is derived by:

- L'hospital in 1696 in basic form
- Euler in 1765
- Savary in 1841

Euler-Savary equation relates $A(r,\psi)$ to its center of curvature $C(r_C,\psi)$ by $d\theta/ds$ which is only a function of the motion of the moving plane.

$$\vec{v}_A = \vec{v}_P + \vec{v}_{A/P}$$

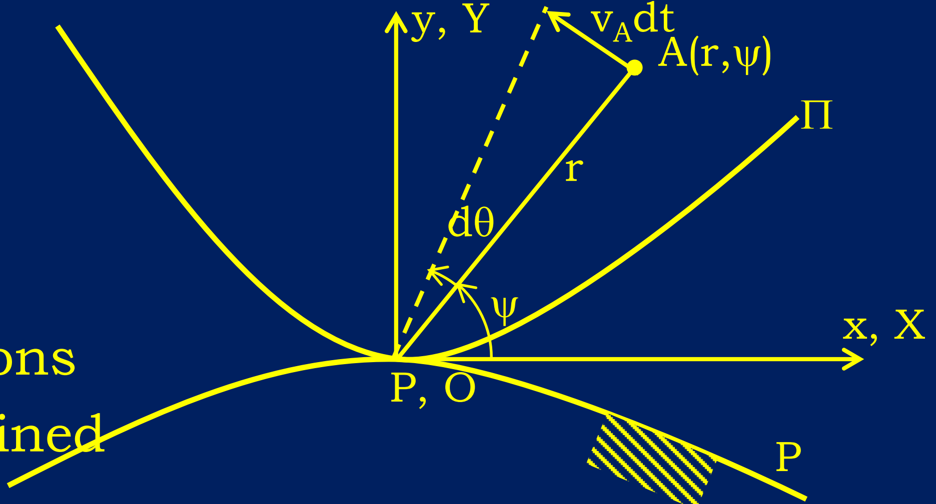
$$\vec{v}_P = \vec{0}, \vec{v}_A = \vec{v}_{A/P}$$

$$\vec{v}_A = \vec{v}_{A/P} = \vec{\omega} \times \vec{r}$$

$$tand\theta = \frac{v_A dt}{r} = \frac{\omega r dt}{r} = \omega dt$$

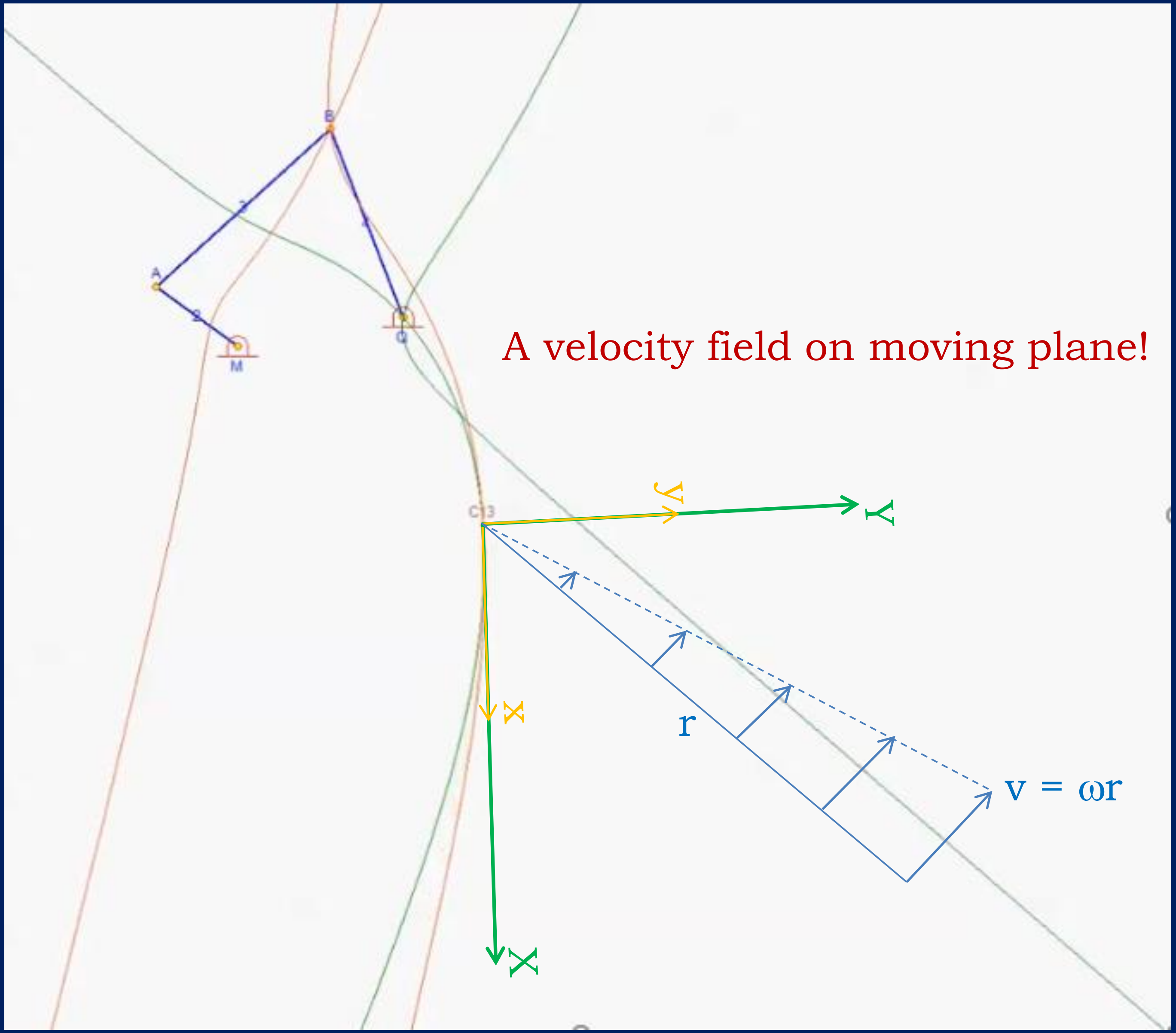
$$sind\theta = tand\theta = d\theta$$

$$\omega = \frac{d\theta}{dt}$$



Two infinitesimally separated positions of a moving plane is entirely determined by the pole.

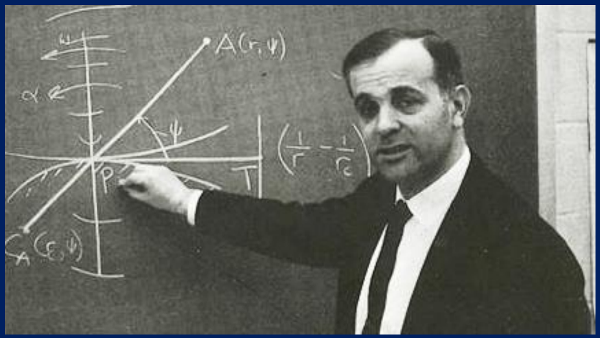
Motion of Coupler (Link 3) of Four-Bar

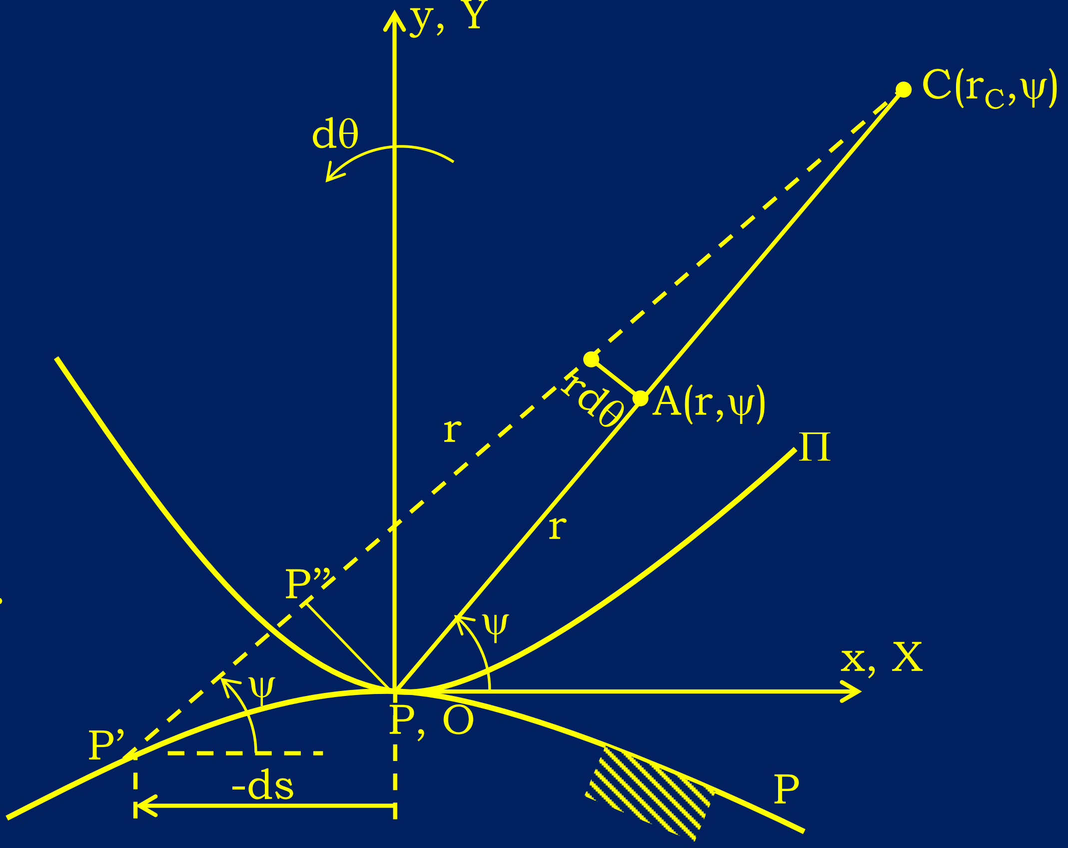


Derivation of Euler-Savary Equation

Geometric Approach

- C, center of curvature of point A for the position shown is on pole ray since $\vec{v}_A \perp$ pole ray.
- dθ is the infinitesimal rotation of the moving centrode Π on the fixed centrode P.
- ds is the arc length on moving and fixed centrodes.



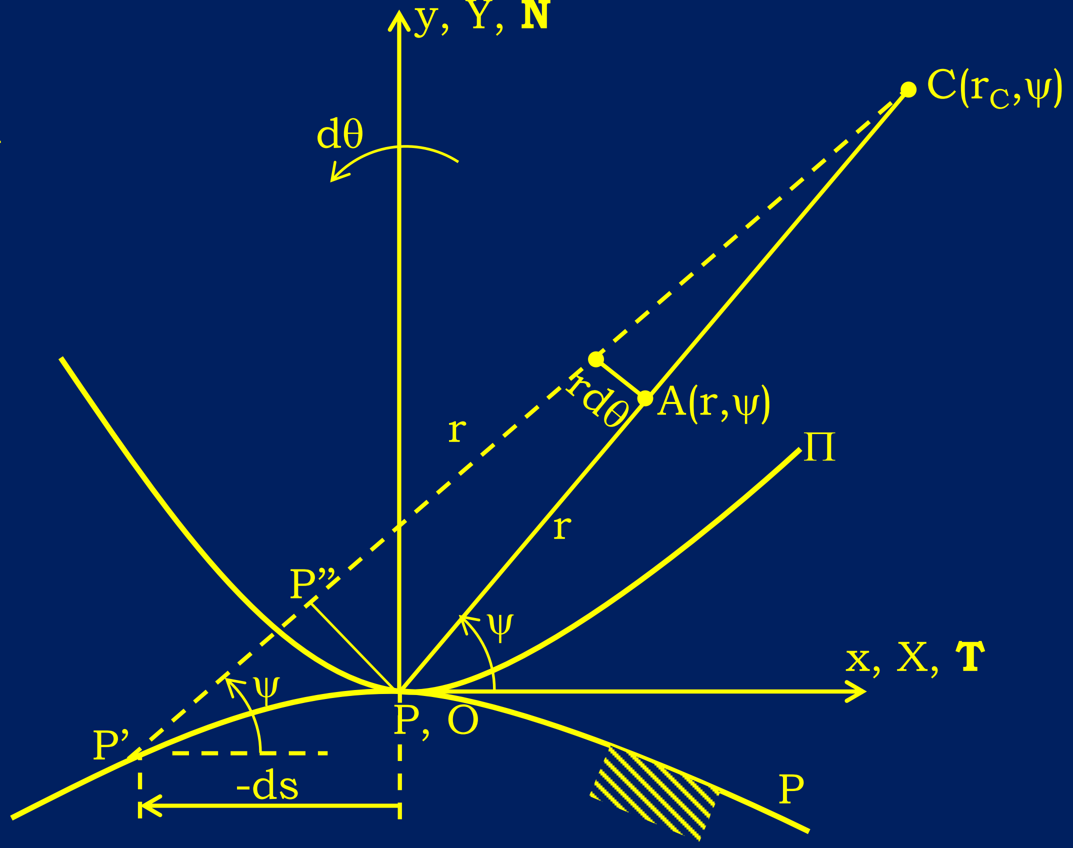


Derivation of Euler-Savary Equation

Geometric Approach

Euler-Savary equation is valid under:

- During infinitesimal motion about the position shown $d\theta/ds$ is finite and non-zero (i.e. $d\theta \neq 0 \wedge ds \neq 0$).
- Points A and P are not coincident.
- |AP| is finite.



Derivation of Euler-Savary Equation

Geometric Approach

$$\Delta CAA' \sim \Delta CPP'' \therefore \frac{|AA'|}{|PP''|} = \frac{|CA|}{|CP|}$$

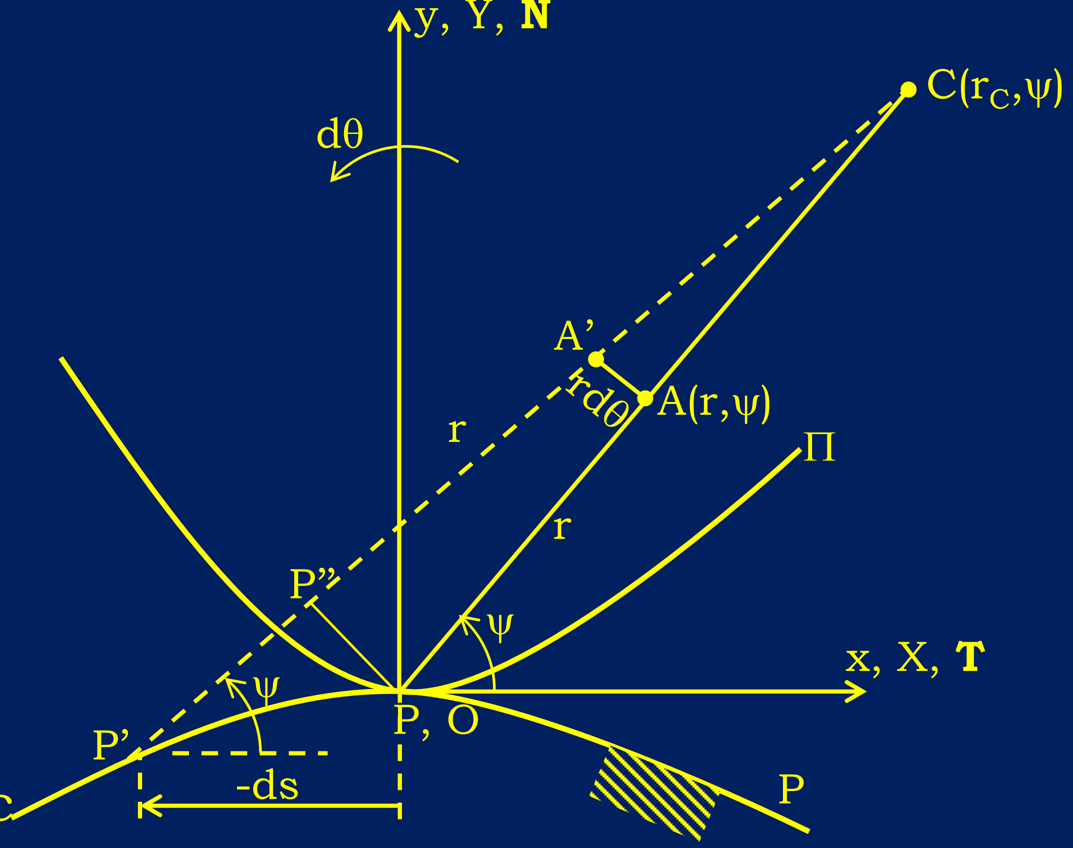
$$|AA'| = rd\theta$$
$$|PP''| = -dssin\Psi$$
$$|CA| = r_C - r$$
$$|CP| = r_C$$

Substitution yields

$$\frac{rd\theta}{-dssin\Psi} = \frac{r_C - r}{r_C}$$

$$\left(\frac{1}{r} - \frac{1}{r_c}\right) sin\psi = -\frac{d\theta}{ds}$$

Euler-Savary equation in basic form.



Derivation of Euler-Savary Equation

Geometric Approach

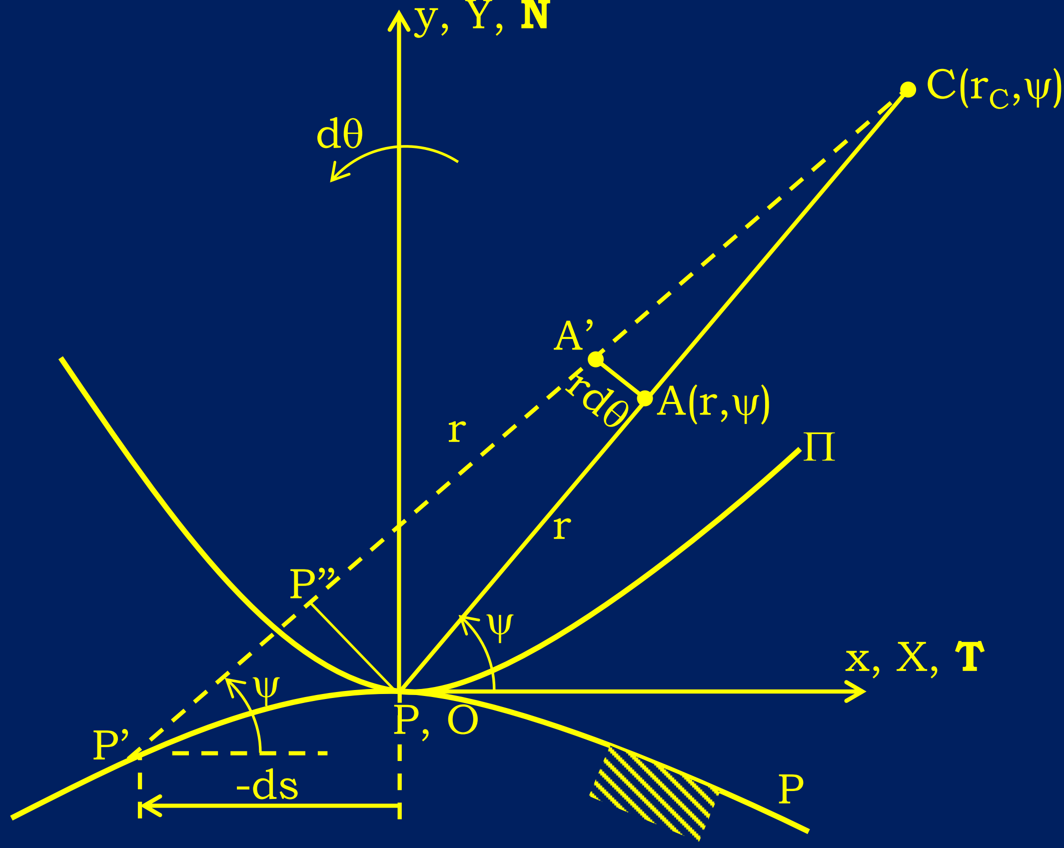
$$\left(\frac{1}{r}-\frac{1}{r_c}\right)sin\psi=-\frac{d\theta}{ds}$$

$$\frac{d\theta}{ds}=\frac{d\theta/dt}{ds/dt}=\frac{\omega}{v_P}$$

v_P is the pole velocity. Please note that at the instant considered pole is stationary but its location is changing on moving and fixed centrodes. Therefore v_P is the rate of change of location of the pole.

$$\left(\frac{1}{r}-\frac{1}{r_c}\right)sin\psi=-\frac{\omega}{v_P}$$

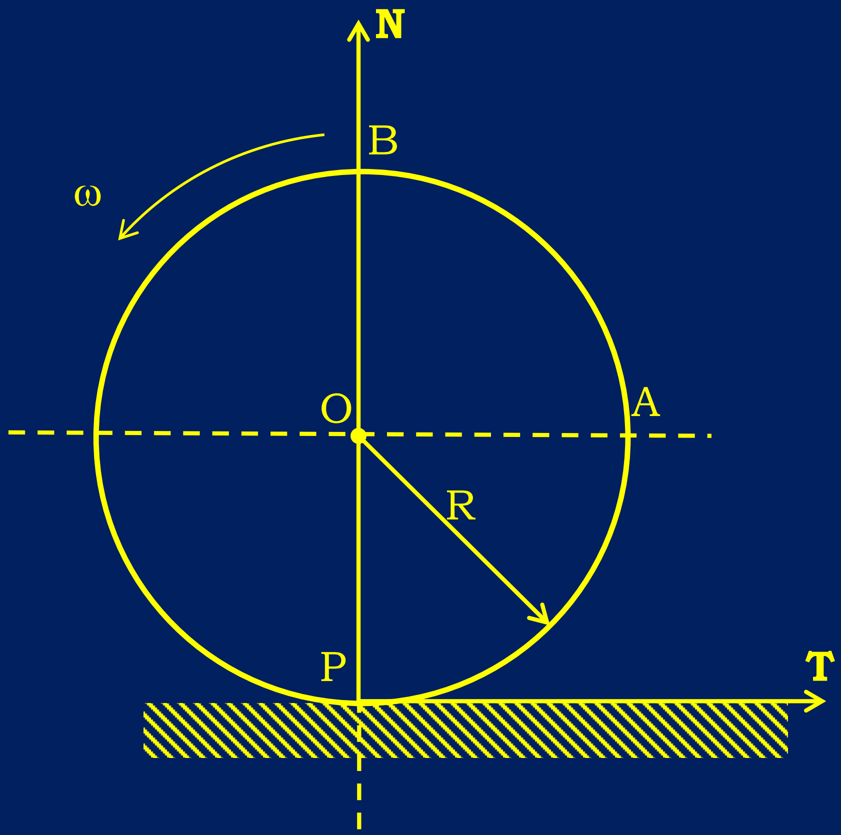
Euler-Savary equation in kinematic form.



Derivation of Euler-Savary Equation

Example

A cylinder of radius R rolls on a straight surface. Determine the center of curvature of points A, B and O.

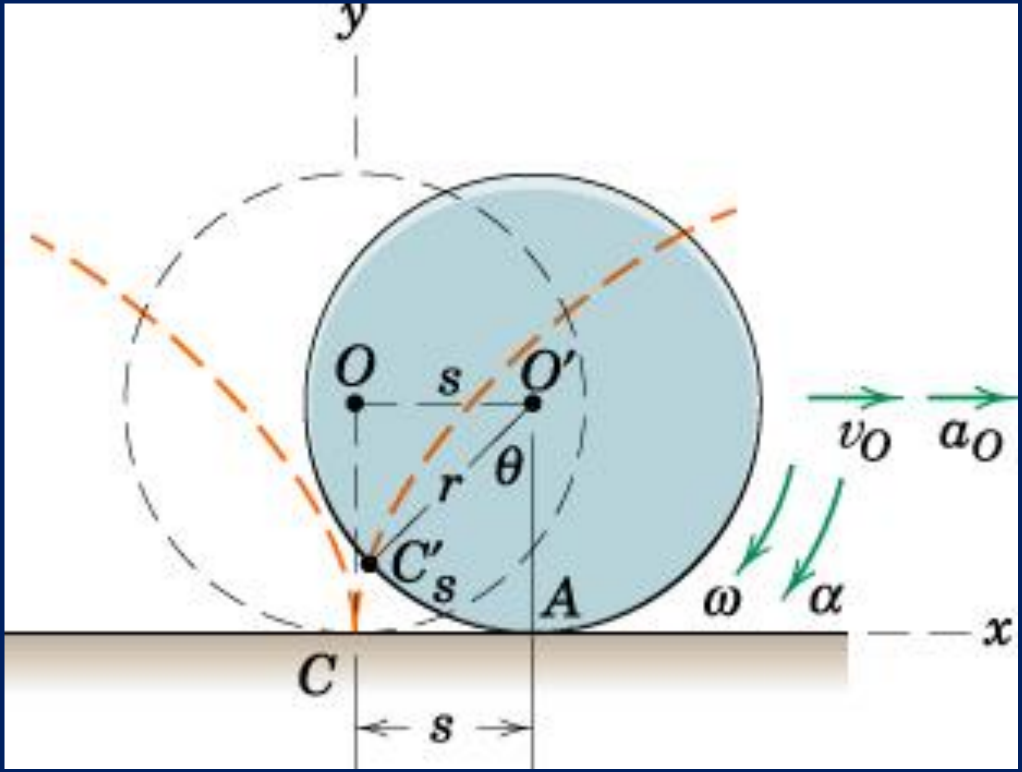


Derivation of Euler-Savary Equation

ME 208 Dynamics

Sample Problem 5/4 (Meriam 4th, 5th, 6th, 7th and 8th editions)

A wheel of radius r rolls on a flat surface without slipping. Determine the angular motion of the wheel in terms of the linear motion of its center O. Also determine the acceleration of a point on the rim of the wheel as the point comes into contact with the surface on which the wheel rolls.



Derivation of Euler-Savary Equation

Example

A cylinder of radius R rolls on a straight surface. Determine the center of curvature of points A, B and O.

$A(\sqrt{2}R, 45°), B(2R, 90°), O(R, 90°)$

$v_P = -\omega R$

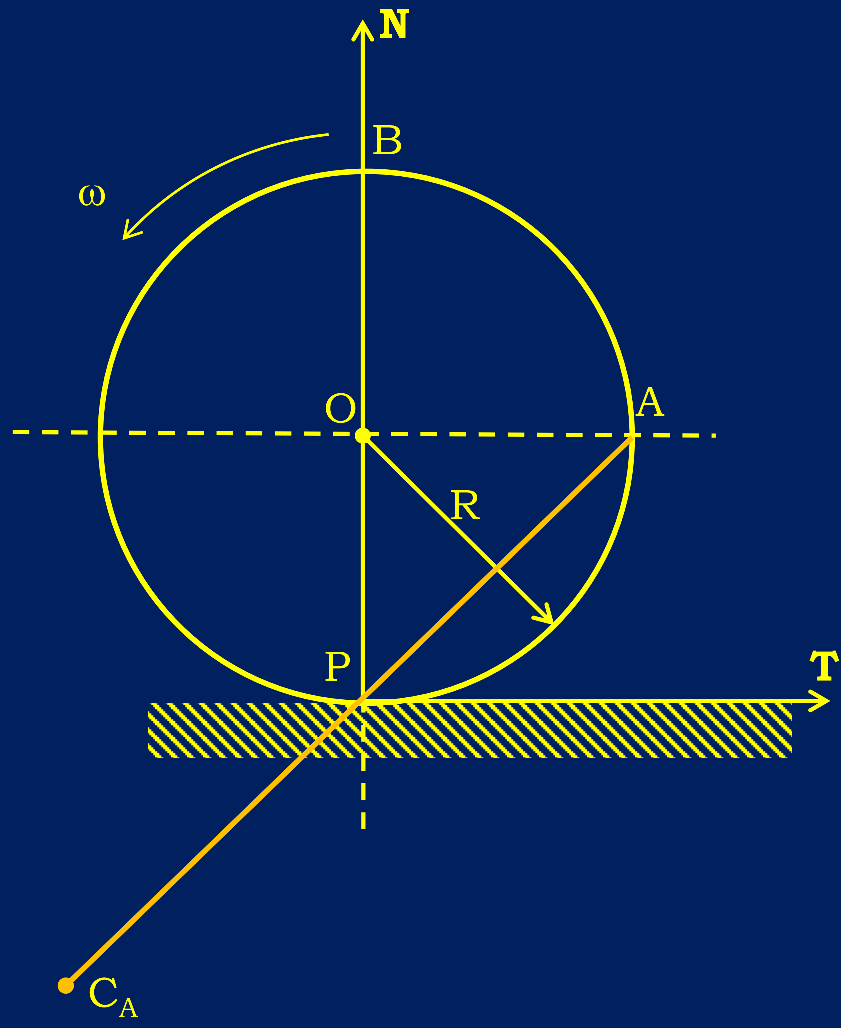
$$\left(\frac{1}{r} - \frac{1}{r_c}\right) sin\psi = -\frac{\omega}{v_P}$$

For A

$$\left(\frac{1}{\sqrt{2}R} - \frac{1}{r_c}\right) sin45° = -\frac{\omega}{-\omega R} = \frac{1}{R}$$

$r_c = -\sqrt{2}R$

$C_A(-\sqrt{2}R, 45°)$ or $C_A(\sqrt{2}R, 225°)$



Derivation of Euler-Savary Equation

Example (cont'ed)

$B(2R, 90°), O(R, 90°)$

For B

$$\left(\frac{1}{2R} - \frac{1}{r_c}\right) sin90° = \frac{1}{R}$$

$r_c = -2R$

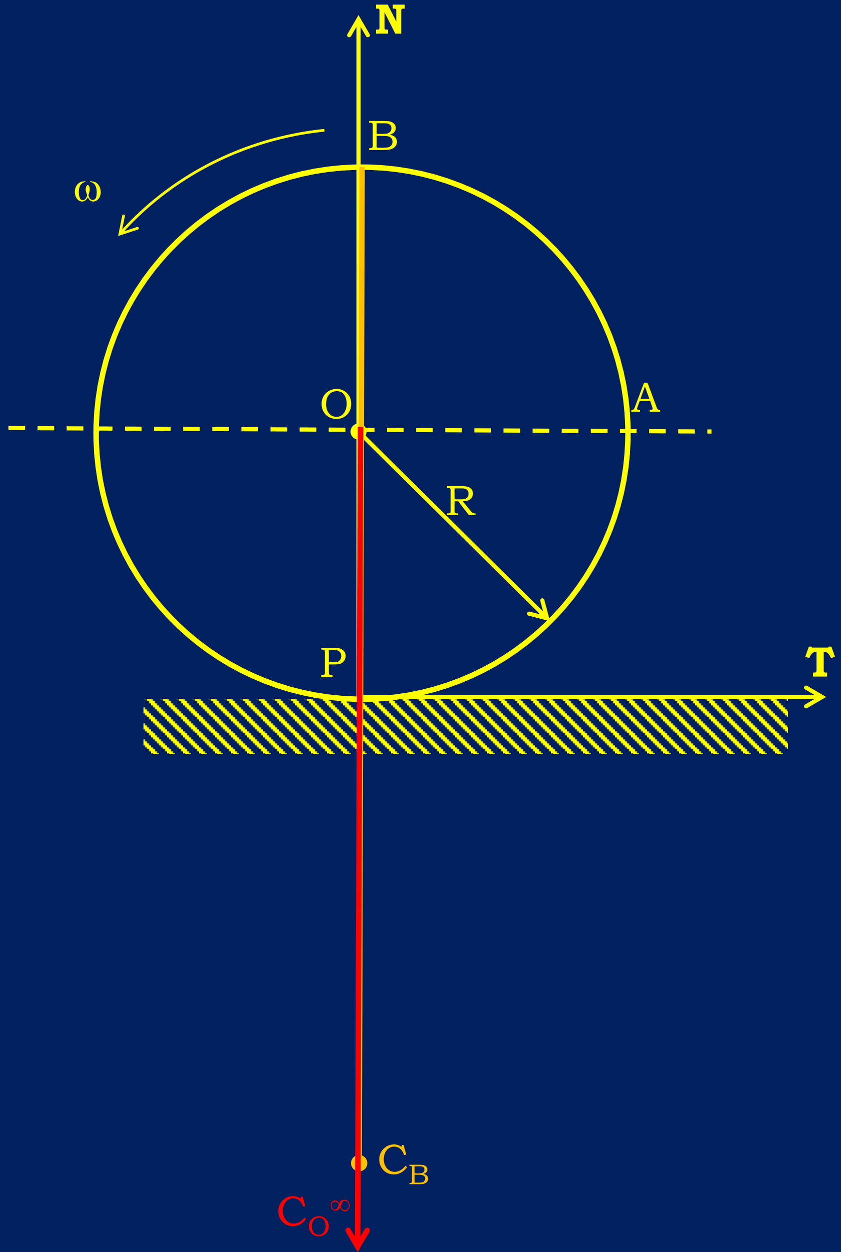
$C_B\left(-\sqrt{2}R, 45°\right) or C_B\left(\sqrt{2}R, 225°\right)$

For O

$$\left(\frac{1}{R} - \frac{1}{r_c}\right) sin90° = \frac{1}{R}$$

$$\frac{1}{r_c} = 0 \therefore r_c \to \infty$$

It is obvious that O travels on a straight path so its center of curvature is at infinity, perpendicular to its path.



Derivation of Euler-Savary Equation

Analytic Approach

Recall radius of curvature from calculus:

$$\frac{1}{\kappa} = \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

In fixed plane

$$\frac{dY}{dX} = \frac{dY/d\phi}{dX/d\phi} = \frac{Y'}{X'}, \frac{d^2Y}{dX^2} = \frac{Y''X' - X''Y'}{X'^2}$$

$$X' = \frac{dX}{d\phi} = a' - xsin\phi - ycos\phi$$

$$Y' = \frac{dY}{d\phi} = b' + xcos\phi - ysin\phi$$

$$X'' = \frac{d^2X}{d\phi^2} = a'' - xcos\phi + ysin\phi$$

$$Y'' = \frac{d^2Y}{d\phi^2} = b'' - xsin\phi - ycos\phi$$

In canonical reference frame $c = 0, a' = a'' = b' = 0, \phi = 0$ but $b'' \neq 0$

Derivation of Euler-Savary Equation

Analytic Approach

$X = x, Y = y$

$X' = -y, Y' = x$

$X'' = -x, Y'' = b'' - y$

$$\frac{1}{\kappa} = \rho = \frac{(X'^2 + Y'^2)^{3/2}/X'^2}{(Y''X' - X''Y')/X'^2} = \frac{(X'^2 + Y'^2)^{3/2}}{Y''X' - X''Y'} = \frac{(x^2 + y^2)^{3/2}}{(b'' - y)(-y) - (-x^2)}$$

$$\frac{1}{\kappa} = \rho = \frac{(x^2 + y^2)^{3/2}}{-by'' + y^2 + x^2} = \frac{(x^2 + y^2)^{3/2}}{-by'' + (x^2 + y^2)}$$

$\rho = r_C - r$

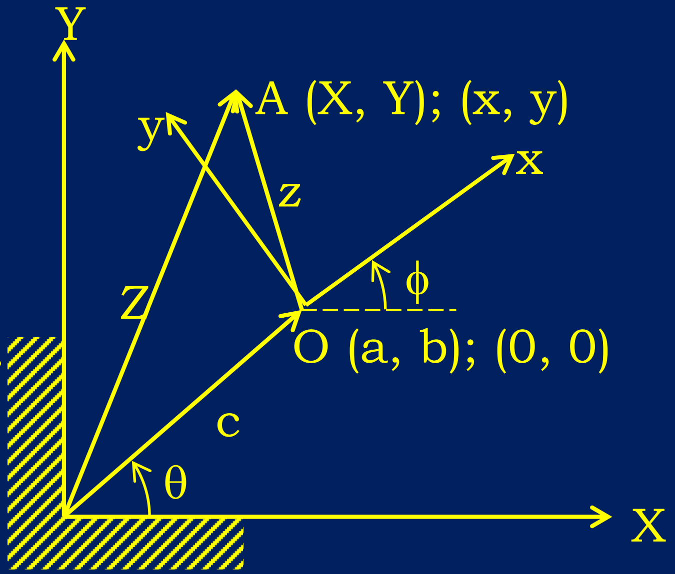
$x^2 + y^2 = r^2$

$y = rsin\psi$

Substitution and simplification yields:

$$\left(\frac{1}{r} - \frac{1}{r_c}\right) sin\psi = \frac{1}{b''}$$

This is Euler-Savary equation in its basic form.



Recall

$$b'' = \frac{d^2 b}{d\phi^2} = -\frac{ds}{d\phi}$$

Infection Point, Infection and Return Circles

Determine the points on the moving plane for which radius of curvature is infinite (i.e. momentarily moving on a straight path).

$$\frac{1}{r_C} = 0$$

Substitution into Euler-Savary equation yields:

$$\frac{1}{r}\ sin\psi = -\frac{ds}{d\phi}$$

Let

$$-\frac{ds}{d\phi} = \delta$$

$$r_W = \delta sin\psi$$

where r_W is the locus of *inflection points*.

On every pole ray *there is only one* inflection point, W.

Infection Point, Infection and Return Circles

$r_W = \delta sin\psi$

is in polar form.

Transforming it into Cartesian coordinates:

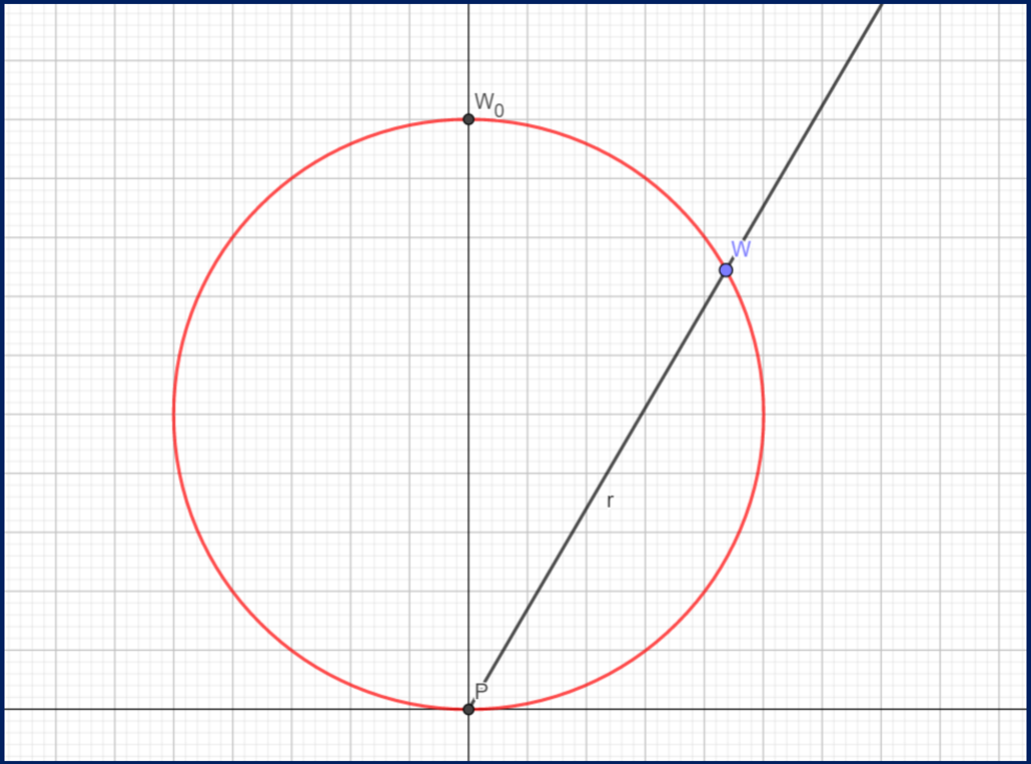
$$sin\psi = \frac{y_W}{r_W}$$

$$r_W = \sqrt{{x_W}^2 + {y_W}^2}$$

$$X_W = x_W = r_W cos\psi = \delta sin\psi cos\psi$$

$$Y_W = y_W = r_W sin\psi = \delta sin^2\psi$$

$$r_W = \delta sin\psi = \delta \frac{y_W}{r_W}$$



Rearranging

$${r_W}^2 - \delta y_W = 0$$

$${x_W}^2 + {y_W}^2 - \delta y_W = 0$$

or

$${x_W}^2 + \left(y_W - \frac{\delta}{2}\right)^2 = \left(\frac{\delta}{2}\right)^2$$

$C(0, \delta/2), R = \delta/2$

This is known as inflection circle.

Infection Point, Infection and Return Circles

Recall

$\rho = r_C - r, r_C = \rho + \mathrm{r}$

substituting into Euler Savary equation

$$\left(\frac{1}{r} - \frac{1}{\rho + \mathrm{r}}\right) sin\psi = \delta$$

simplification yields:

$$\rho = \frac{r^2}{\delta sin\psi - r}$$

Quadratic form of Euler-Savary equation.

Infection Point, Infection and Return Circles

$$\rho = \frac{r^2}{\delta sin\psi - r}$$

Normalizing this equation by $\delta sin\psi$

$$r^* = \frac{r}{\delta sin\psi}, \rho^* = \frac{\rho}{\delta sin\psi}$$

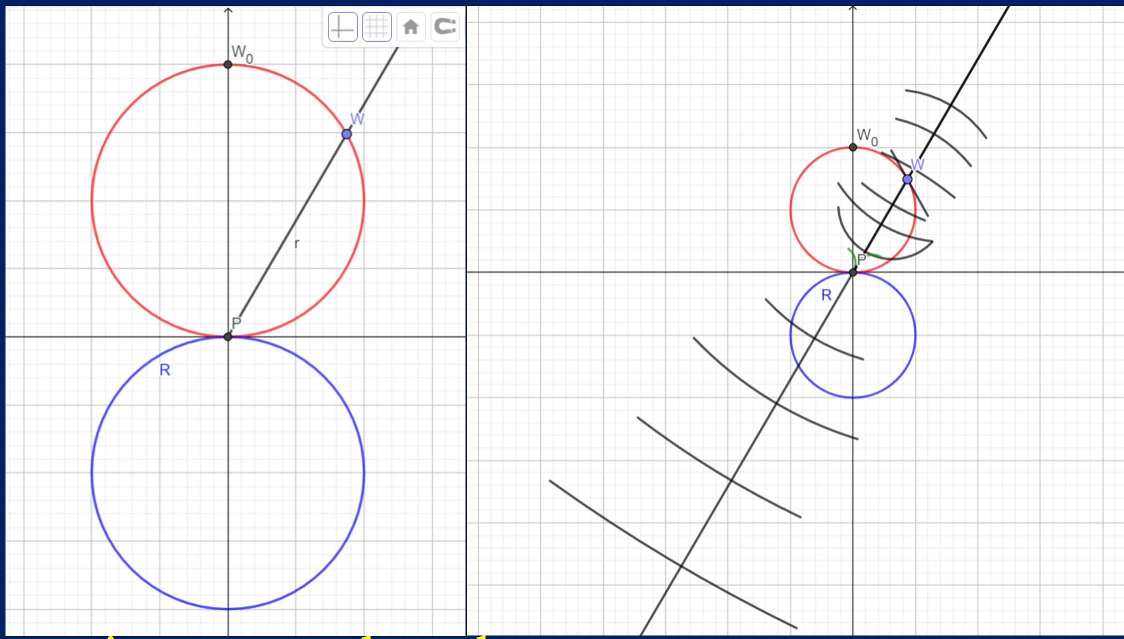
$$\rho^* = \frac{{r^*}^2}{1 - r^*}$$

Normalized quadratic form of Euler-Savary equation.

Infection Point, Infection and Return Circles

$$\rho = \frac{r^2}{\delta sin\psi - r}$$

$$\rho^* = \frac{{r^*}^2}{1 - r^*}$$



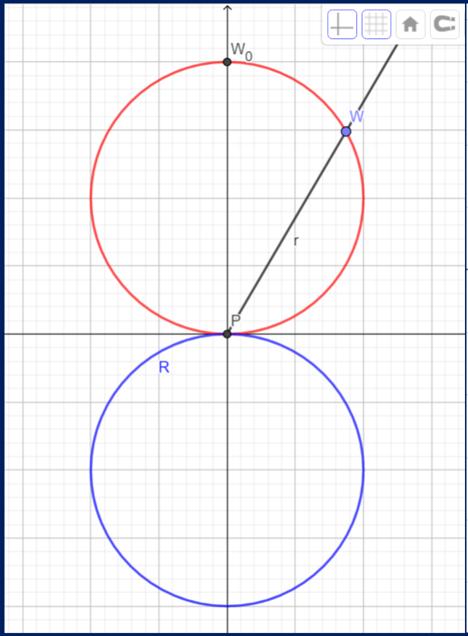
Consequences:

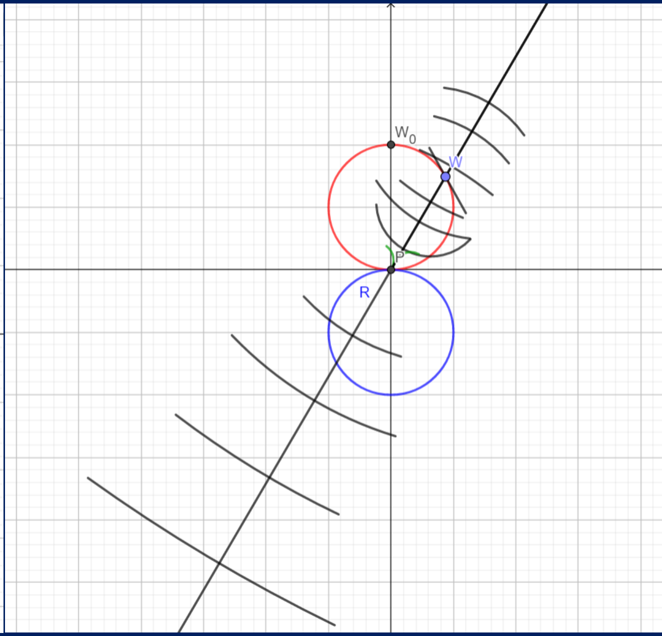
1. There is only one inflection point on each pole ray.
2. When $r^* > 1$ (i.e. r > $\delta sin\psi$) center of curvature is "below" (i.e. $\rho^* < 0$) and radius of curvature increases as r increases.
3. When $r^* < 1$ (i.e. r < $\delta sin\psi$) center of curvature is "above" (i.e. $\rho^* > 0$) and radius of curvature decreases as r decreases to 0. At $r^* = 0$ (r = 0) $\rho^* = 0$ ($\rho = 0$) there is a *cusp*. For $r^* < 0$ ρ starts increasing.
4. When $\frac{1}{r} = 0$ Euler-Savary equation takes the form $r_C = -\delta sin\psi$ which is the locus of *return circle*. It is the locus of center of curvature of *points at infinity*.

Infection Point, Infection and Return Circles

$$\rho = \frac{r^2}{\delta sin\psi - r}$$

$$\rho^* = \frac{r^{*2}}{1 - r^*}$$





Please note that curvatures are always concave when viewed from the inflection point, W.

Substituting

$$r_W = \delta sin\psi$$

into Euler-Savary equation in basic form

$$\frac{1}{r} - \frac{1}{r_C} = \frac{1}{r_W}$$

geometric form of Euler-Savary equation is obtained.

Infection Point, Infection and Return Circles

Please note that one term in Euler-Savary equation, which is only a function of motion, is not evaluated up to now which is the differential coefficient or the inflection circle diameter.

$$\delta = -\frac{ds}{d\theta} = -\frac{v_P}{\omega} = \frac{1}{b''}$$

Utilizing arc lengths:

$$-ds = r_P d\theta_P = r_\Pi d\theta_\Pi$$

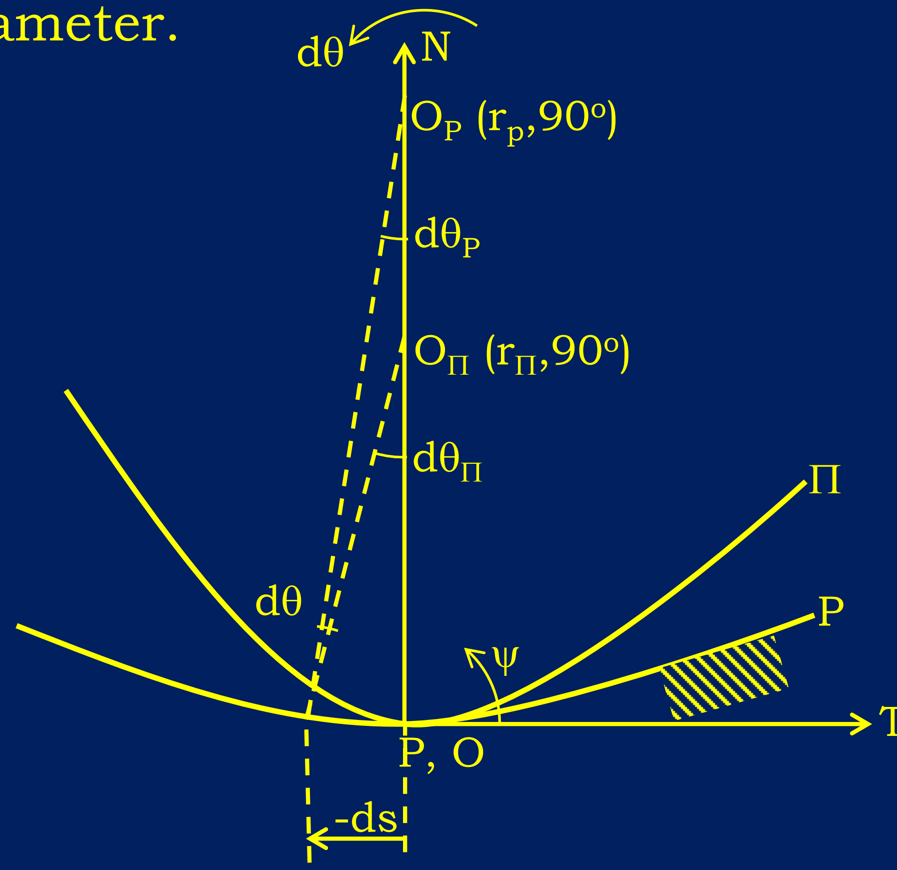
yielding

$$\frac{r_P}{r_\Pi} = \frac{d\theta_\Pi}{d\theta_P}$$

Recall

$$-\frac{d\theta}{ds} = \frac{1}{\delta}, ds = -\delta d\theta$$

$$\frac{1}{\delta} = \frac{1}{r_\Pi} - \frac{1}{r_P}$$



The centers of curvature of the fixed and moving centrodes are conjugate points and this equation is analogous to Euler-Savary equation for ψ = 90°. If centers of curvature of centrodes are known the differential coefficient can be evaluated.

Infection Point, Infection and Return Circles

For matching two motions in three infinitesimally separated positions (i.e. both motions to have the same path, path tangent and path curvature at the design point):

- the poles of two motions should be superimposed,
- the pole tangents should be aligned in the same direction,
- scale one motion so that the inflection circle diameters are the same.

Inverted Motion

For the inverted motion dθ reverses its direction, fixed and moving centrodes, and, inflection and return circles change their roles.

Infection Point, Infection and Return Circles

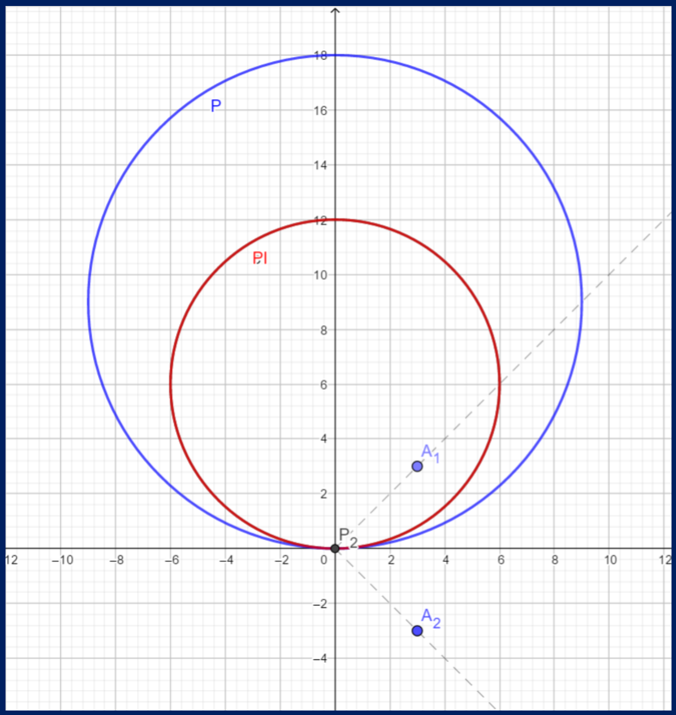
Example:

A cylinder of radius 6 cm rolls inside a fixed cylindrical hole of radius 9 cm without slipping at ω = 1 rad/s CCW. Two points on the moving plane, $A_1(3\sqrt{2}, 45°)$ and $A_2(-3\sqrt{2}, 135°)$ are given.

a. Determine the centers of curvature for A_1 and A_2,

b. Determine the inflection circle,

c. Determine the pole velocity.



Infection Point, Infection and Return Circles

Example:

$$\frac{1}{\delta} = \frac{1}{r_\Pi} - \frac{1}{r_P} = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}, \delta = 18\ cm$$

$$v_P = -\omega\delta = -1 * 18 = -18\ cm/s$$

Euler-Savary equation

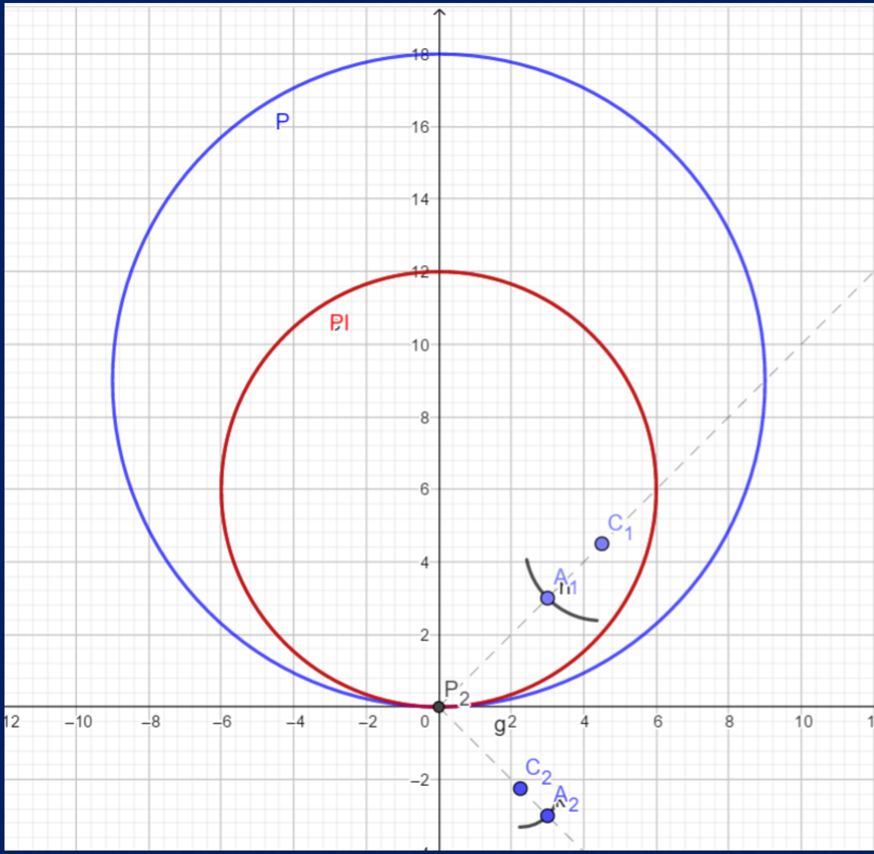
$$\left(\frac{1}{r} - \frac{1}{r_c}\right) sin\psi = \frac{1}{\delta}$$

for point A_1

$$\left(\frac{1}{3\sqrt{2}} - \frac{1}{r_c}\right) sin45° = \frac{1}{18}, r_c = \frac{9\sqrt{2}}{2} cm$$

for point A_2

$$\left(\frac{1}{-3\sqrt{2}} - \frac{1}{r_c}\right) sin135° = \frac{1}{18}, r_c = \frac{-9\sqrt{2}}{4} cm$$



Infection Point, Infection and Return Circles

Example: Long Period Pendulum with Small Size

For a simple pendulum with a massless rod and point mass

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\ell\dot{\theta}\right)^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$V = mg\ell(1 - cos\theta) \cong mg\ell\frac{\theta^2}{2}$$

$$\dot{T} + \dot{V} = \mathbb{P}_{in} - \mathbb{P}_{dis}$$

$$m\ell^2\dot{\theta}\ddot{\theta} + mg\ell\theta\dot{\theta} = 0$$

$$\ell\ddot{\theta} + g\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{\ell}}, \mathrm{T} = 2\pi\sqrt{\frac{\ell}{g}}$$

Longer the pendulum length, ℓ, larger the period of undamped free oscillations of the pendulum. However space restrictions may limit the length of the pendulum!

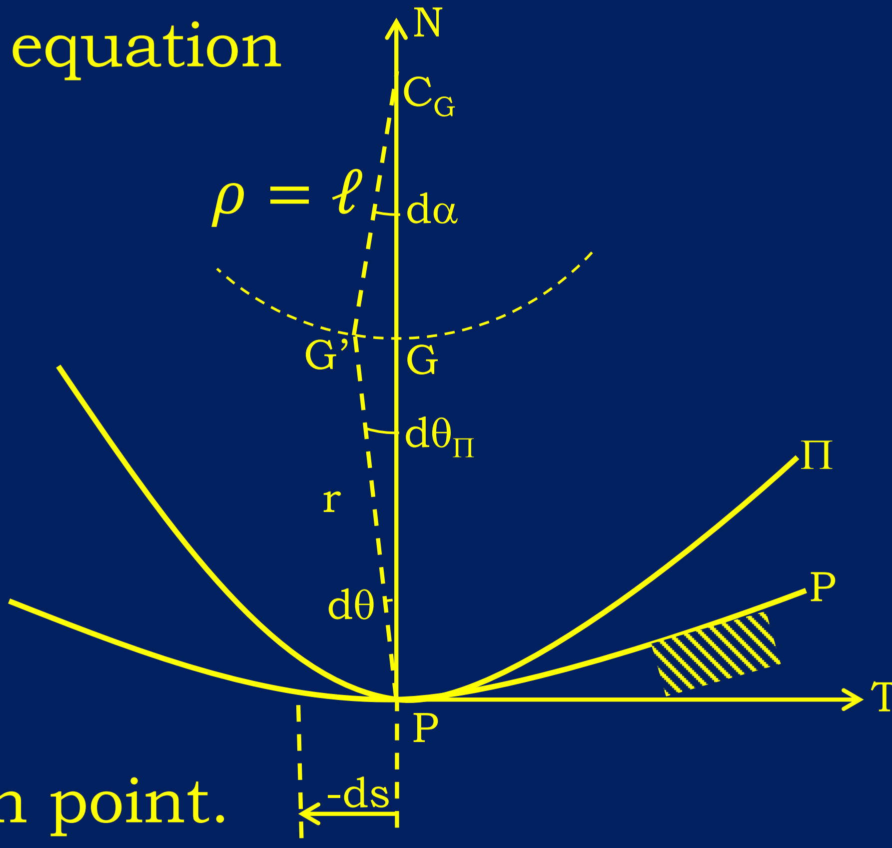
Infection Point, Infection and Return Circles

Example: Long Period Pendulum (cont'ed)

Quadratic form of Euler-Savary equation

$$\rho = \ell = \frac{r^{2}}{\delta - r}$$

$$T = 2\pi \sqrt{\frac{\rho}{g}} = 2\pi r \sqrt{\frac{1}{g(\delta - r)}}$$



To increase period, T, $(\delta - r)$ should be reduced.

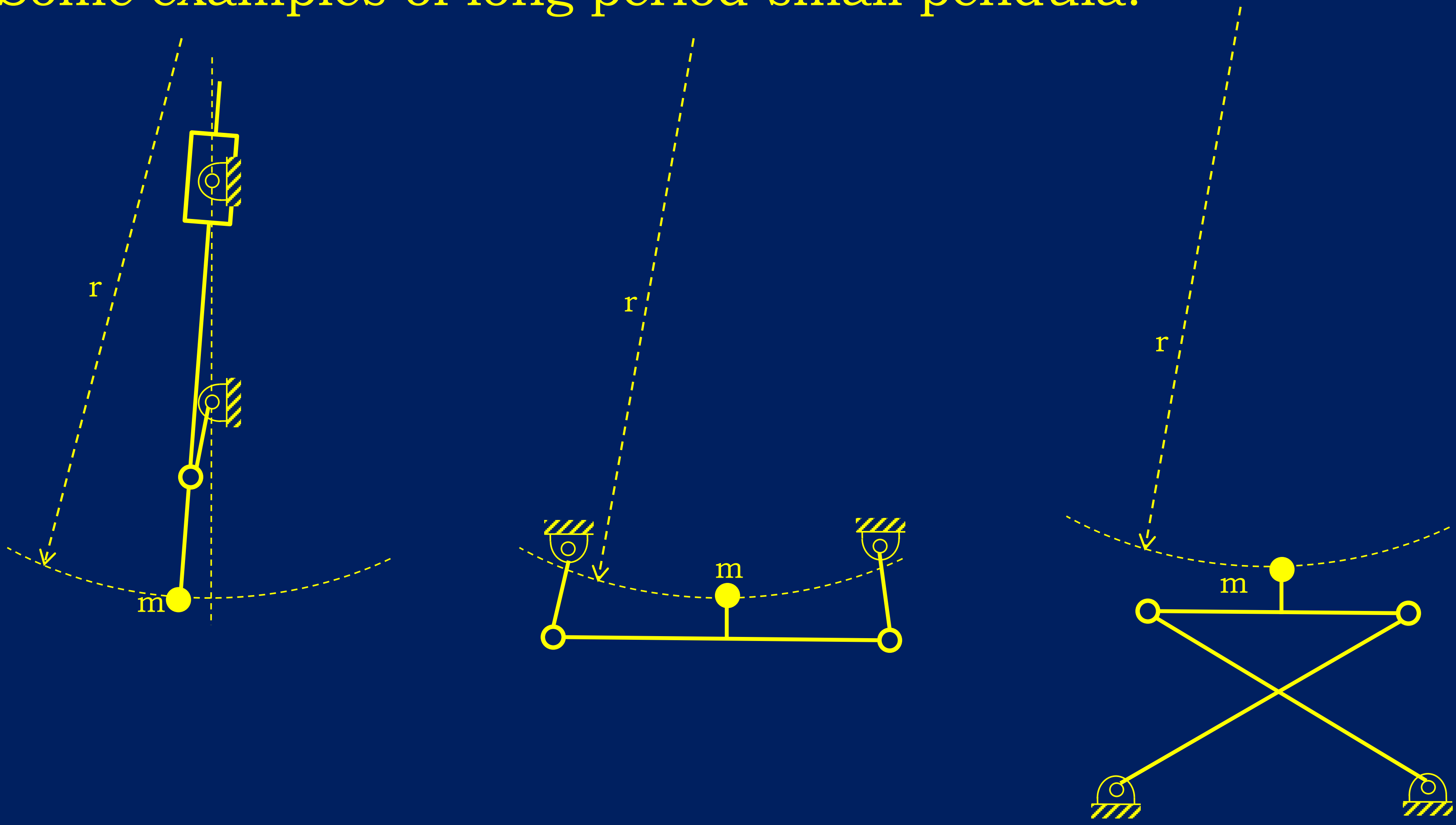
This can be achieved by moving G close (but below) the inflection point.

Curvature is concave up *in the infinitesimal neighborhood of the design point*. Stability should be checked for the entire range even for small (but *finite*) motion of the pendulum.

Infection Point, Infection and Return Circles

Example: Long Period Pendulum (cont'd)

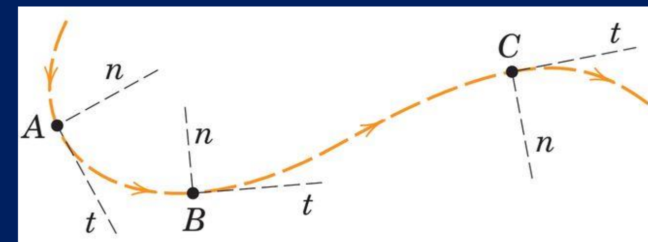
Some examples of long period small pendula:



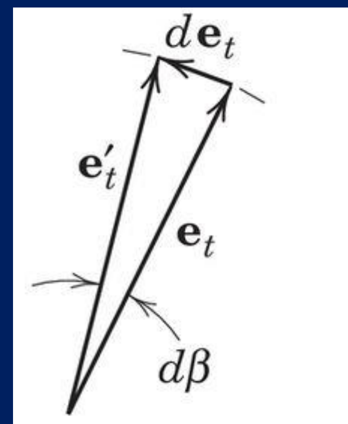
Acceleration Analysis Using Centrodes

Review of n-t Coordinates (ME 208 Dynamics)

$$ds = \rho d\beta, \frac{ds}{dt} = \rho\frac{d\beta}{dt} + \frac{d\rho}{dt}d\beta, v = \rho\dot{\beta}$$

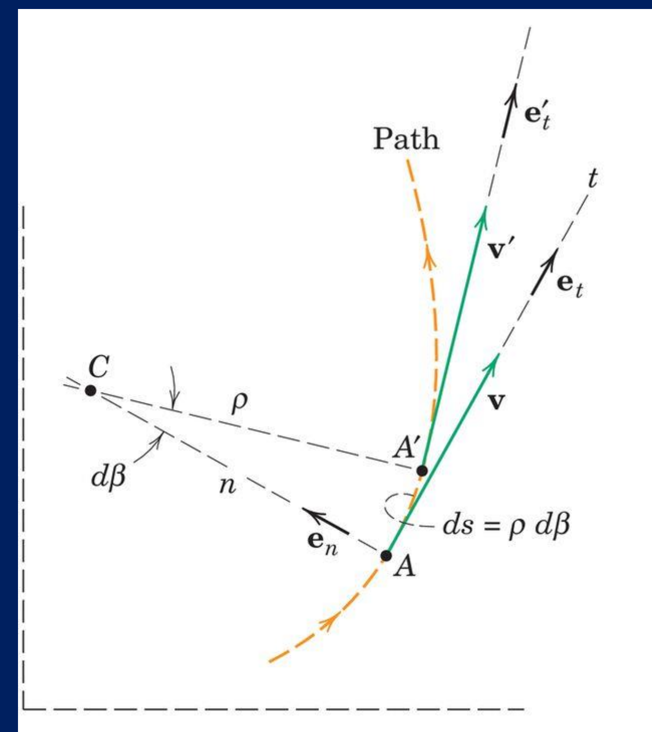


$$\boxed{\vec{v} = v\hat{e}_t}, \hat{e}_t \equiv \frac{\vec{v}}{v}$$



$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{v}\hat{e}_t + v\dot{\hat{e}}_t$$

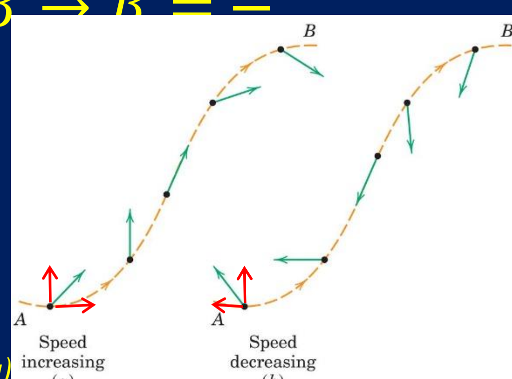
$$\dot{v} = \frac{d|\vec{v}|}{dt}$$



$$d\hat{e}_t = d\beta\hat{e}_n, \dot{\hat{e}}_t = \frac{d\hat{e}_t}{dt} = \frac{d\beta}{dt}\hat{e}_n = \dot{\beta}\hat{e}_n$$

$$\vec{a} = \dot{v}\hat{e}_t + v\dot{\beta}\hat{e}_n, v = \rho\dot{\beta} \rightarrow \dot{\beta} = \frac{v}{}$$

$$\boxed{\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n}$$



Article 2.5 of Dynamics by Meriam (Wiley)

Acceleration Analysis Using Centrodes

Consider A on the moving plane with angular velocity ω and angular acceleration α.

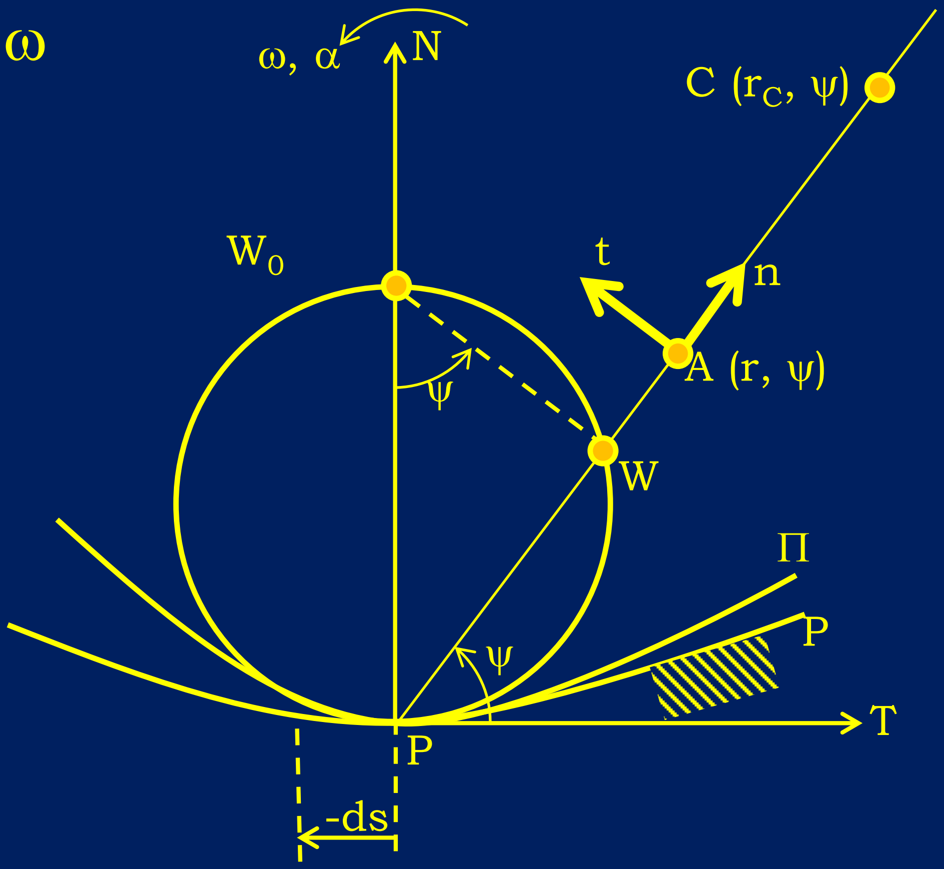
$$\vec{a}_A = \vec{a}_A{}^t + \vec{a}_A{}^n = \dot{v}_A \hat{e}_t + \frac{v_A{}^2}{\rho}\hat{e}_n$$

$$v_A = \omega|PA| = \omega r$$

$$\rho = \frac{r^2}{\delta sin\psi - r}$$

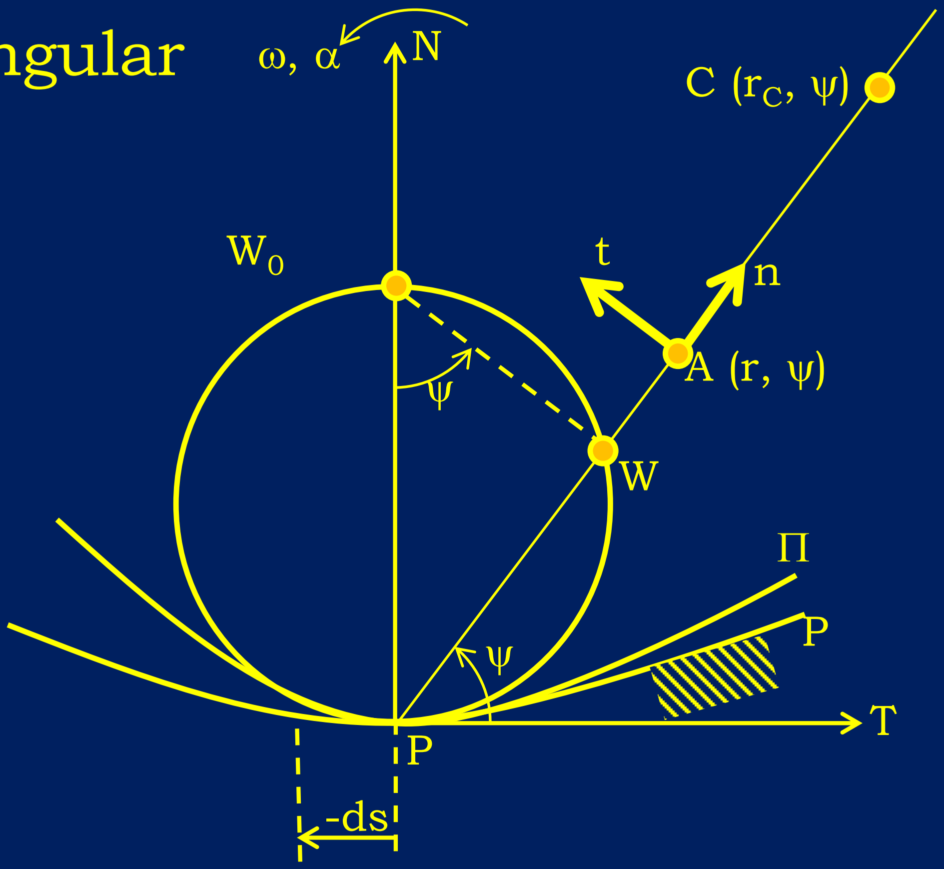
$$a_A{}^n = \frac{\omega^2 r^2}{r^2/(\delta sin\psi - r)}$$

$$a_A{}^n = \omega^2(\delta sin\psi - r)$$



Acceleration Analysis Using Centrodes

Consider A on the moving plane with angular velocity ω and angular acceleration α.



$$\vec{a}_A = \vec{a}_A{}^t + \vec{a}_A{}^n = \dot{v}_A \hat{e}_t + \frac{v_A{}^2}{\rho}\hat{e}_n$$

$$a_A{}^n = \omega^2(\delta sin\psi - r)$$

$$a_A{}^t = \frac{d}{dt}(r\omega) = r\alpha + \omega\frac{dr}{dt}$$

$$dr = -ds cos\psi$$

$$\frac{dr}{dt} = -\frac{ds}{d\theta}\frac{d\theta}{dt}cos\psi = \delta\omega cos\psi$$

$$a_A{}^t = r\alpha + \delta\omega^2 cos\psi$$

$$\vec{a}_A = (r\alpha + \delta\omega^2 cos\psi)\hat{e}_t + \omega^2(\delta sin\psi - r)\hat{e}_n$$

Acceleration Analysis Using Centrodes

$$\vec{a}_A = (r\alpha + \delta\omega^2 cos\psi)\hat{e}_t + \omega^2(\delta sin\psi - r)\hat{e}_n$$

Rearrange

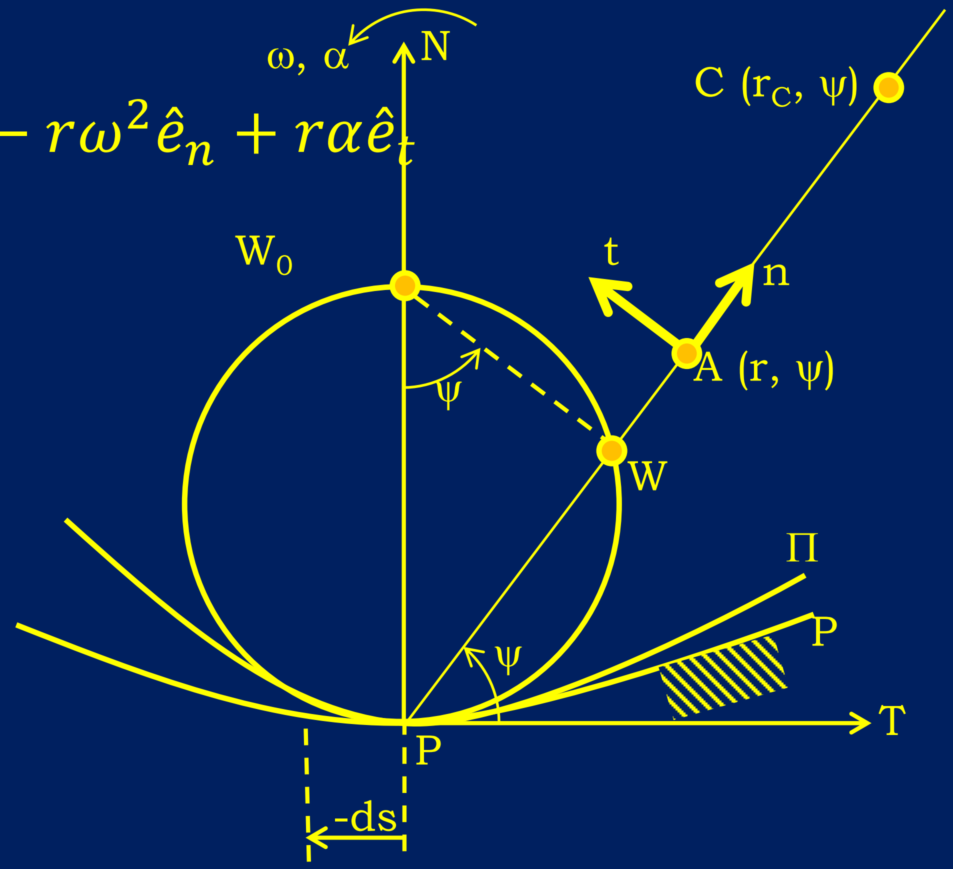
$$\vec{a}_A = \omega^2(\delta sin\psi\hat{e}_n + \delta\omega^2 cos\psi\hat{e}_t) - r\omega^2\hat{e}_n + r\alpha\hat{e}_t$$

$$\vec{a}_A = \omega^2\left(\overrightarrow{PW_0} - r\hat{e}_n\right) + r\alpha\hat{e}_t$$

Recall

$$r\hat{e}_n + \overrightarrow{AW_0} = \overrightarrow{PW_0}$$

$$\boxed{\vec{a}_A = \overrightarrow{AW_0}\omega^2 + r\alpha\hat{e}_t}$$

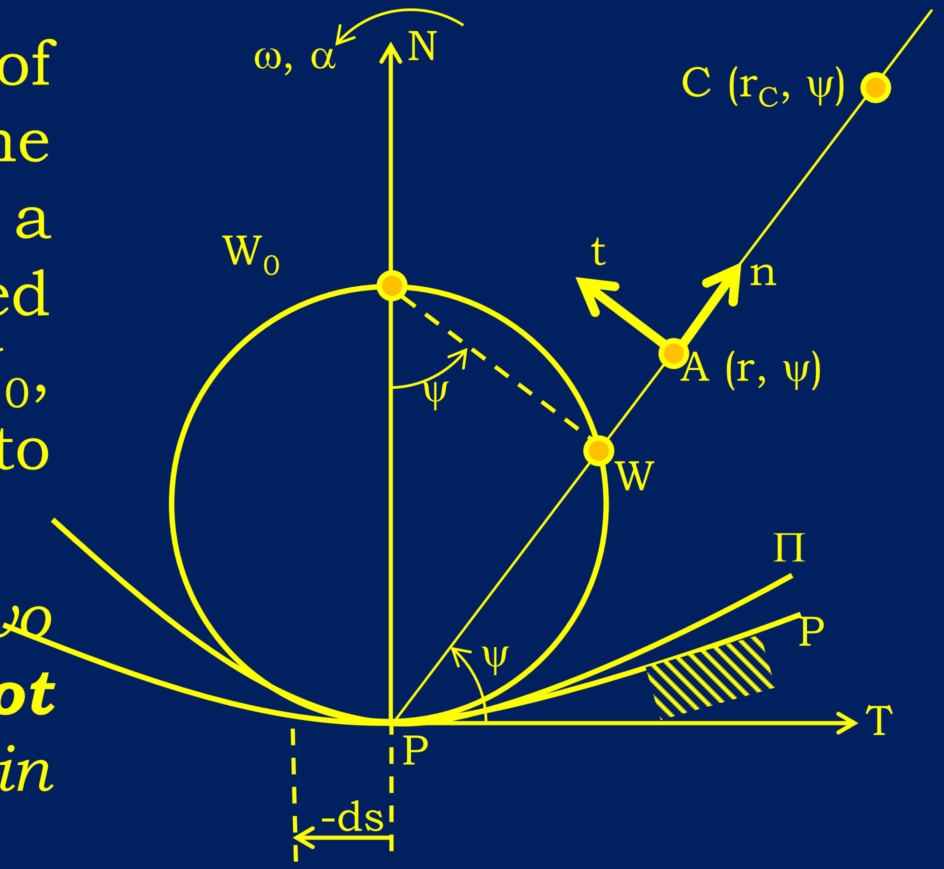


Acceleration Analysis Using Centrodes

$\vec{a}_A = \overrightarrow{AW_0}\omega^2 + r\alpha\hat{e}_t$

Theorem: The acceleration of any point on the moving plane is the vector sum of a component $|AW_0|\omega^2$ directed towards the inflection pole, W_0, and a component rα tangent to its path.

Please note that these two components are ***not*** *perpendicular to each other in general.*



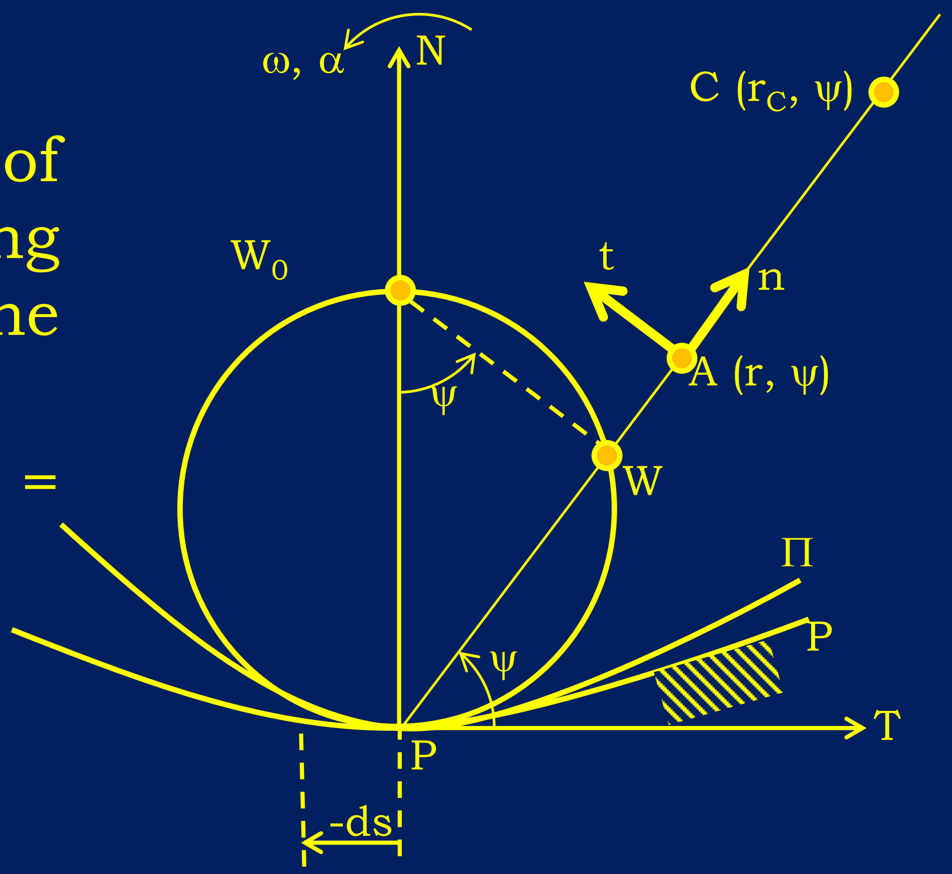
Acceleration Analysis Using Centrodes

$\vec{a}_A = \overrightarrow{AW_0}\omega^2 + r\alpha\hat{e}_t$

Consequences:

- For $\alpha = 0$ the acceleration of every point on the moving plane is towards the inflection pole, W_0.
- Acceleration of the pole (r = 0, path is a *cusp*):

  $\vec{a}_P = \overrightarrow{PW_0}\omega^2$



Acceleration Analysis Using Centrodes

$$\vec{a}_A = (r\alpha + \delta\omega^2 cos\psi)\hat{e}_t + \omega^2(\delta sin\psi - r)\hat{e}_n$$

$$\vec{a}_A = \overrightarrow{AW_0}\omega^2 + r\alpha\hat{e}_t$$

Consequences (cont'ed):

- Points on moving plane with $a^n = 0, \omega \neq 0$

  $\delta sin\psi - r = 0$

  $\delta sin\psi = r$

  Inflection circle, point on a straight path, no curvature no normal acceleration!

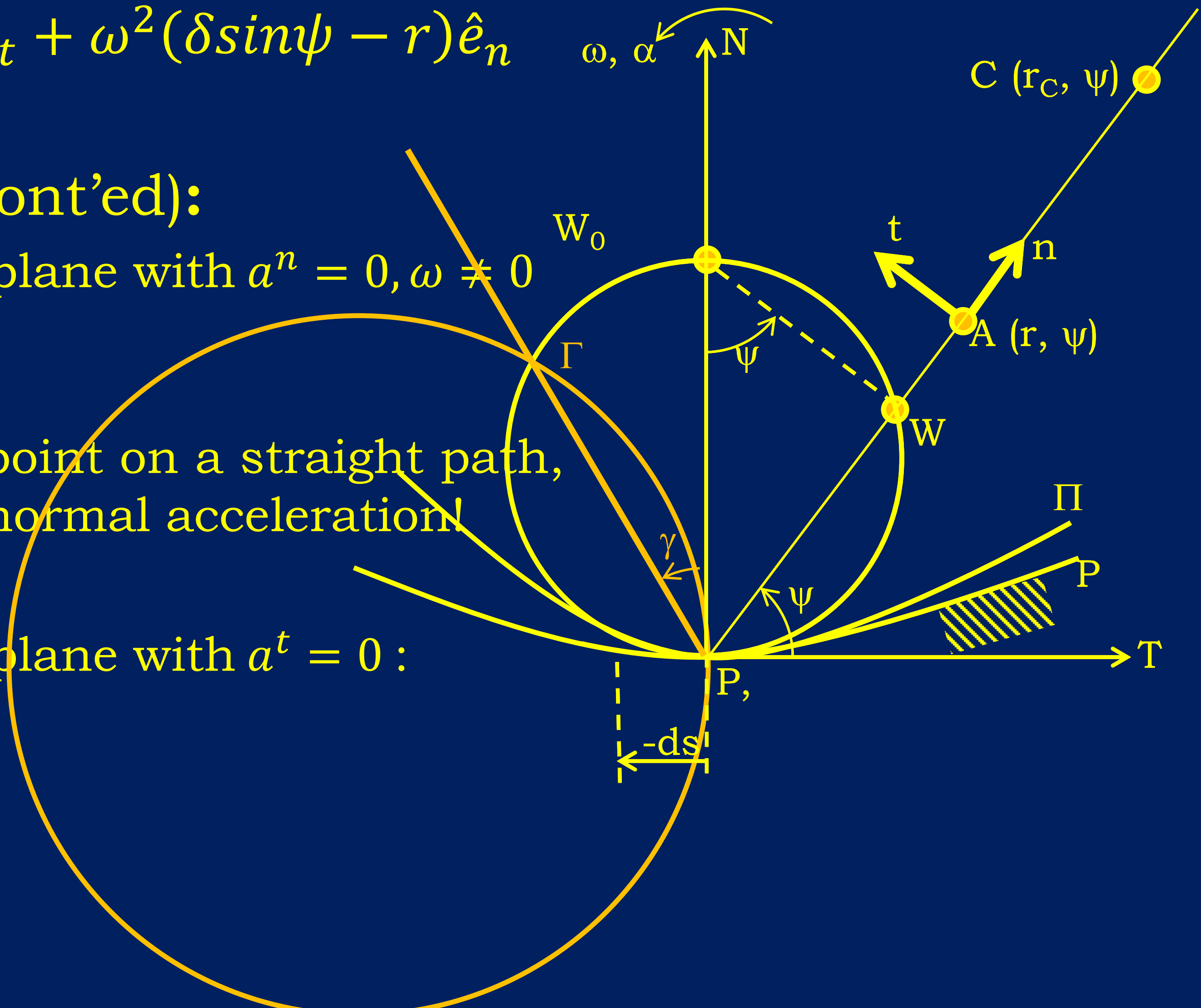
  Circle de la Hire

- Points on moving plane with $a^t = 0$:

  $r\alpha + \delta\omega^2 cos\psi = 0$

  $r = -\frac{\delta\omega^2}{\alpha} cos\psi$

  Bresse Circle



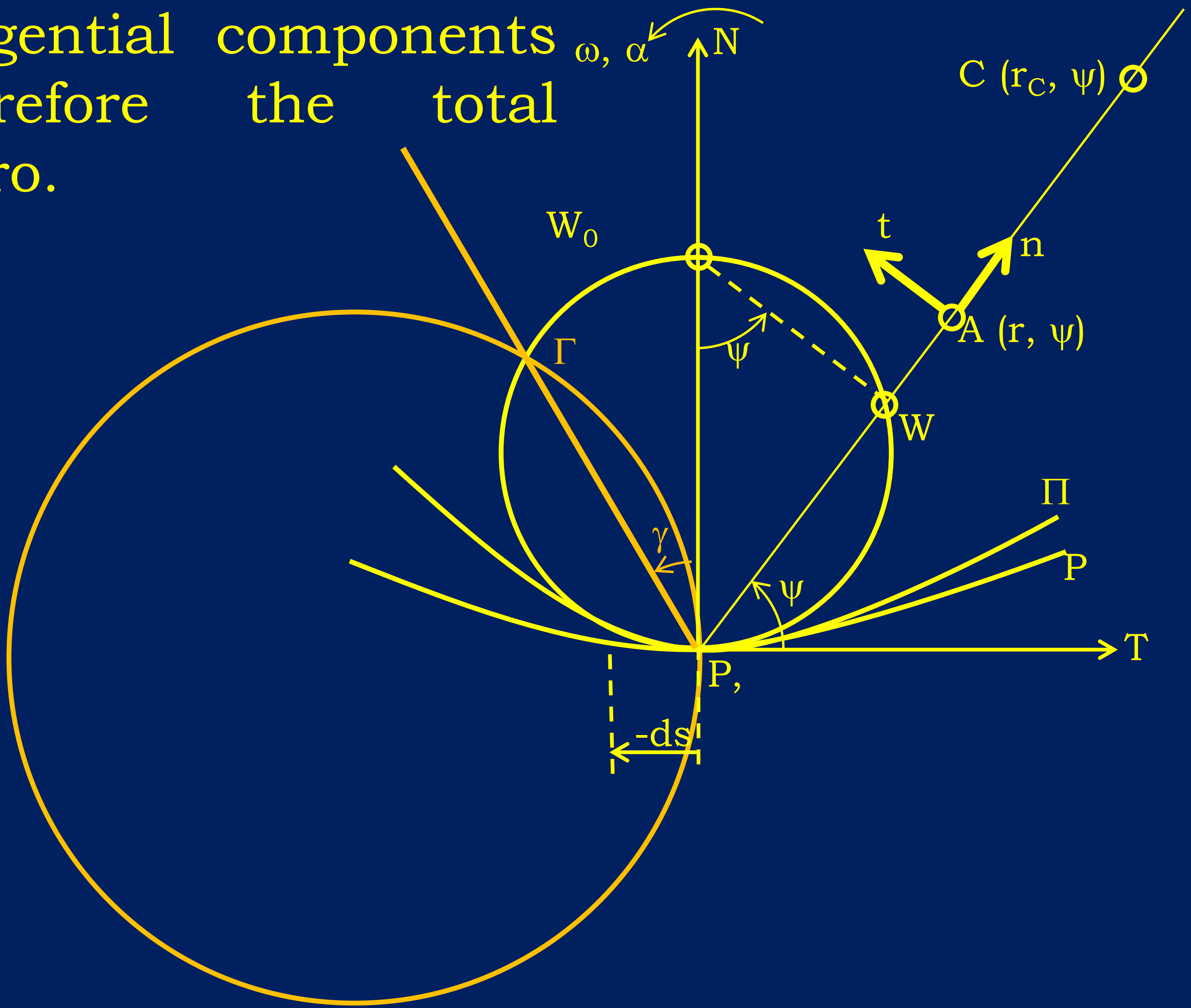
Acceleration Analysis Using Centrodes

Γ is the acceleration pole where both normal and tangential components are zero therefore the total acceleration is zero.

$$\psi^* = \frac{\pi}{2} + \gamma$$

$$tan\gamma = \frac{\alpha}{\omega^2}$$

$$tan\psi^* = \frac{-\omega^2}{\alpha}$$



Acceleration Analysis Using Centrodes

$$a_A = \sqrt{{a_A^t}^2 + {a_A^n}^2}$$

$$a_A = \sqrt{(r\alpha + \omega^2 \delta cos\psi)^2 + \omega^4(\delta sin\psi - r)^2}$$

$$\alpha = \omega^2 tan\gamma$$

$$a_A = \frac{\omega^2}{cos\gamma}\sqrt{\delta^2 cos^2\gamma + r^2 - 2\delta r cos\gamma sin(\psi - \gamma)}$$

$$|P\Gamma| = \delta cos\gamma$$

$$cos(\measuredangle \Gamma PA) = cos\left(\gamma + \frac{\pi}{2} - \psi\right) = sin(\psi - \gamma)$$

$$a_A = \frac{\omega^2}{cos\gamma}\sqrt{r^2 - |P\Gamma|^2 - 2r|P\Gamma|cos(\measuredangle \Gamma PA)}$$

$$a_A = \frac{\omega^2|A\Gamma|}{cos\gamma}$$

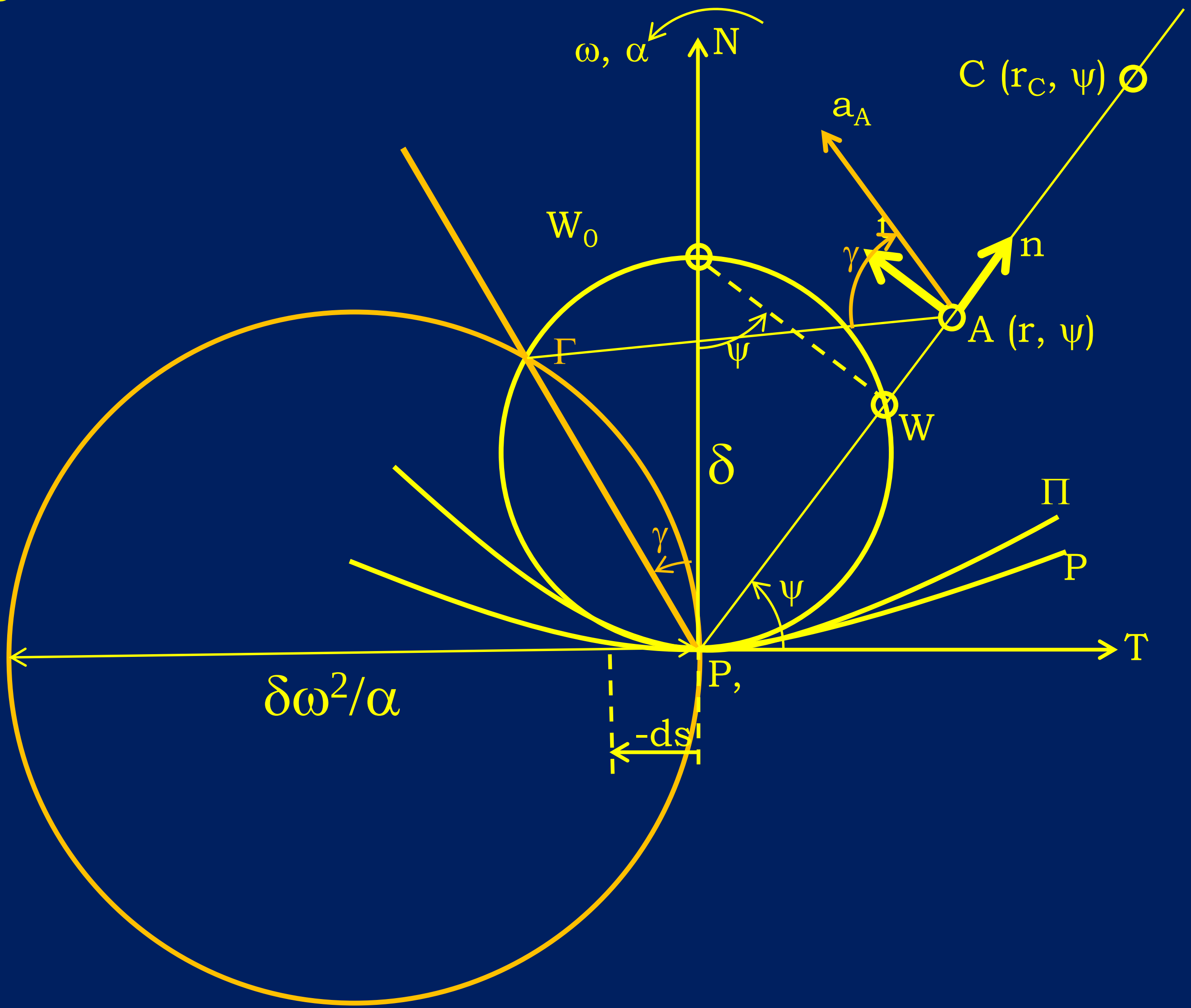
$$cos\gamma = \frac{1}{\sqrt{1 + tan^2\gamma}} = \frac{1}{\sqrt{1 + \alpha^2/\omega^4}} = \frac{\omega^4}{\sqrt{\omega^4 + \alpha^2}}$$

$$a_A = |A\Gamma|\sqrt{\omega^4 + \alpha^2}$$

$$\overline{a_A} = \overline{|A\Gamma|}\sqrt{\omega^4 + \alpha^2}e^{-i\gamma}$$

Acceleration Analysis Using Centrodes

$$\overline{a_A} = \overline{|A\Gamma|}\sqrt{\omega^4 + \alpha^2}e^{-i\gamma}$$



Bobillier's Theorem

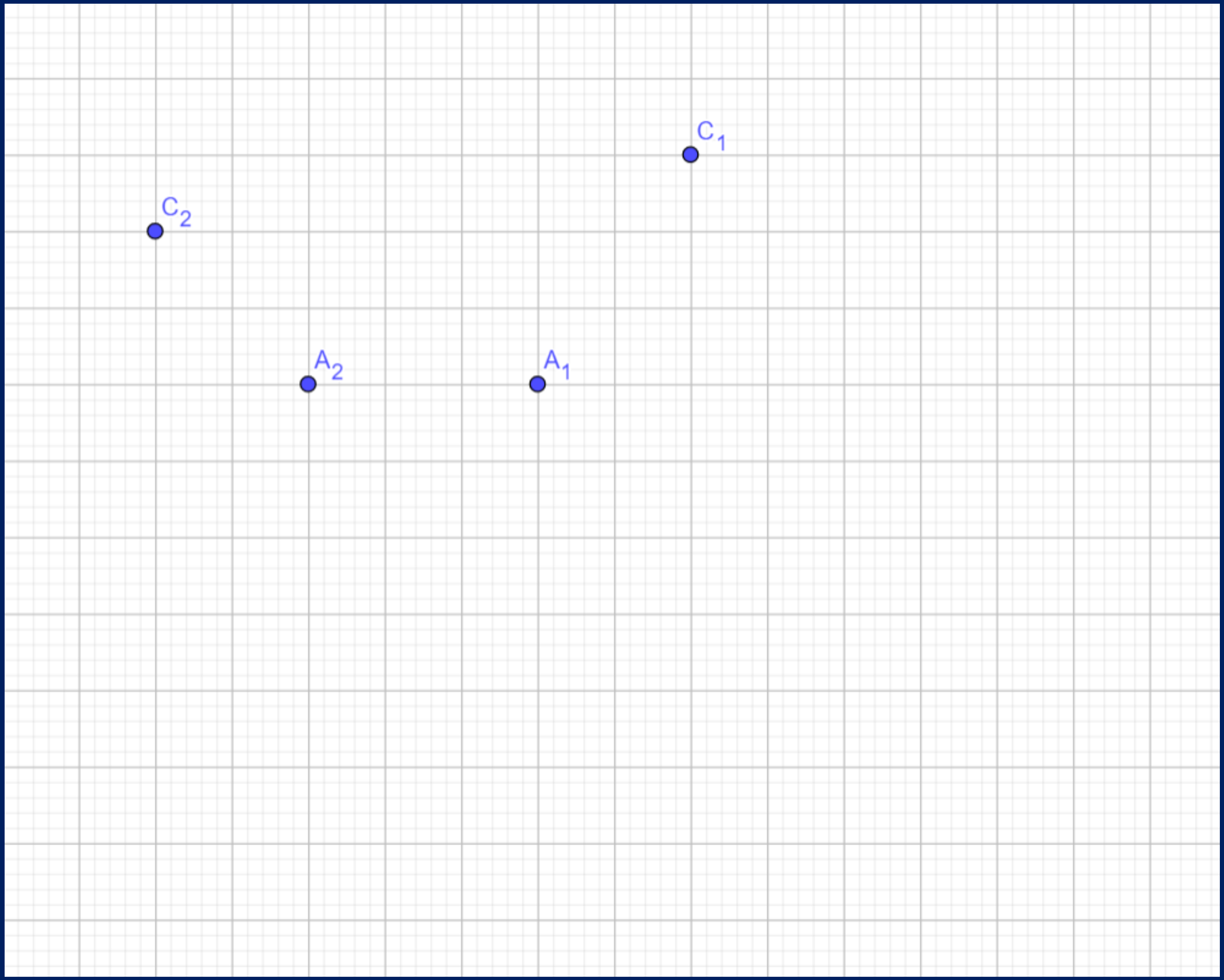
Theorem: The path normals (i.e. pole rays) of two points on the moving plane make equal angles with the *collineation axis* and the pole tangent.

Given the radii of curvature of two points on the moving plane, inflection circle diameter can be determined.

Let A_1 and A_2 be two distinct points on a moving plane with centers of curvature being C_1 and C_2 respectively.

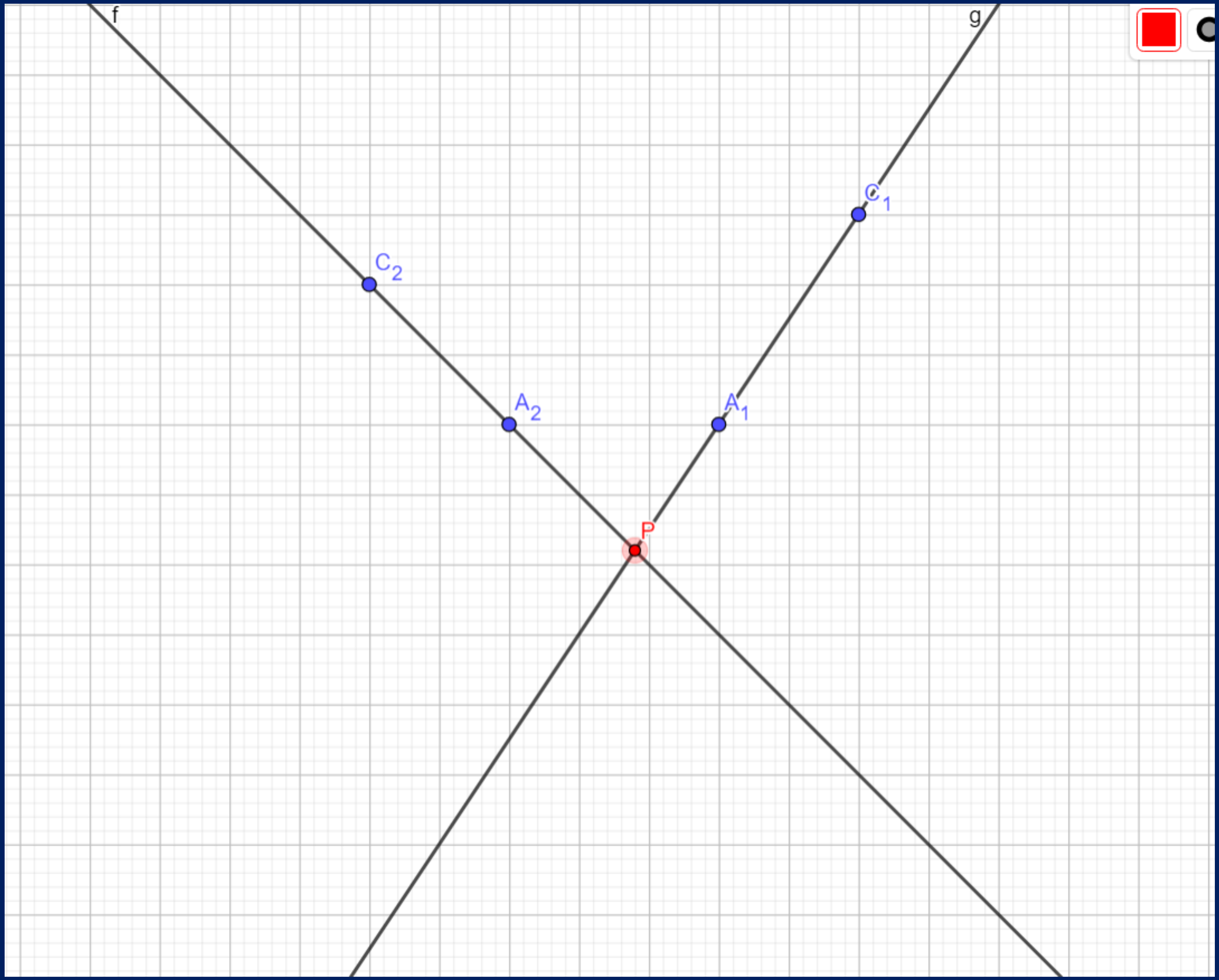
Bobillier's Theorem

Let A_1 and A_2 be two distinct points on a moving plane with centers of curvature being C_1 and C_2 respectively.



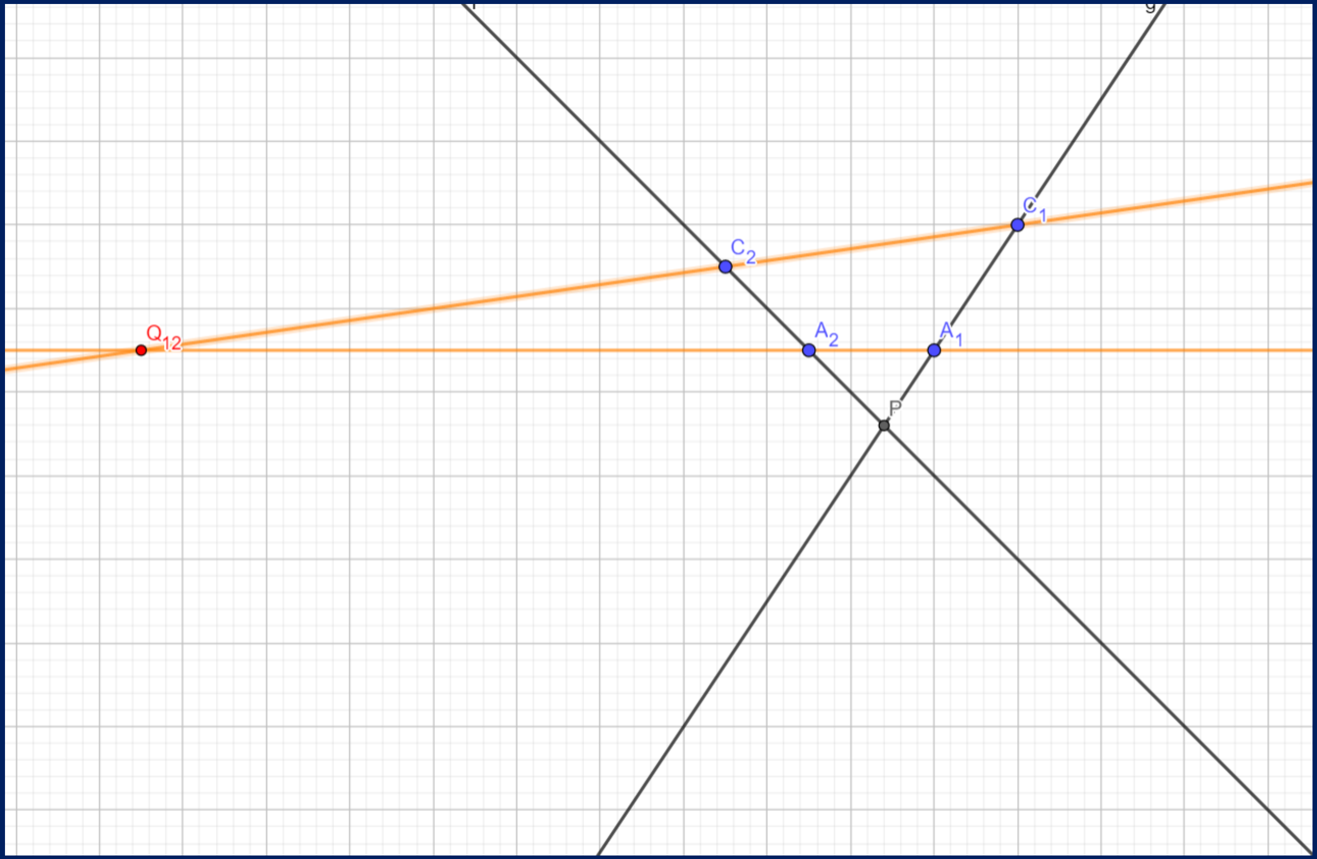
Bobillier's Theorem

1. Draw lines through A_1 C_1 and A_2 C_2, intersection is P.



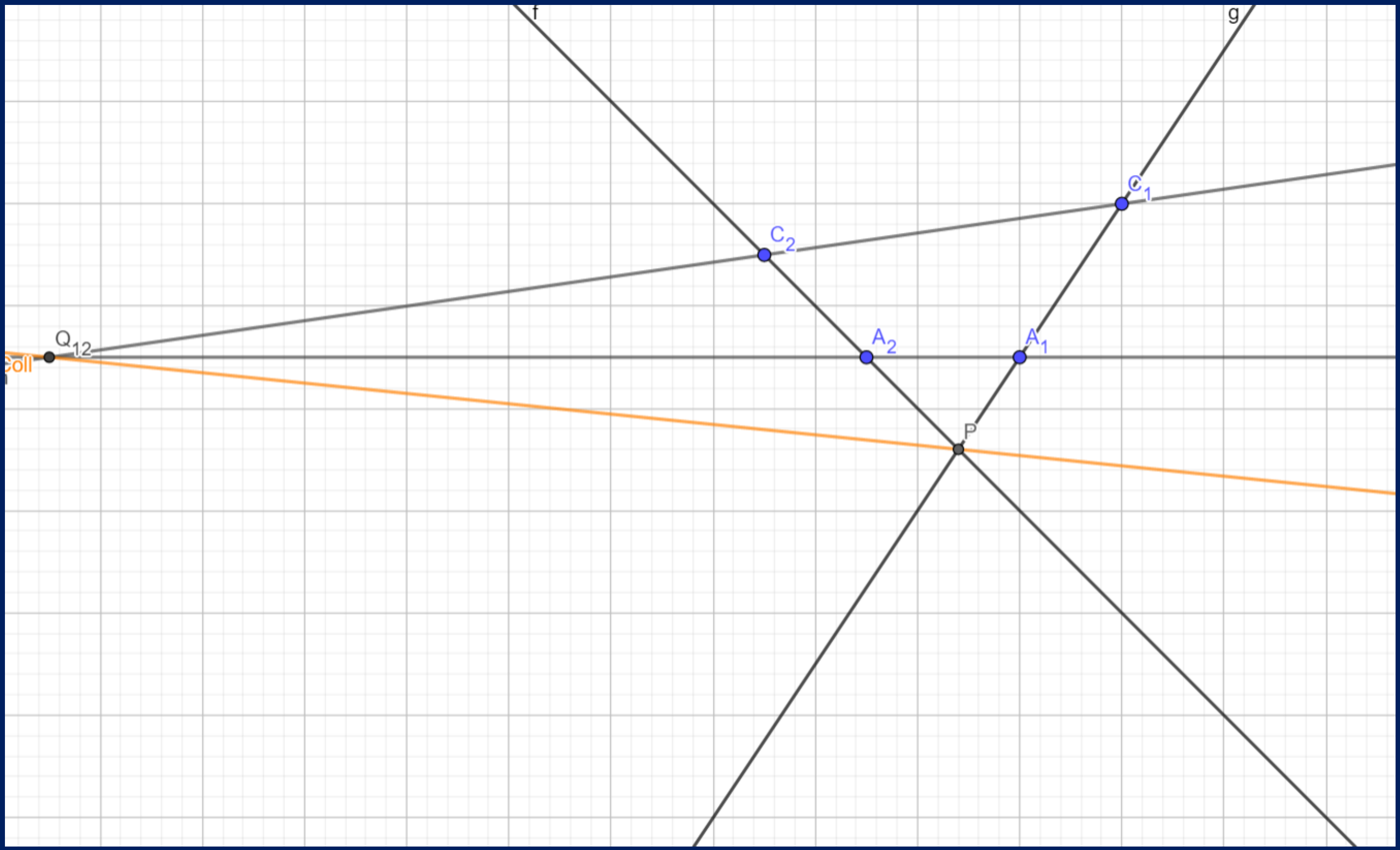
Bobillier's Theorem

2. Draw lines through A_1 A_2 and C_1 C_2, intersection is Q_{12}.



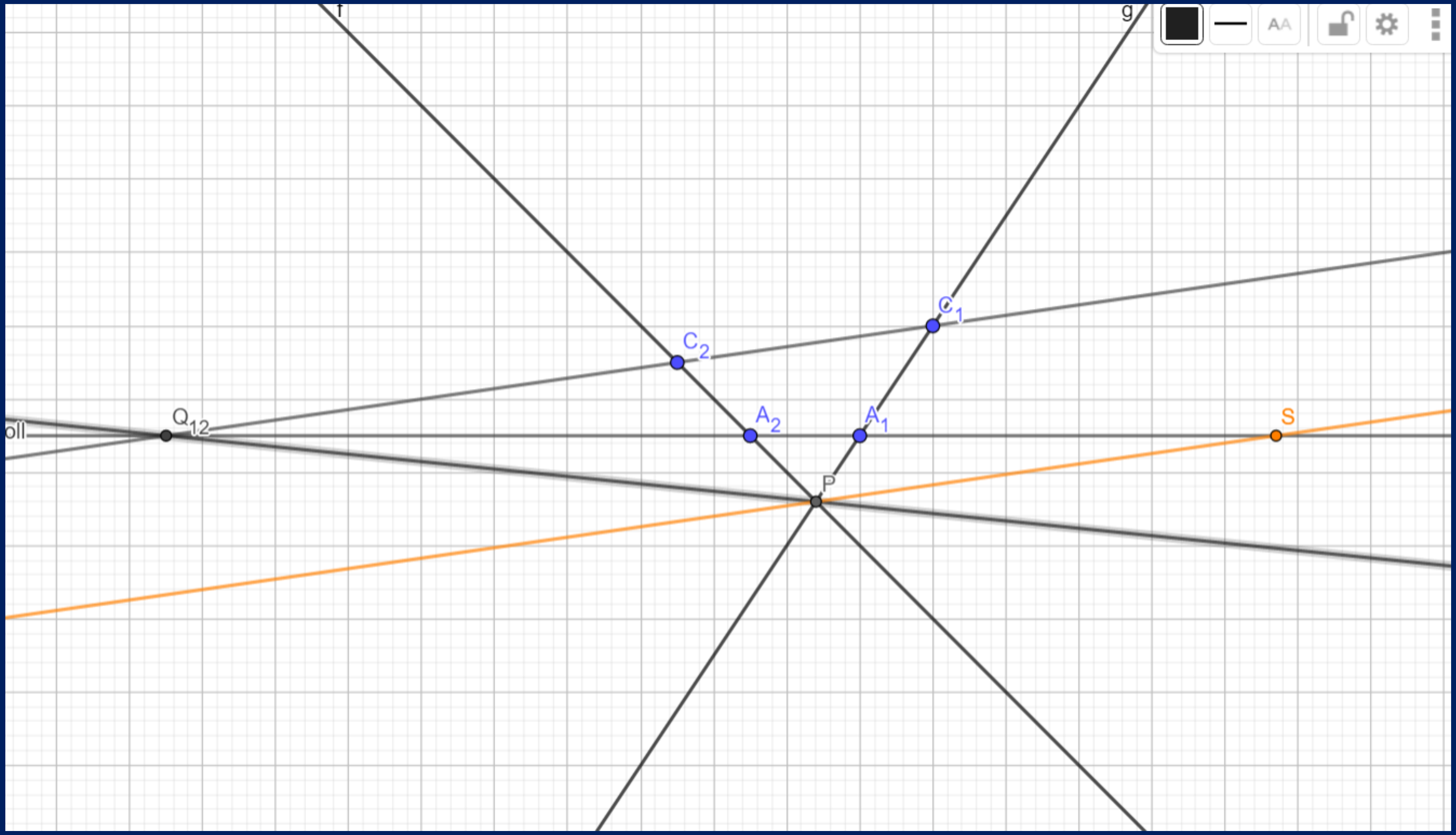
Bobillier's Theorem

3. Line P Q_{12} is *the collineation axis.*



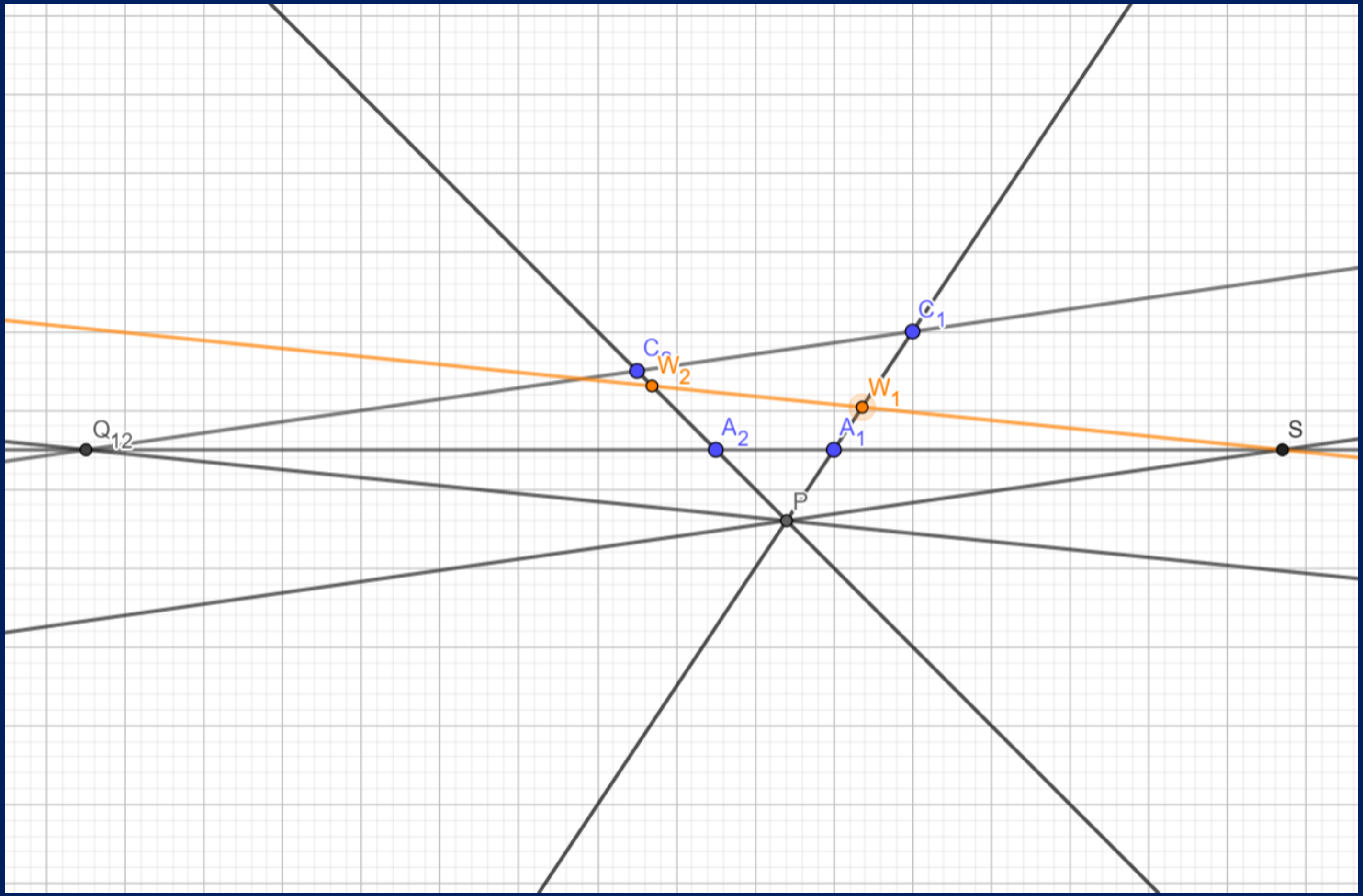
Bobillier's Theorem

4. Draw a line parallel to C_1 C_2 through P, S is the intersection with A_1 A_2.



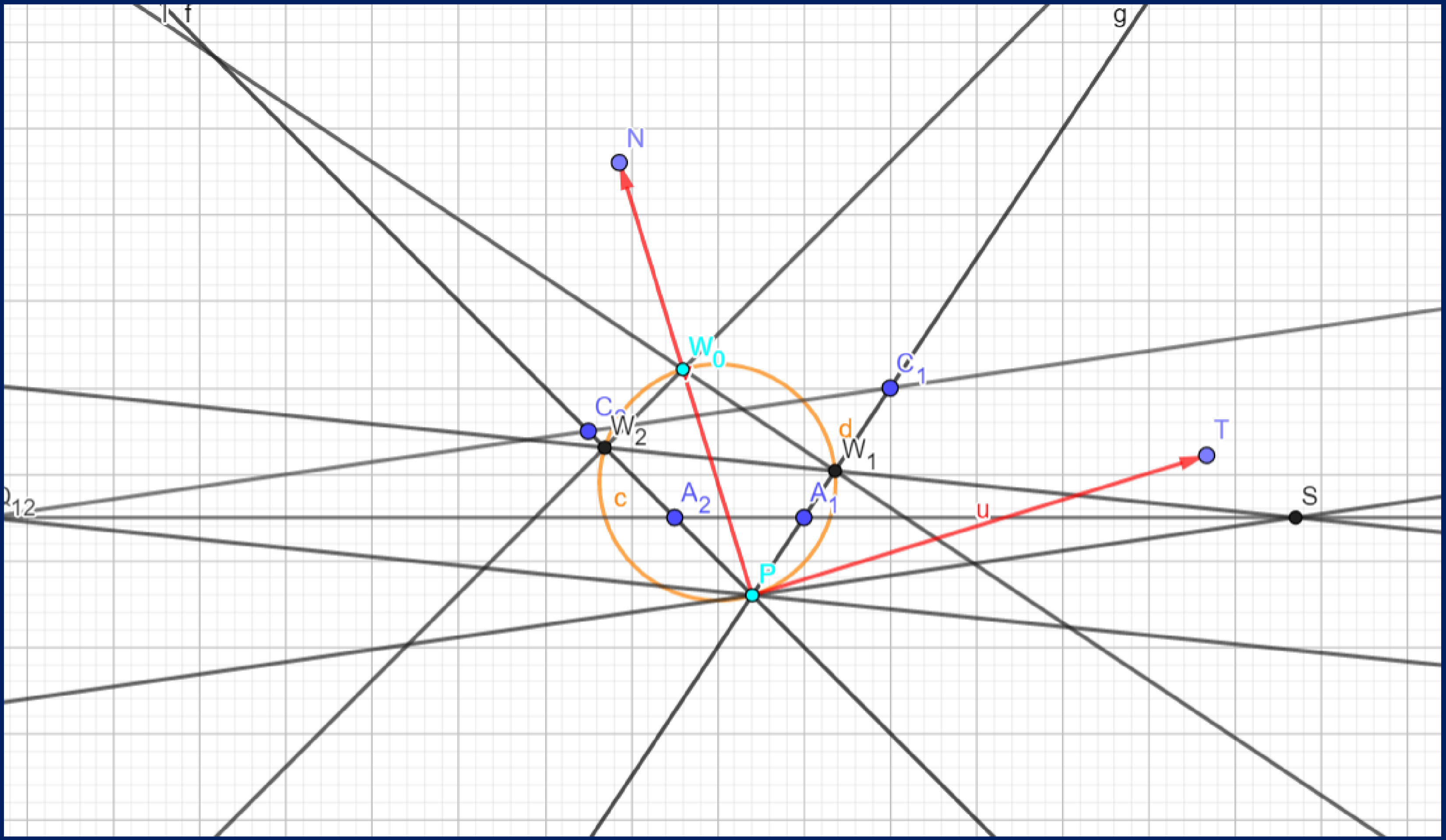
Bobillier's Theorem

5. Draw a parallel to collineation axis through S, S is the intersection with A_1 A_2. The intersection with pole rays 1 and 2 are W_1 and W_2 respectively.



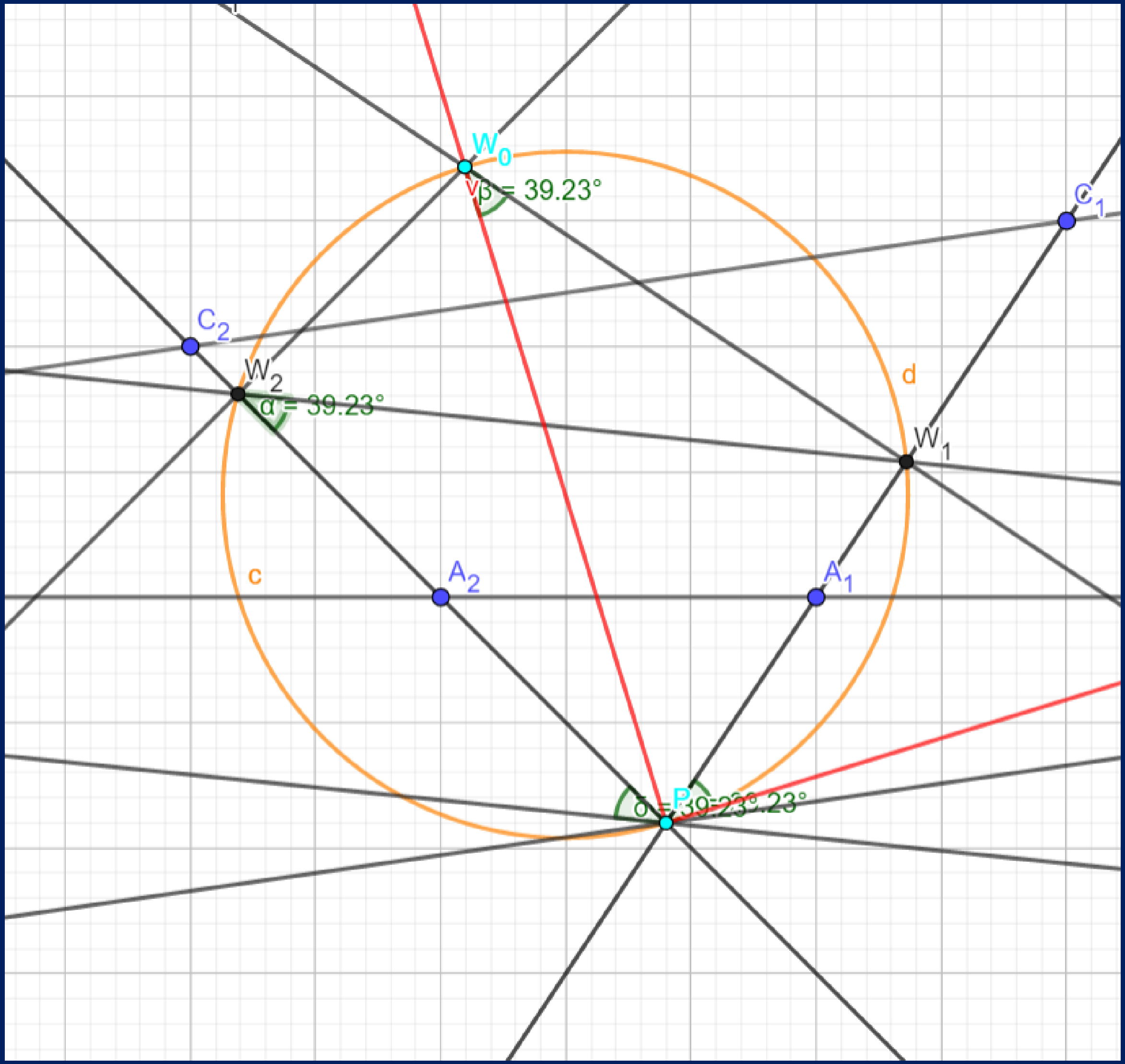
Bobillier's Theorem

7. W_0, and P define the inflection circle therefore pole tangent and normal are known.



Bobillier's Theorem

Notice $\sphericalangle PW_2W_1 = \sphericalangle PW_0W_1 = \sphericalangle TPW_1 = \sphericalangle W_2PQ_{12} = \beta$



Bobillier's Theorem

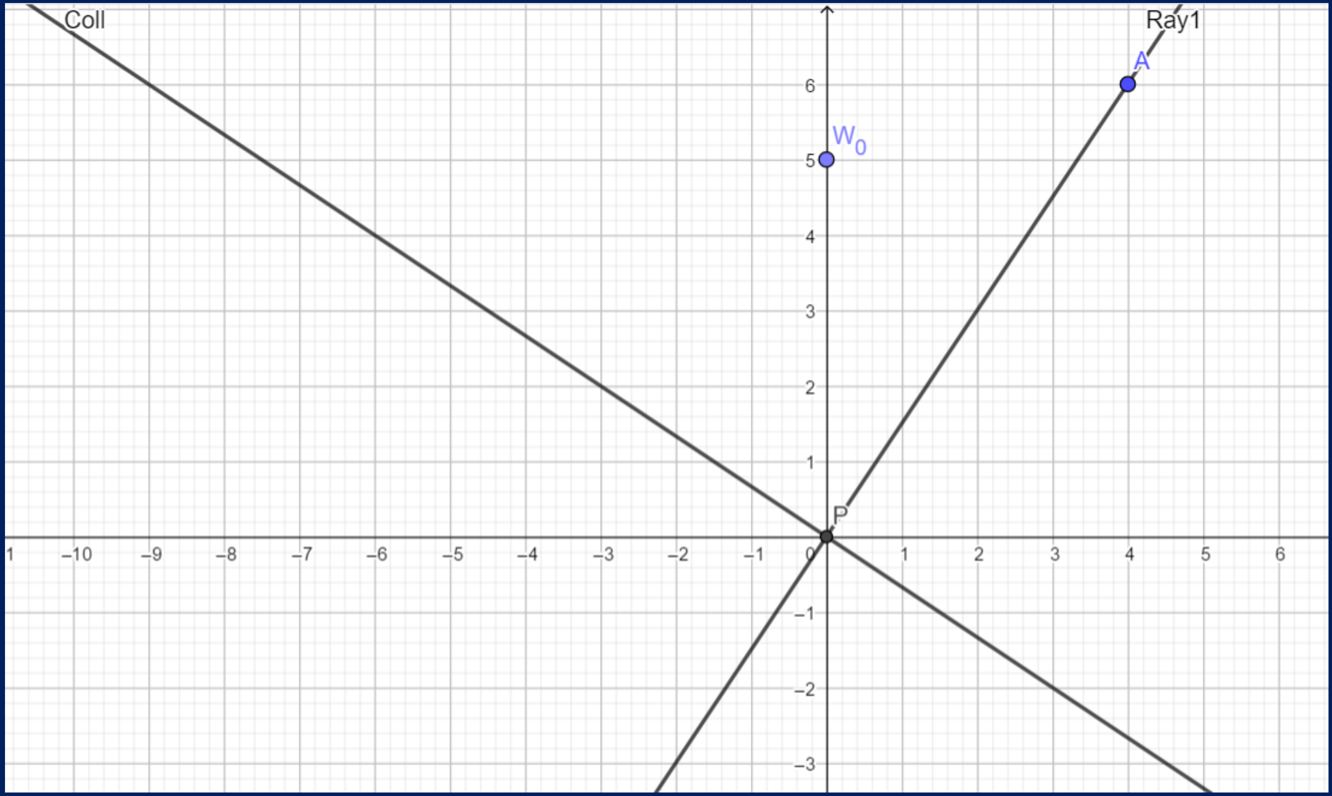
Please note that the location of collineation axis does not depend on selection of points A_1 and A_2 but on the location of inflection points W_1 and W_2, therefore on the inflection circle and instantaneous motion of the moving plane.

Application Examples of Bobillier's Theorem

1. Known pole, P, pole tangent, T, inflection pole W_0, for a point A on moving plane determine its center of curvature C_A.

PA is ray 1, PW_0 is ray 2.

PT is ⊥ PN so collineation axis is ⊥ PA

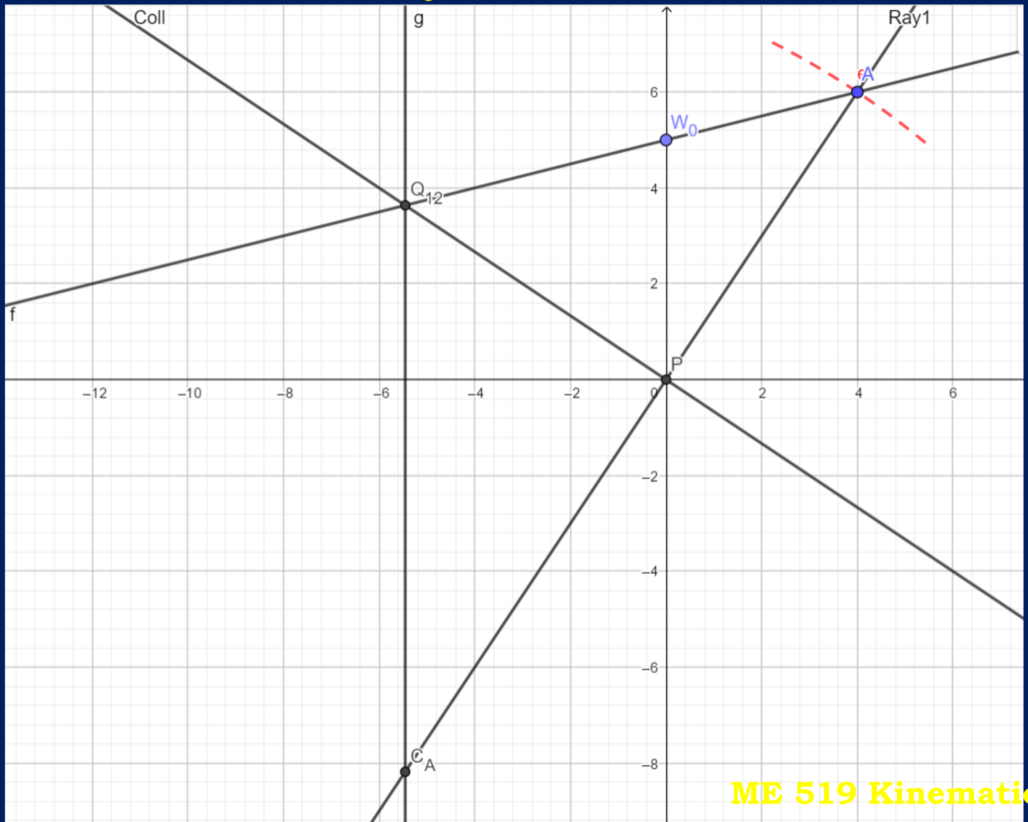


Application Examples of Bobillier's Theorem

1. Known pole, P, pole tangent, T, inflection pole W_0, for a point A on moving plane determine its center of curvature C_A (cont'ed)

W_0 has its center of curvature at ∞ along N (ray 2), AW_0 intersects collineation axis at Q_{12}.

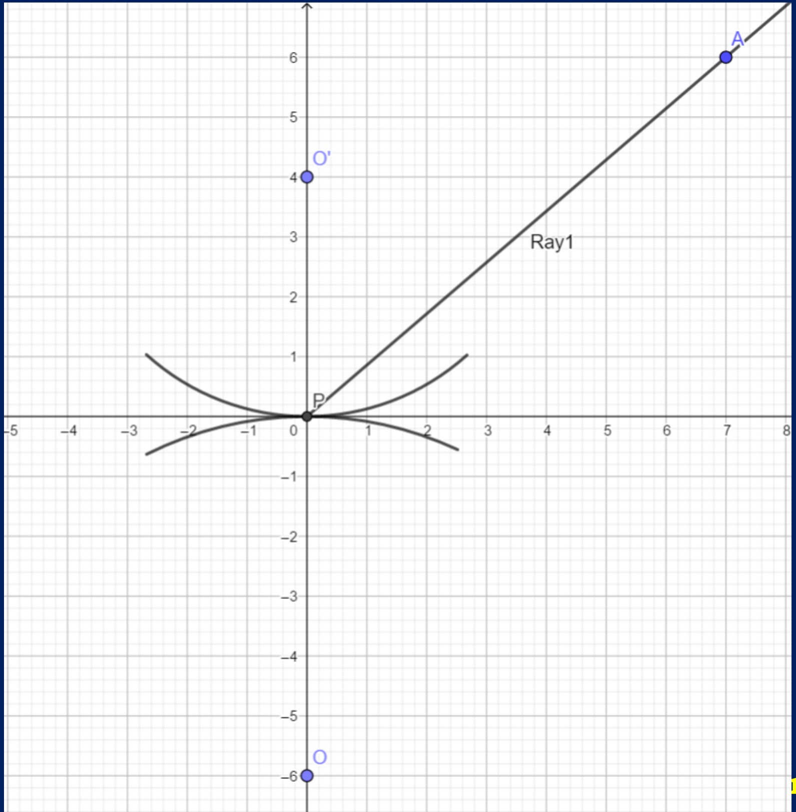
A line ∥ PN (which is towards C_{W_0}) through Q_{12} passes through C_A.



Application Examples of Bobillier's Theorem

2. Known pole, P, pole tangent, T, and conjugate point pairs on pole normal (OO'), for a point A on moving plane determine its center of curvature C_A. This corresponds to known radii of curvature of centrodes (e.g. planetary gear trains).

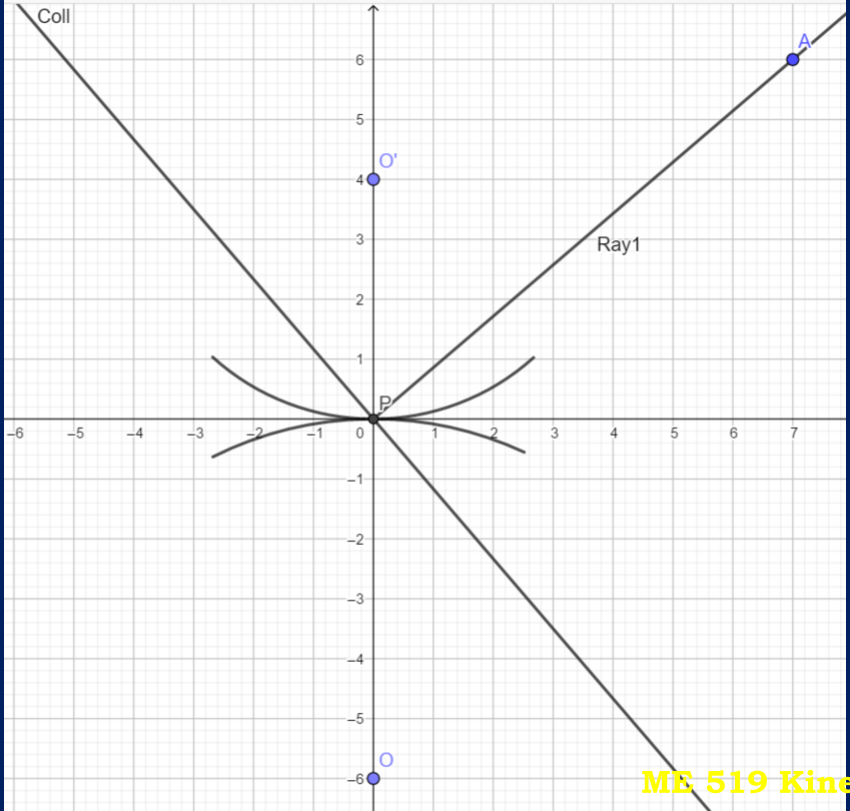
AP is on Ray 1, OO' defines PN (Ray 2).



Application Examples of Bobillier's Theorem

2. Known pole, P, pole tangent, T, and conjugate point pairs on pole normal (OO_2'), for a point A on moving plane determine its center of curvature C_A. This corresponds to known radii of curvature of centrodes (e.g. planetary gear trains) (Cont'ed).

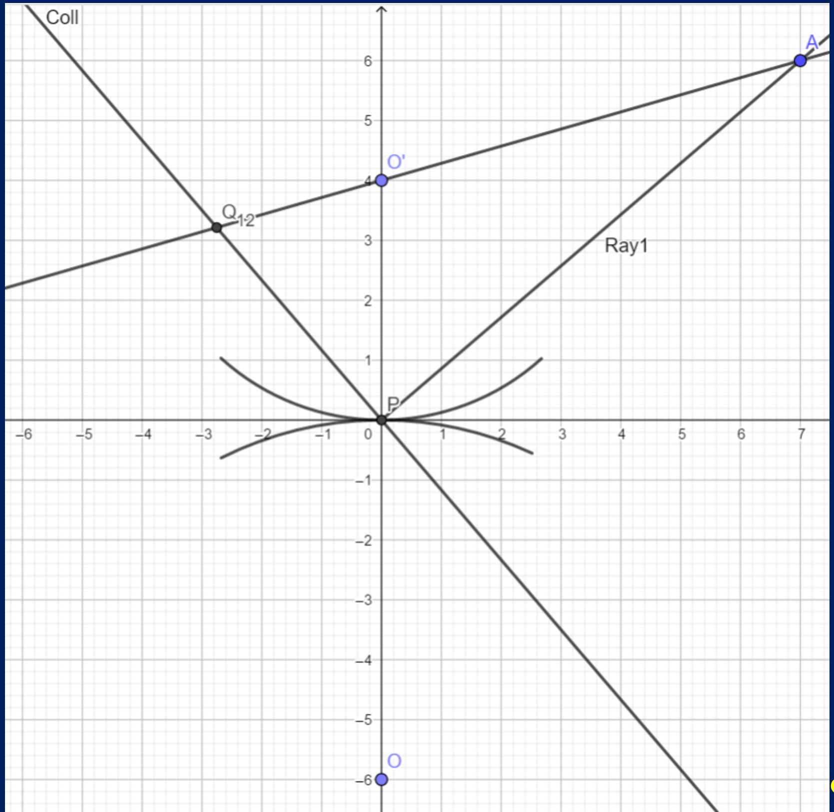
Collineation axis is ⊥ Ray 1, since PT is ⊥ Ray 2.



Application Examples of Bobillier's Theorem

2. Known pole, P, pole tangent, T, and conjugate point pairs on pole normal (OO_2'), for a point A on moving plane determine its center of curvature C_A. This corresponds to known radii of curvature of centrodes (e.g. planetary gear trains) (Cont'ed).

Draw AO', intersection with collineation axis is Q_{12}.



Application Examples of Bobillier's Theorem

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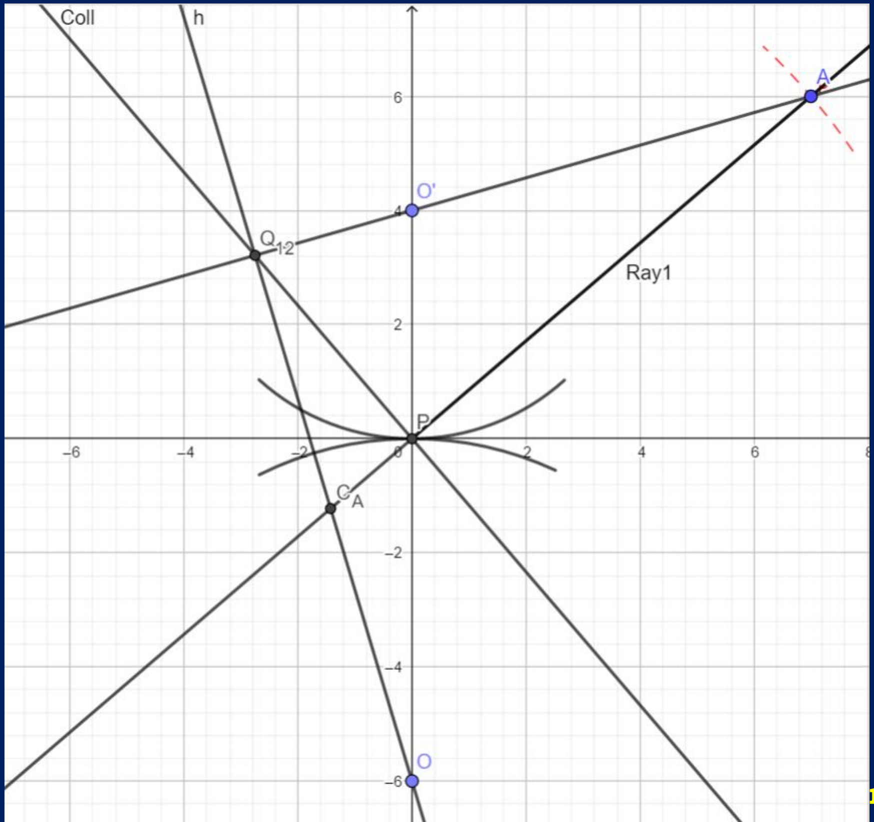
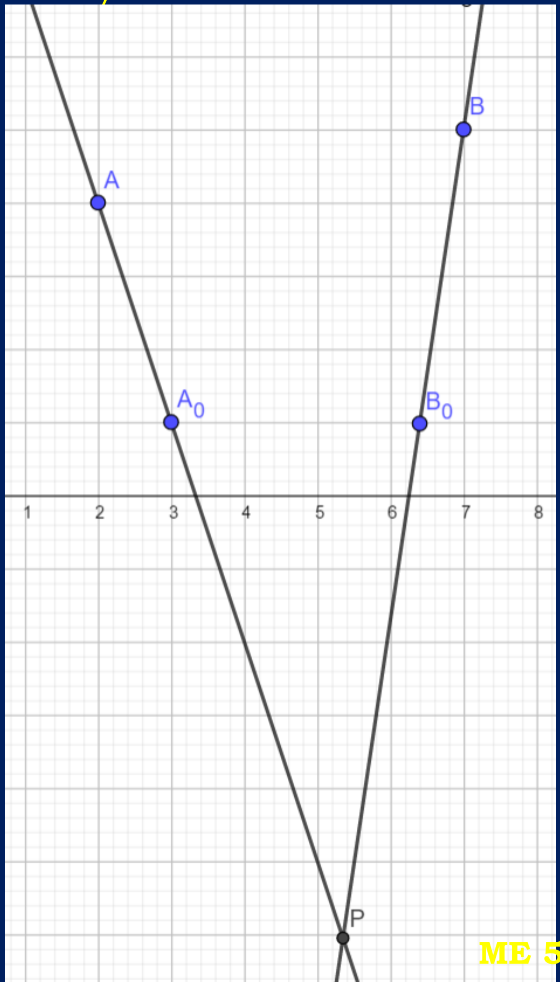
Since O is center of curvature of O' intersection of $Q_{12}O$ with Ray 1 is C_A.



Application Examples of Bobillier's Theorem

3. Known two conjugate points (like AA_0 and BB_0 for a four-bar) on two distinct pole rays. Determine center of curvature for another point, E, on the moving plane.

AA_0 and BB_0 form two pole rays therefore P is at the intersection.



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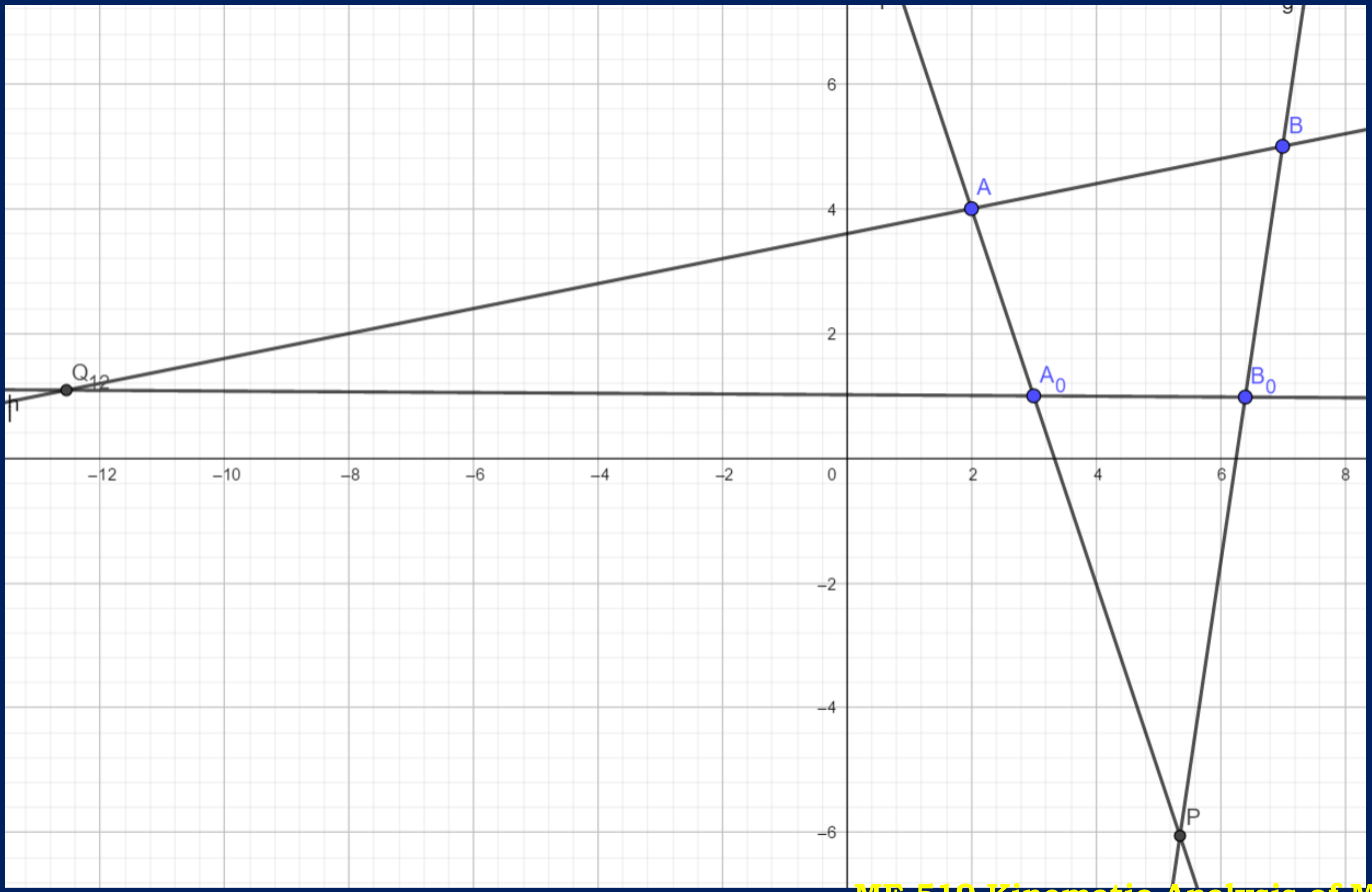
Q_{AB} is at the intersection of AB and A_0B_0.



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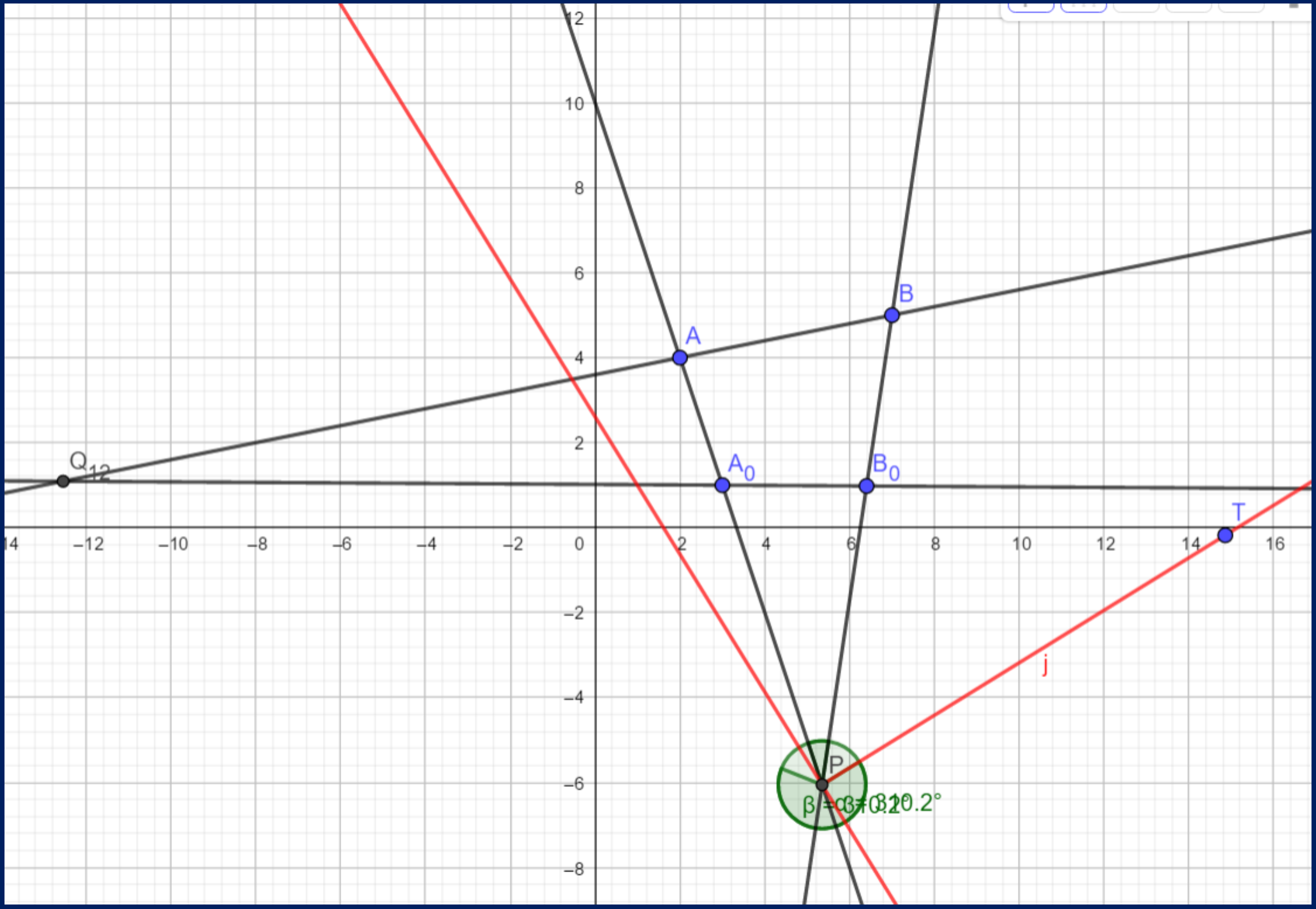
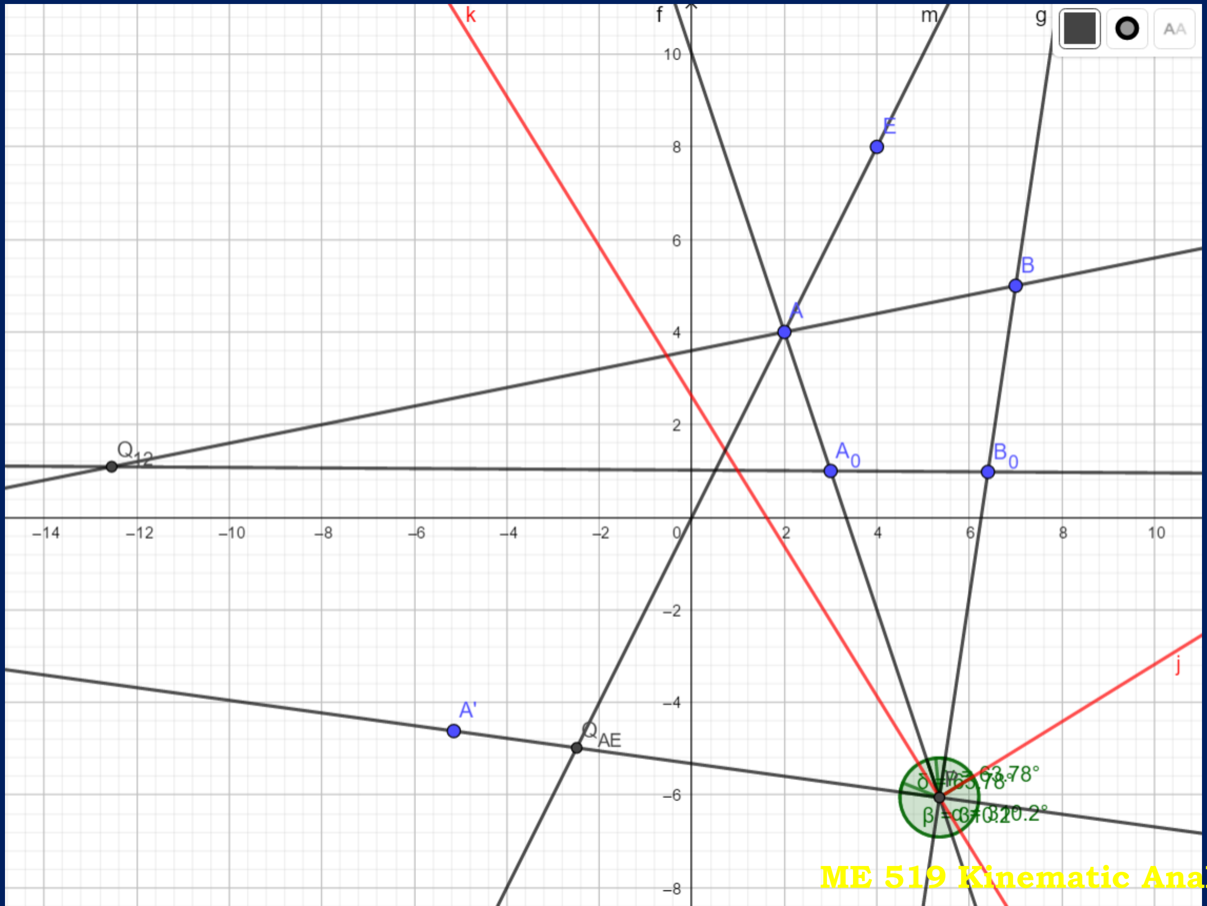
Bobiller's theorem states $\sphericalangle Q_{AB}PA = \sphericalangle BPT = \alpha$



Application Examples of Bobillier's Theorem

3. Known two conjugate points (like AA_0 and BB_0 for a four-bar) on two distinct pole rays. Determine center of curvature for another point, E, on the moving plane.

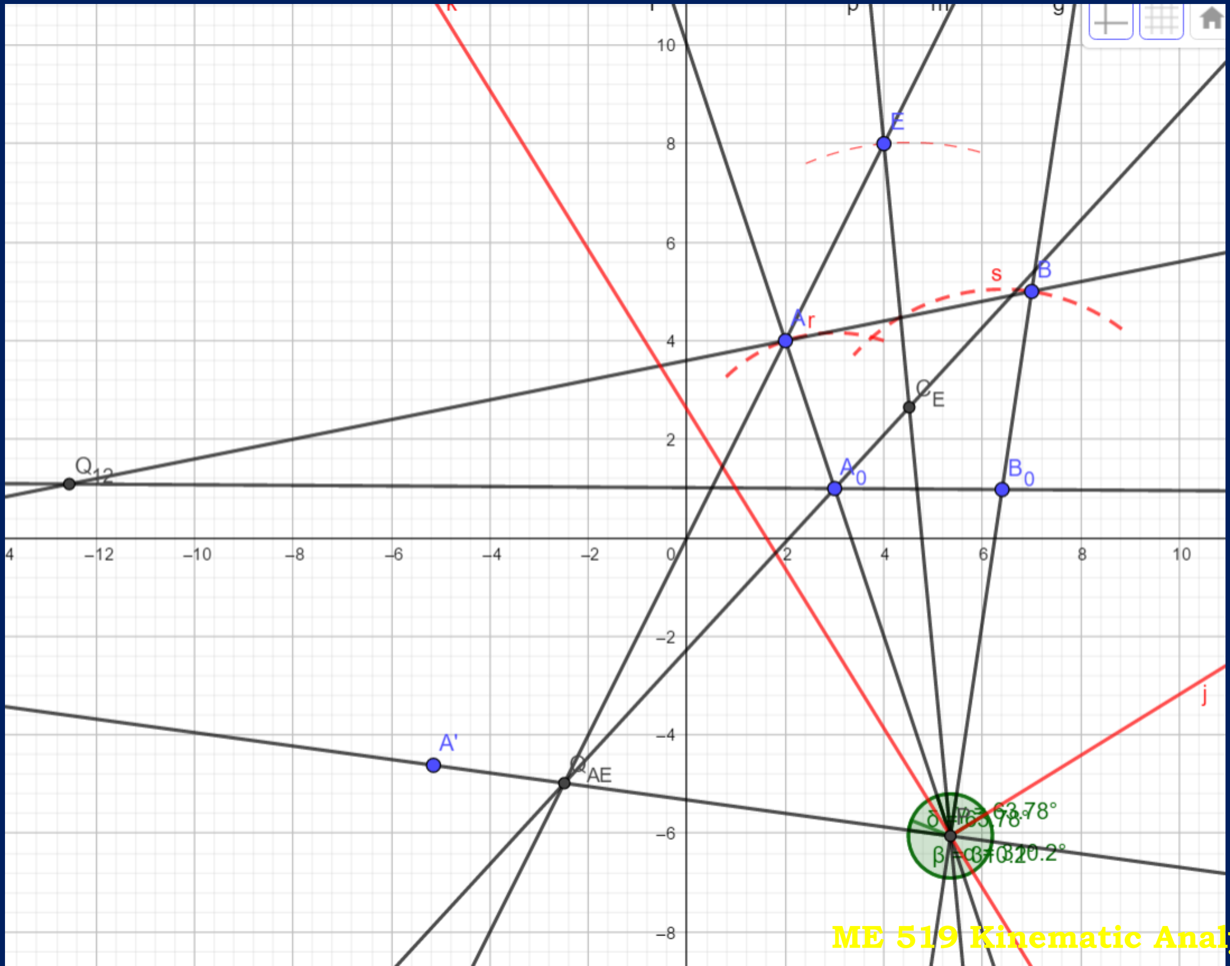
Bobiller's theorem states $\measuredangle TPE = \measuredangle APQ_{AE} = \beta$ Q_{AE} is at the intersection of PQ_{AE} and AE.



Application Examples of Bobillier's Theorem

3. Known two conjugate points (like AA_0 and BB_0 for a four-bar) on two distinct pole rays. Determine center of curvature for another point, E, on the moving plane.

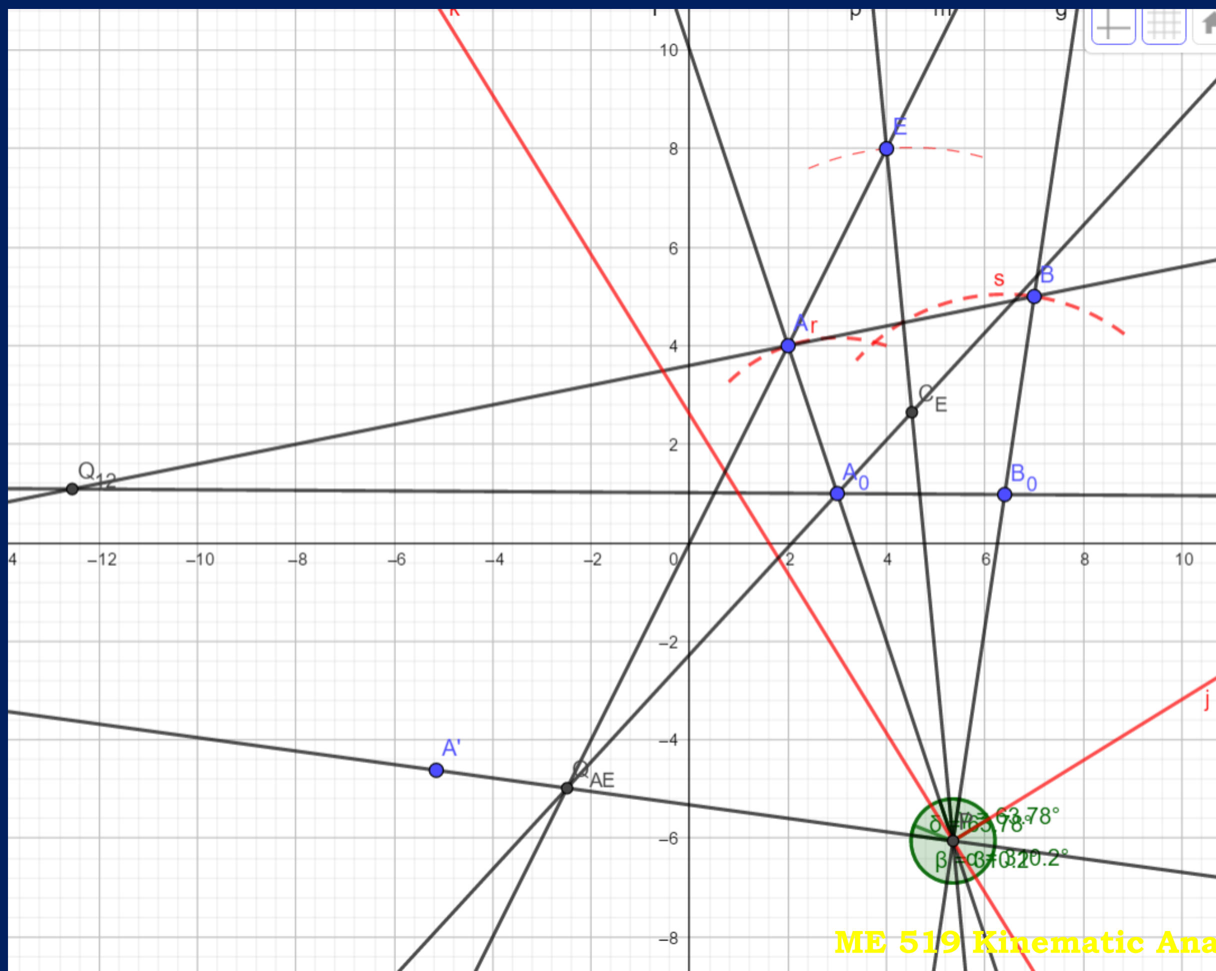
Draw $Q_{AE}A_O$. Intersection with ray PE yields C_E.



Application Examples of Bobillier's Theorem

3. Known two conjugate points (like AA_0 and BB_0 for a four-bar) on two distinct pole rays. Determine center of curvature for another point, E, on the moving plane.

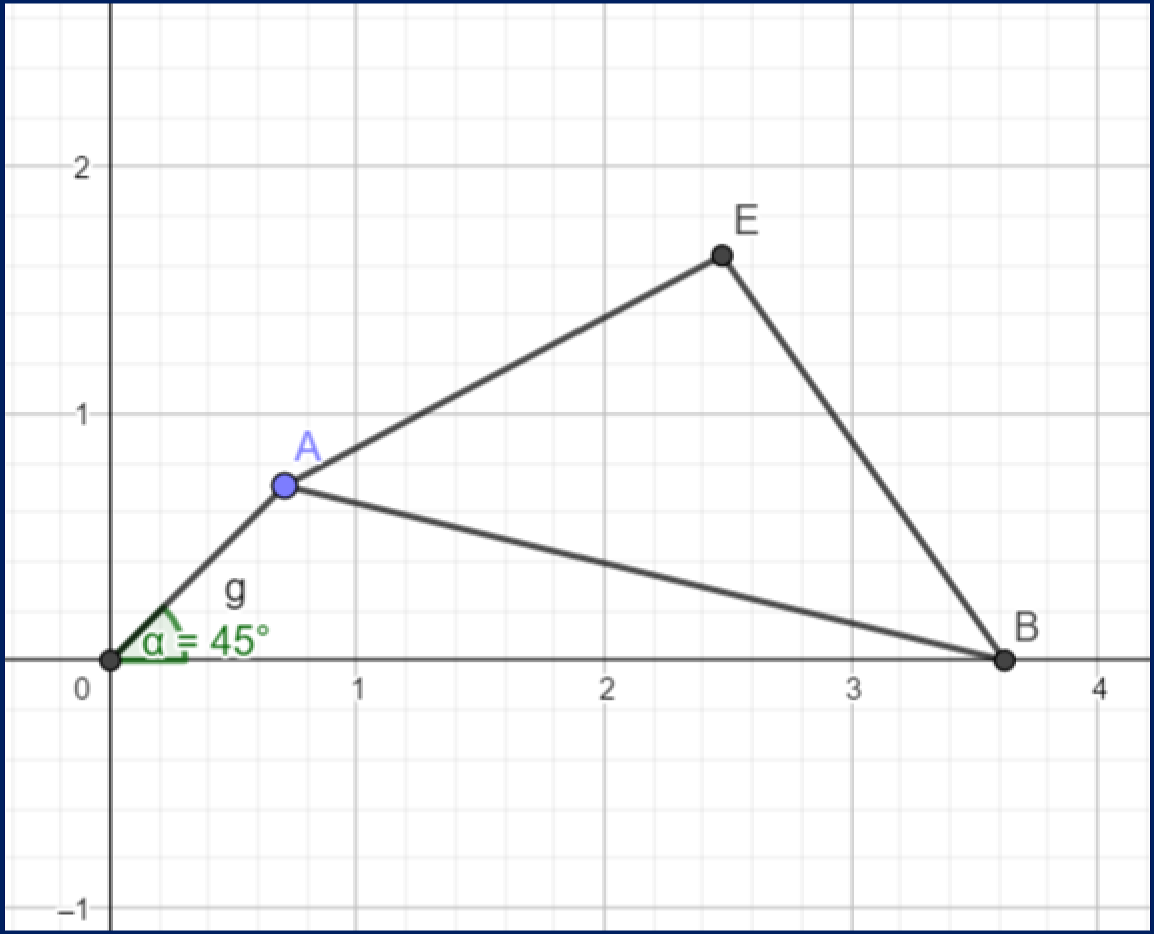
The same result would be obtained if you use B and B_0 instead of A and A_0.



Bobillier's Theorem

Example:

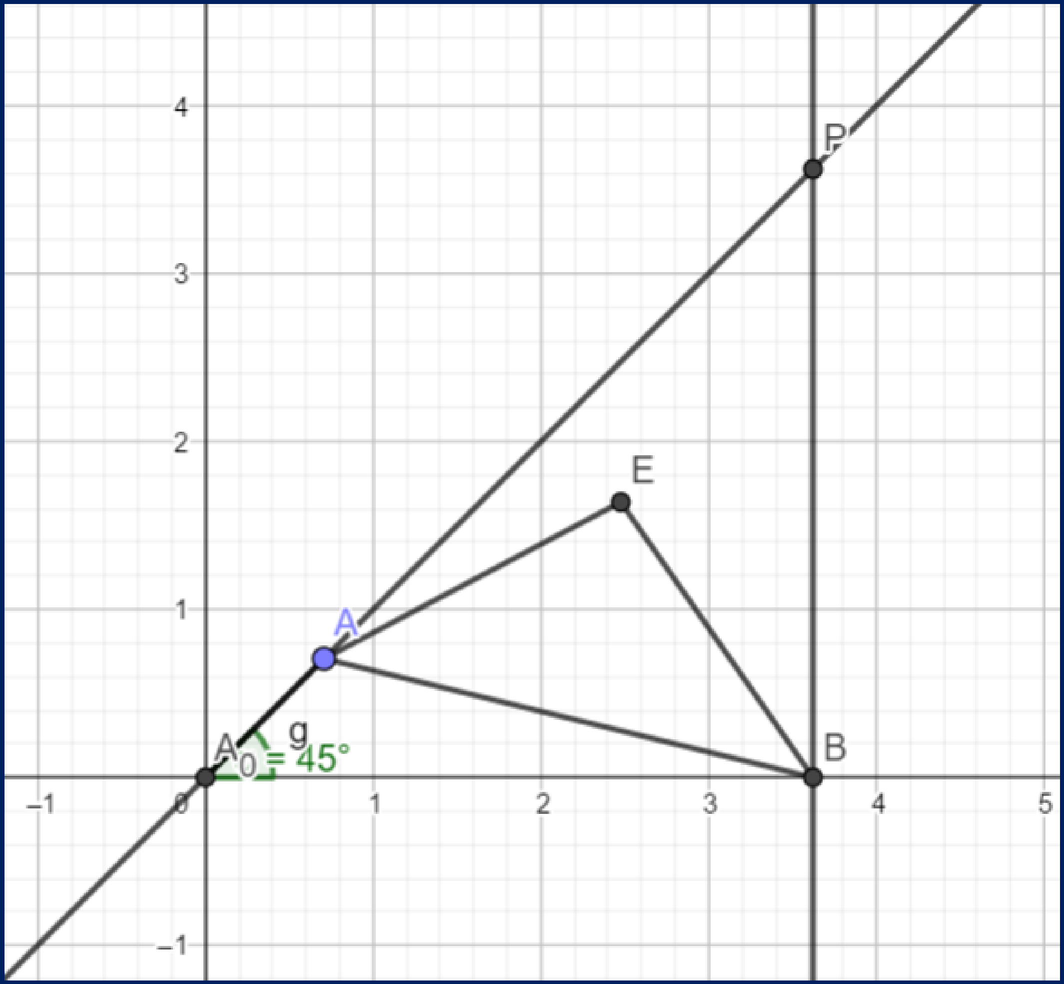
Determine the center of curvature of point E on the coupler of the slider-crank mechanism.



Bobillier's Theorem

Example:

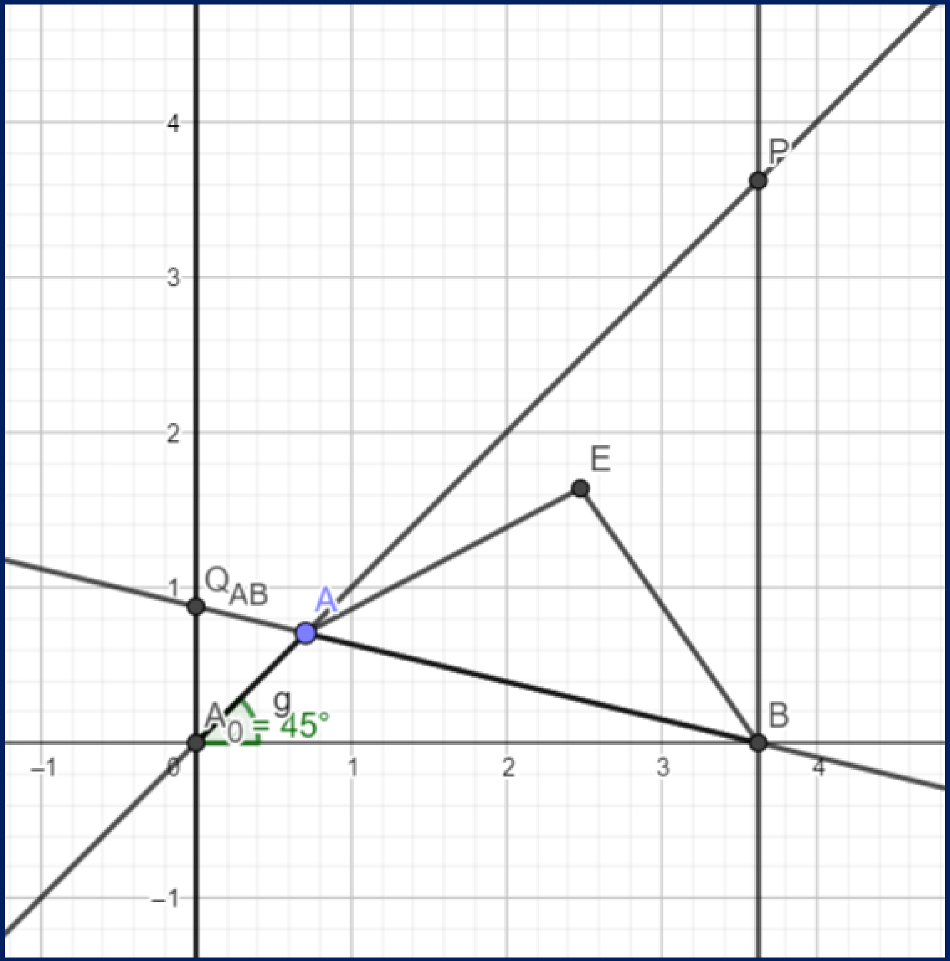
A_0A and B_0B are two pole rays therefore intersection yields the pole.



Bobillier's Theorem

Example:

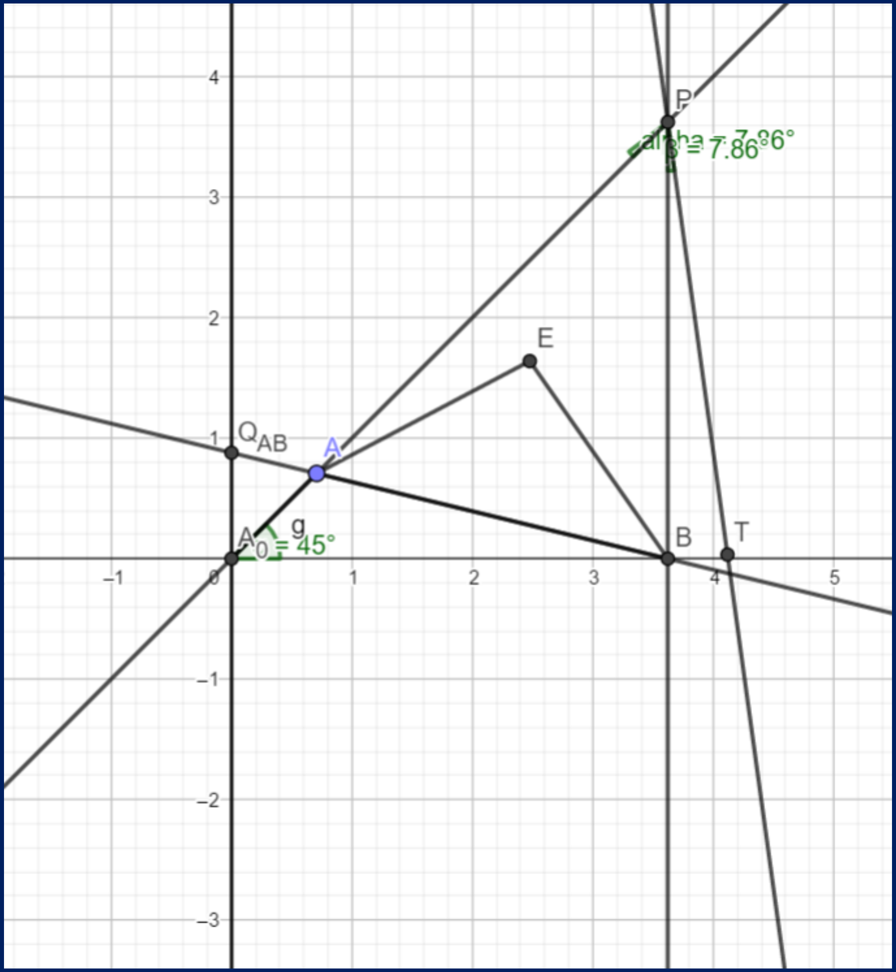
Q_{AB} is at the intersection of A_0B_0 and AB.



Bobillier's Theorem

Example:

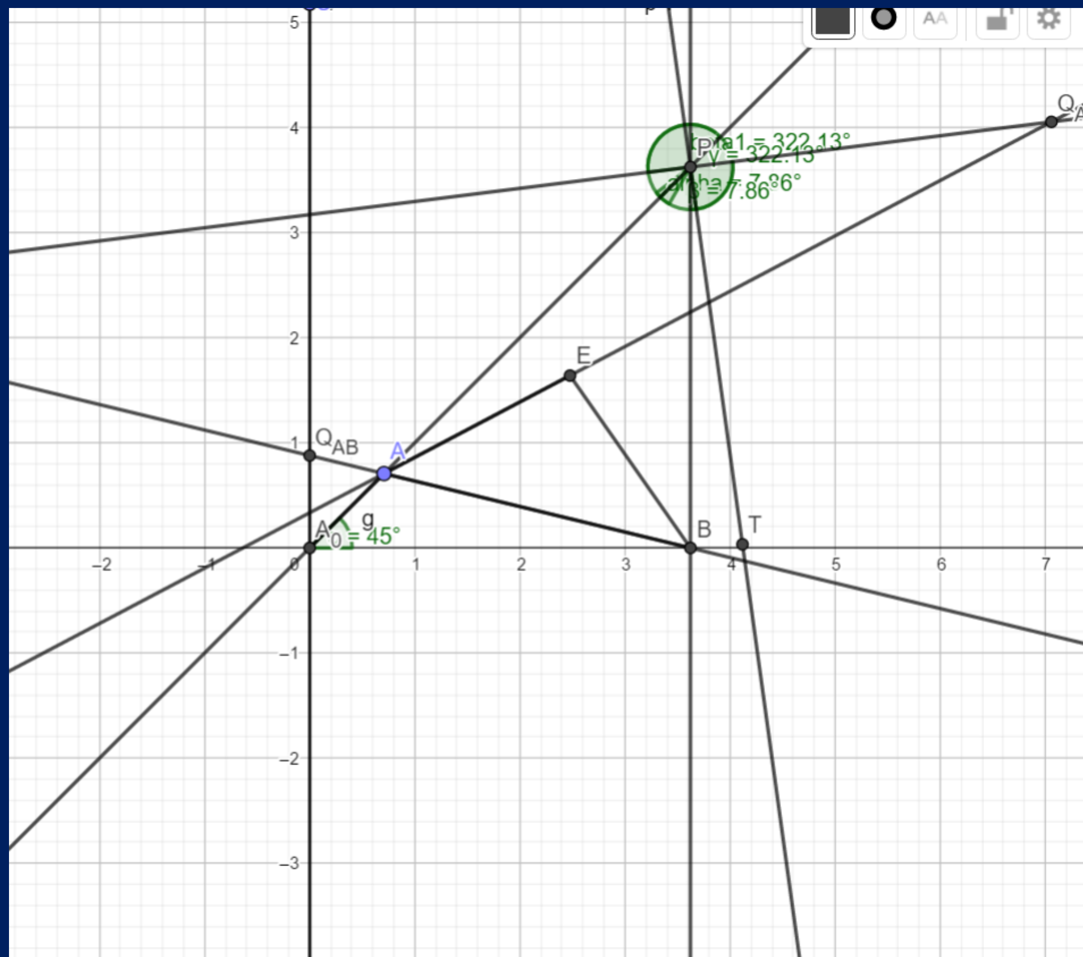
Bobiller's theorem states $\sphericalangle Q_{AB}PA = \sphericalangle BPT = \alpha.$



Bobillier's Theorem

Example:

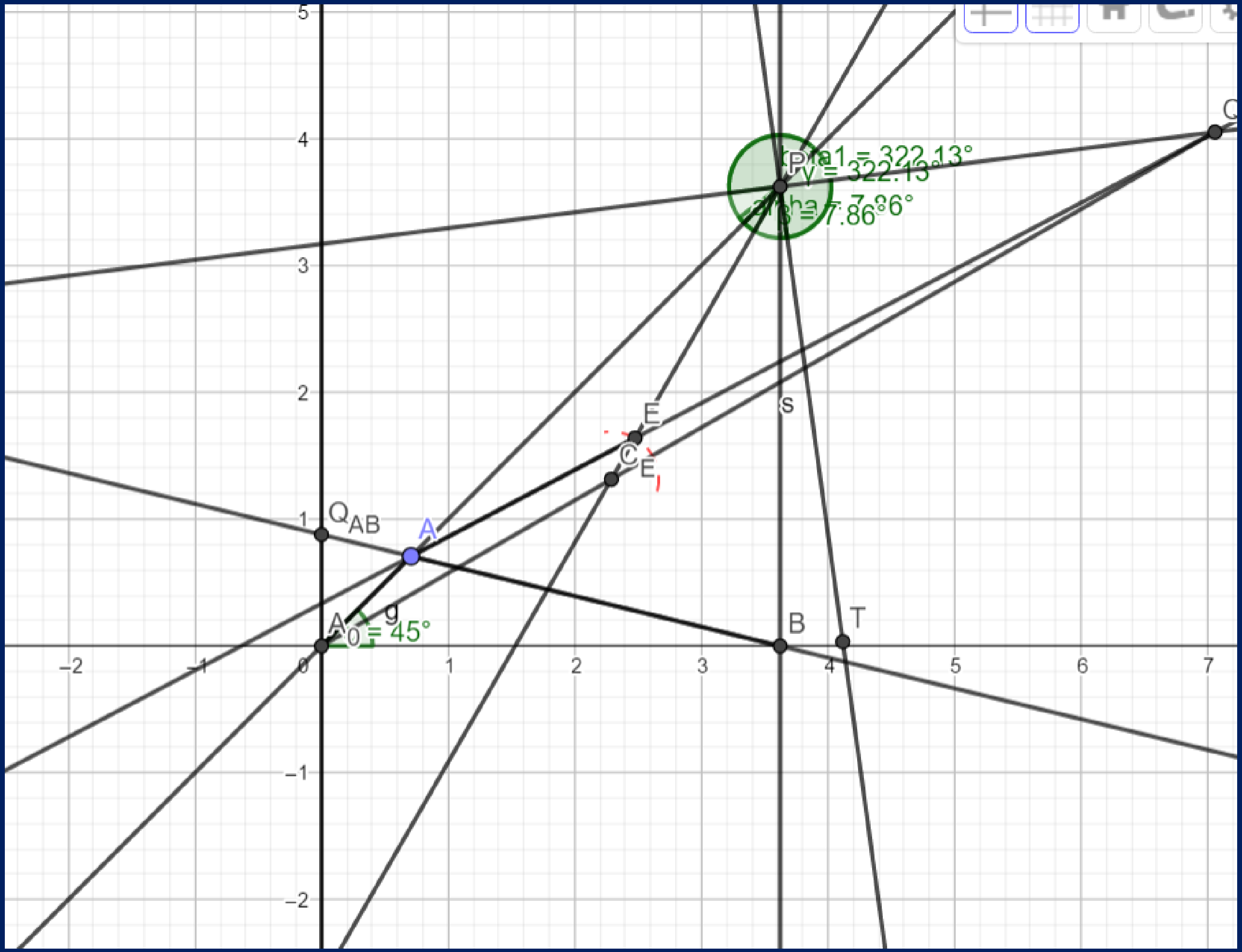
Bobiller's theorem states $\measuredangle TPE = \measuredangle APQ_{AE} = \beta$. Q_{AE} is the intersection of PQ_{AE} and AE.



Bobillier's Theorem

Example:

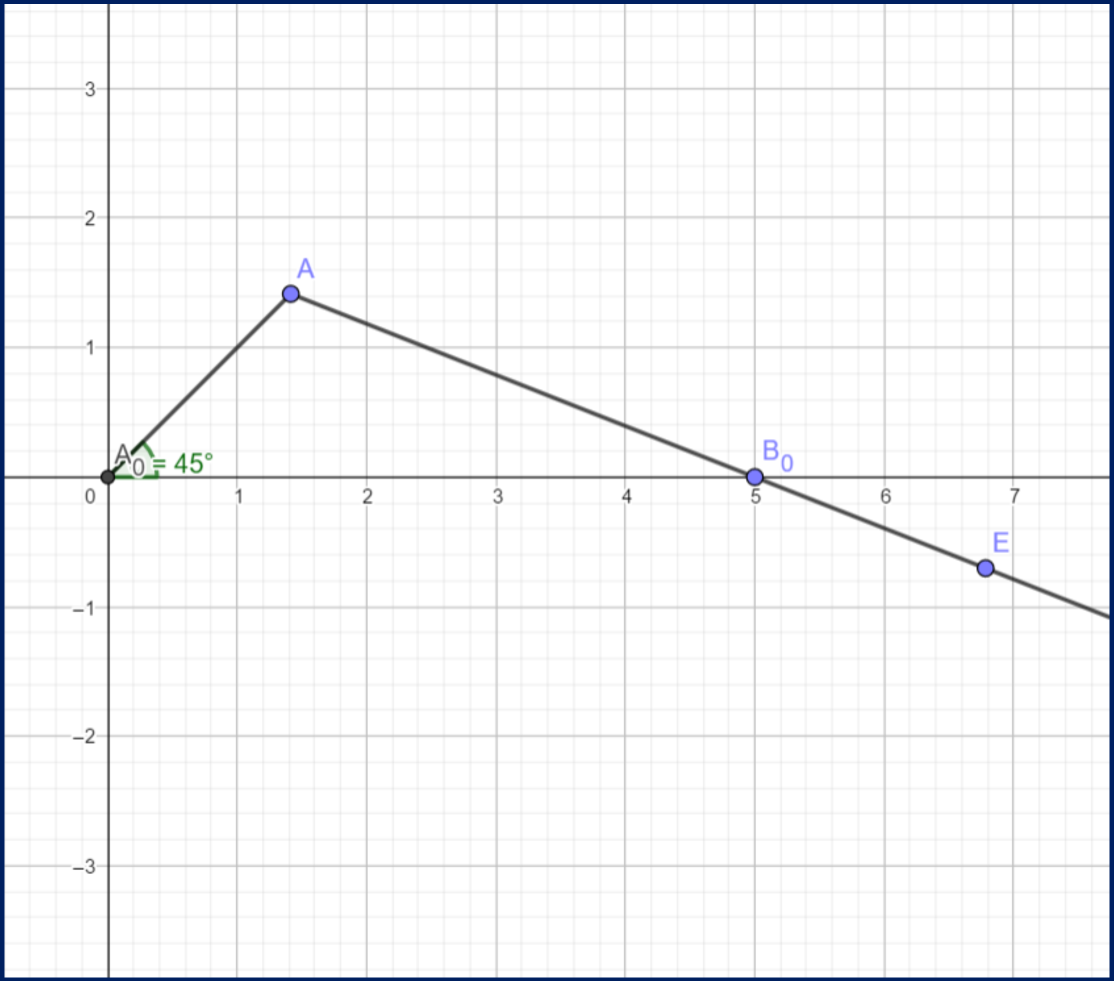
Draw $Q_{AE}A_0$ intersection of pole ray PE yields C_E.



Bobillier's Theorem

Example:

Determine the center of curvature of point E on the coupler of the inverted slider-crank mechanism.

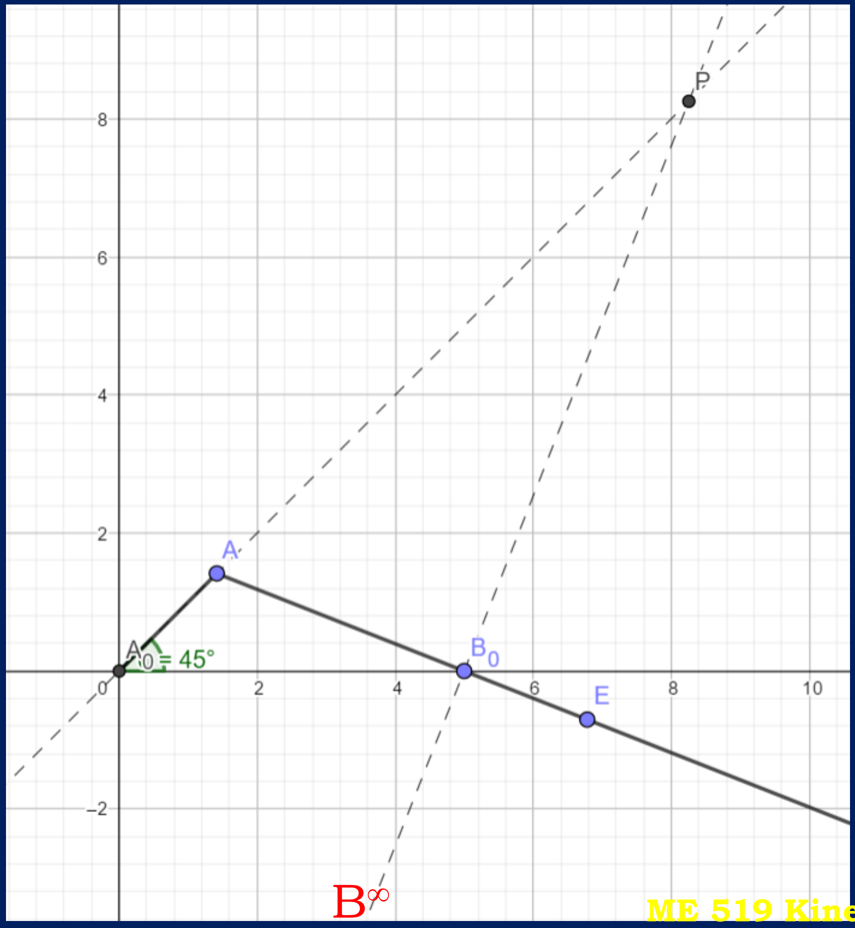


Bobillier's Theorem

Example:

Determine the center of curvature of point E on the coupler of the inverted slider-crank mechanism.

A_0A and B_0B are two pole rays therefore P is at the intersection.

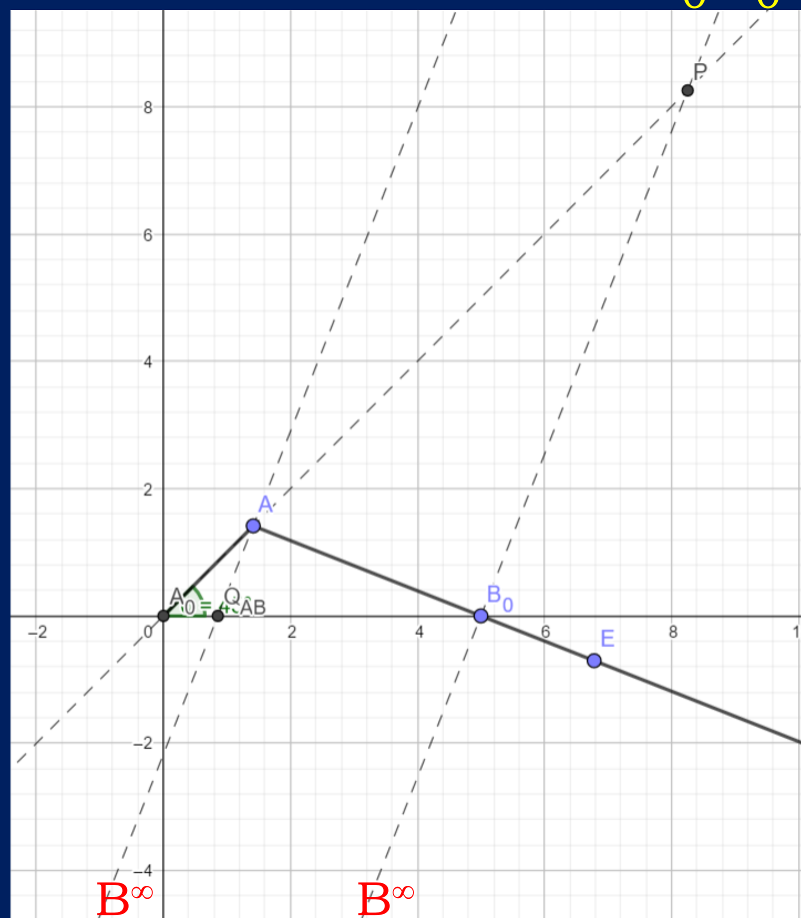


Bobillier's Theorem

Example:

Determine the center of curvature of point E on the coupler of the inverted slider-crank mechanism.

$Q_{AB}A$ is at the intersection of AB and A_0B_0.

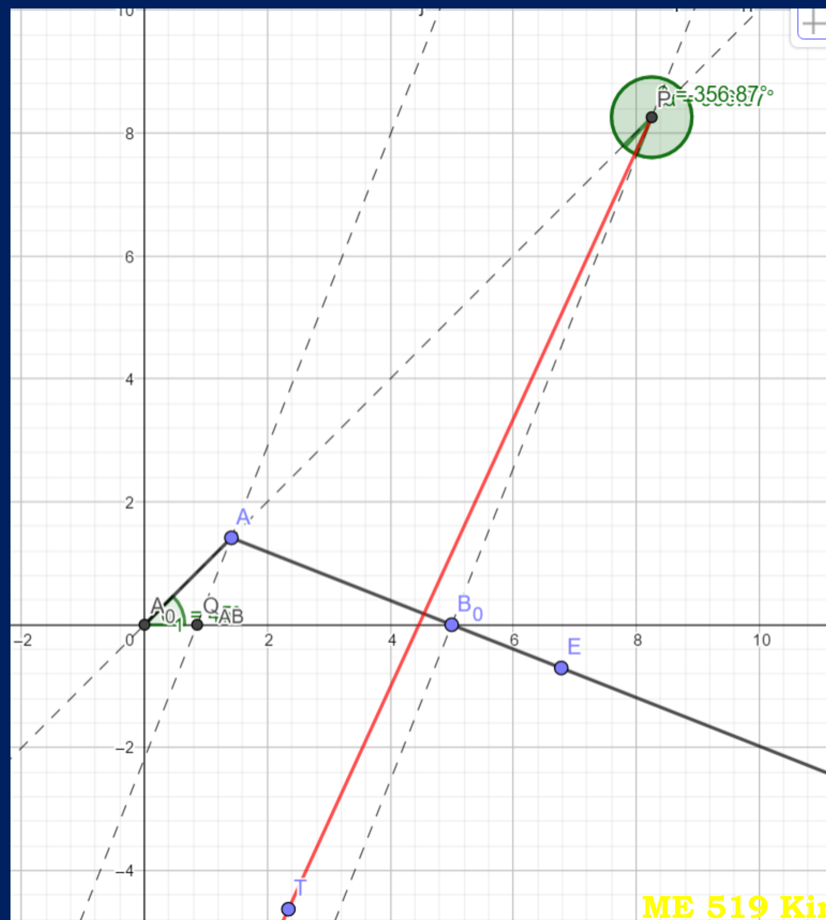


Bobillier's Theorem

Example:

Determine the center of curvature of point E on the coupler of the inverted slider-crank mechanism.

Bobiller's theorem states $\sphericalangle Q_{AB}PA = \sphericalangle BPT = \alpha$.



Bobillier's Theorem

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Determine the center of curvature of point E on the coupler of the inverted slider-crank mechanism.

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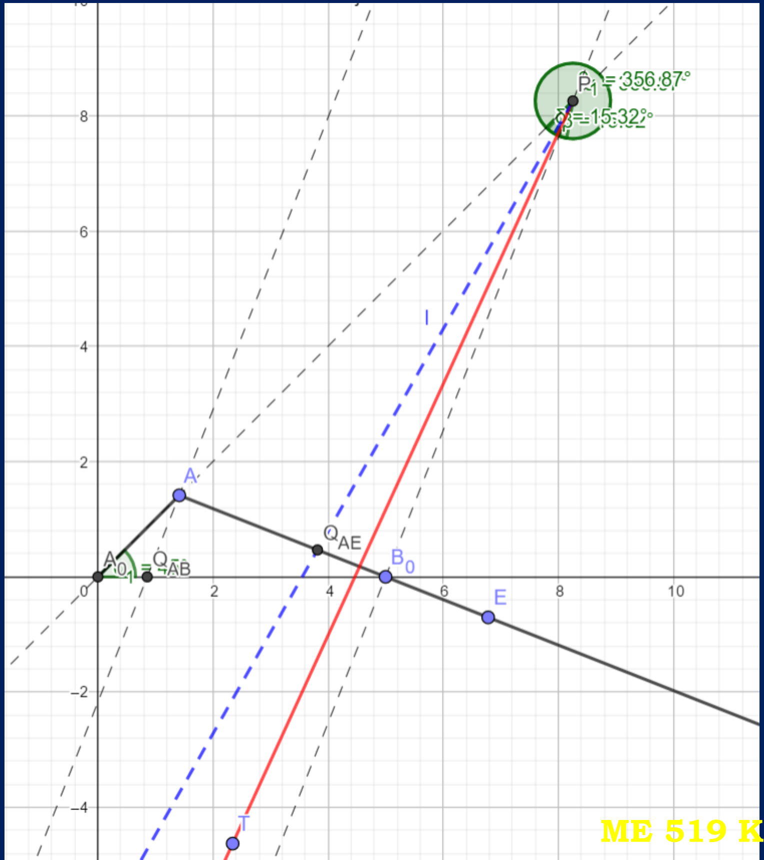

Bobillier's Theorem

Example:

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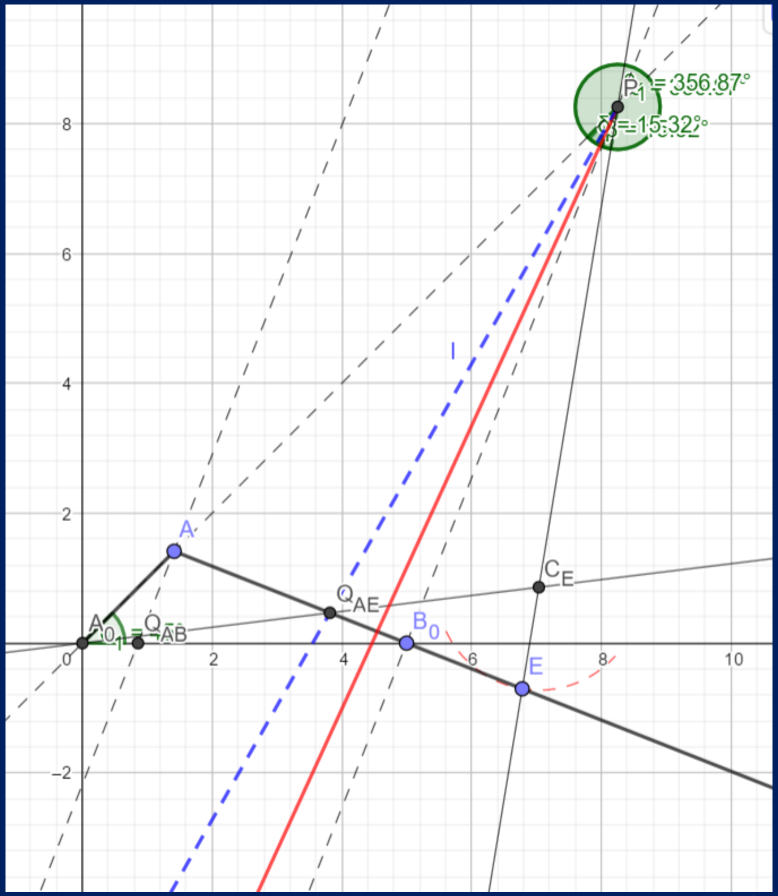
Draw $Q_{AE}A_0$ intersection with PE yields C_E.



Bobillier's Theorem

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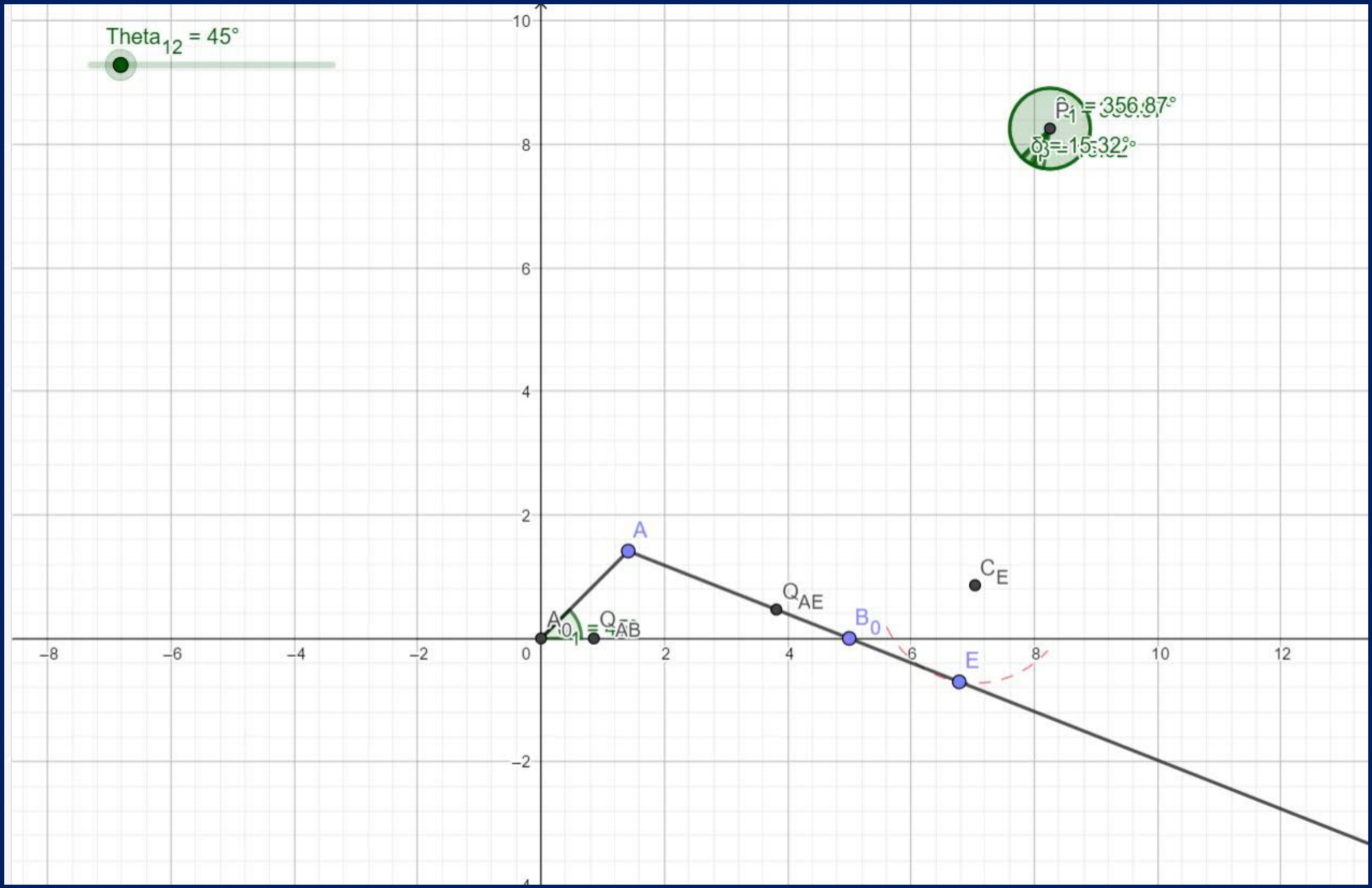


Bobillier's Theorem

Example:

Long period pendulum (revisited):

$|A_0A| = 2\ cm$

$|A_0B_0| = 5\ cm$

$|MA| = 8\ cm$

For ψ = 90° Euler-Savary equation becomes

$$\frac{1}{\delta sin\psi} = \frac{1}{r_W} = \frac{1}{r_A} - \frac{1}{r_{A_0}}$$

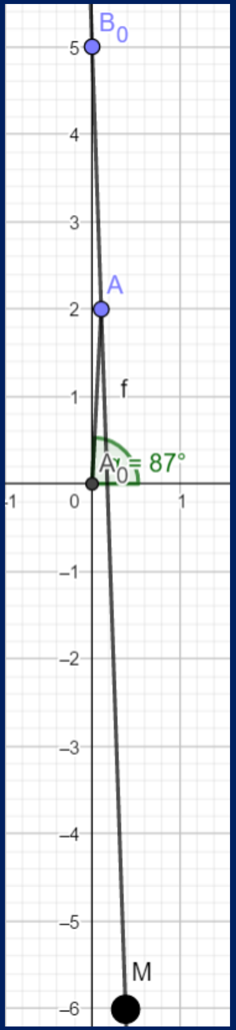
Please recognize P is B_0 for symmetry position

$r_A = 3\ cm, r_{A_0} = 5\ cm\ \therefore r_W = \frac{15}{2}\ cm$

For point M

$$\frac{1}{r_W} = \frac{1}{r_M} - \frac{1}{r_{C_M}} = \frac{2}{15} = \frac{1}{11} - \frac{1}{r_{C_M}} \rightarrow r_{C_M} = -23.6\ cm$$

$\rho_M = 23.6 + 11 = 34.6\ cm$



Bobillier's Theorem

Dwell Mechanisms:

During dwell the output of the mechanism remains stationary for a certain motion of the input crank. The output may not be completely stationary but if very small this may be considered as dwell for many applications.

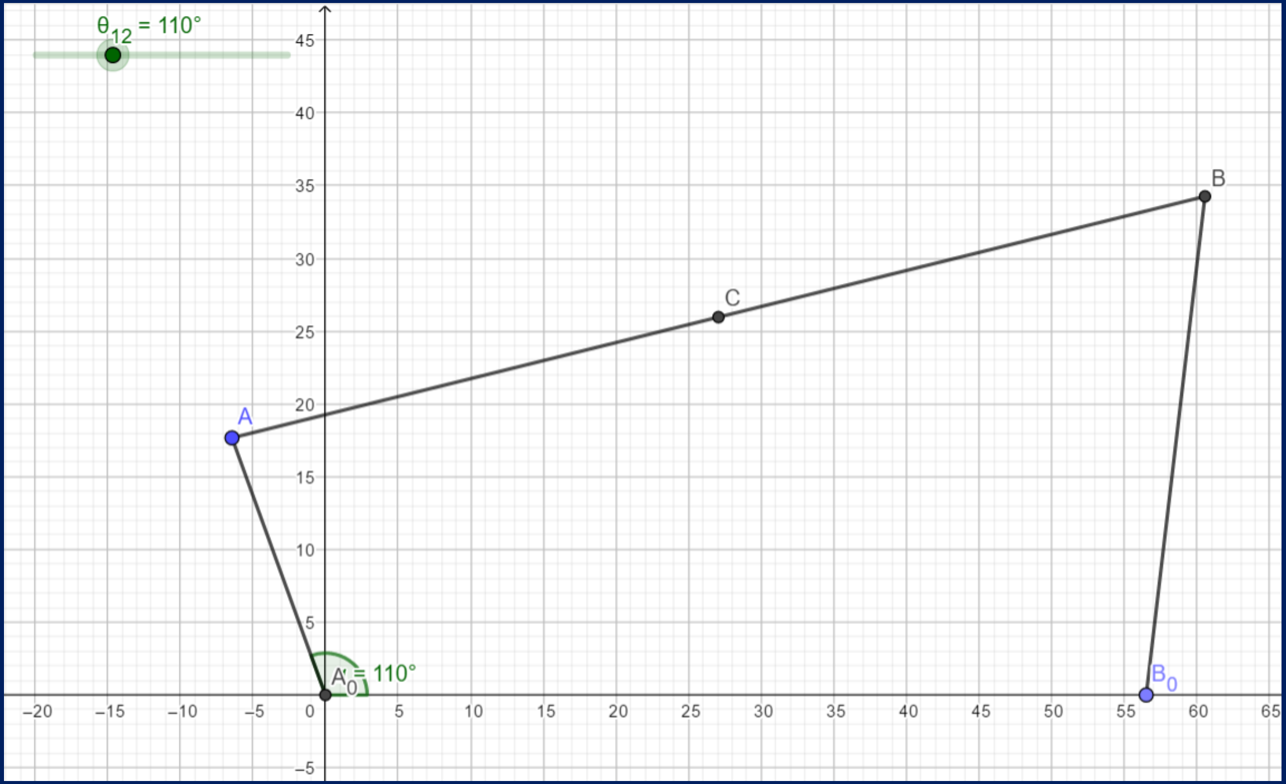
$|A_0B_0| = 56.3\ cm$

$|A_0A| = 18.8\ cm$

$|AB| = 69.0\ cm$

$|AC| = 34.5\ cm$

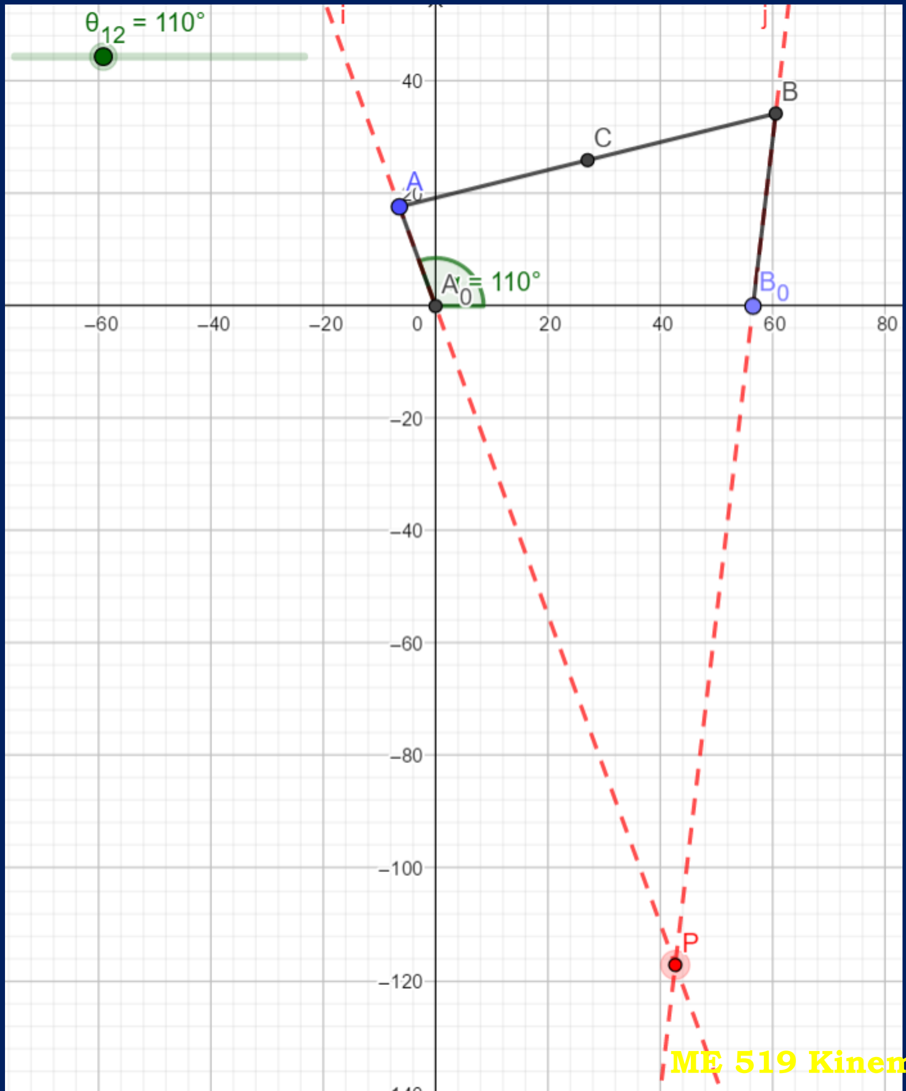
$|B_0B| = 34.5\ cm$



Bobillier's Theorem

Dwell Mechanisms:

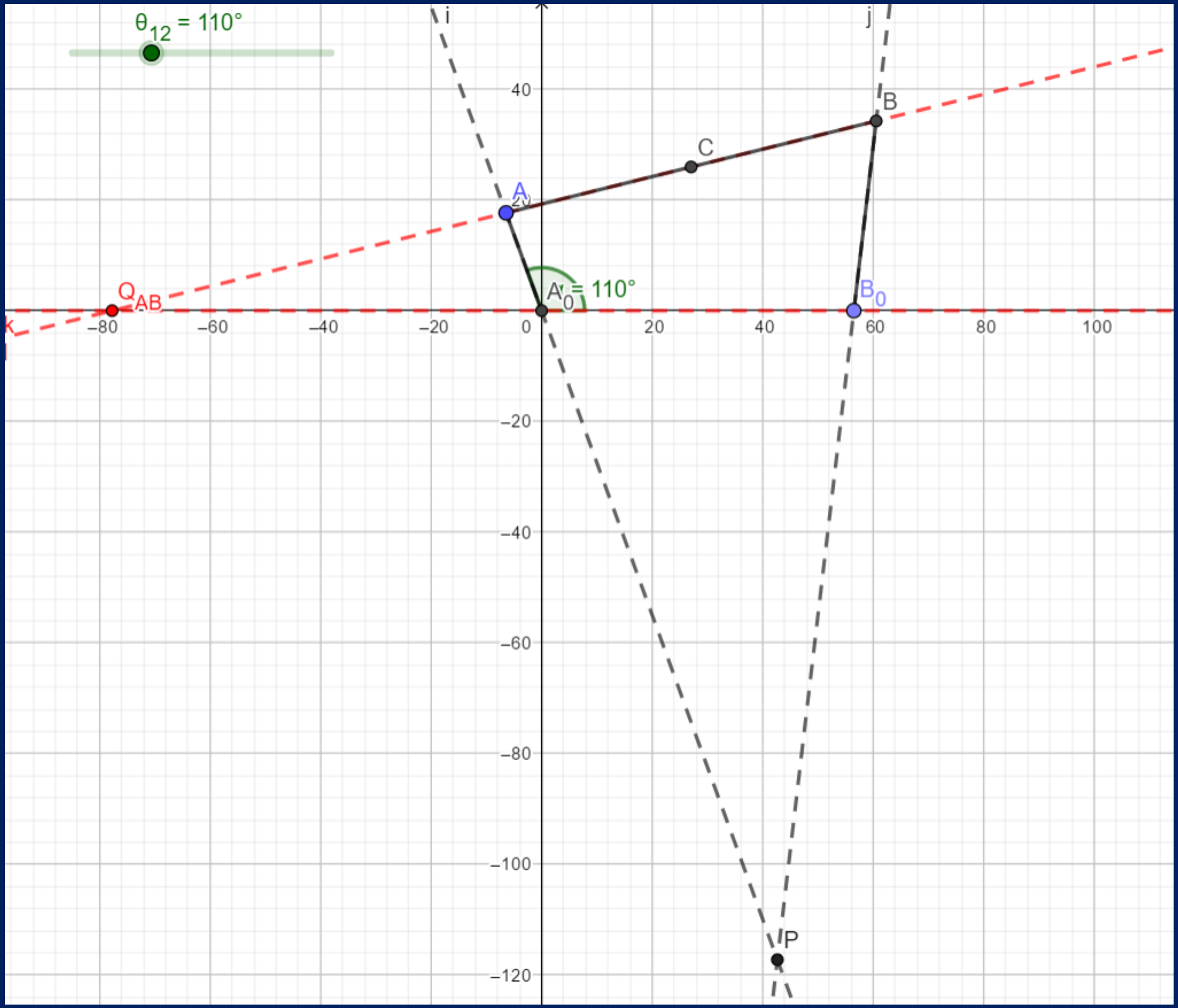
P is at the intersection of A_0A and B_0B.



Bobillier's Theorem

Dwell Mechanisms:

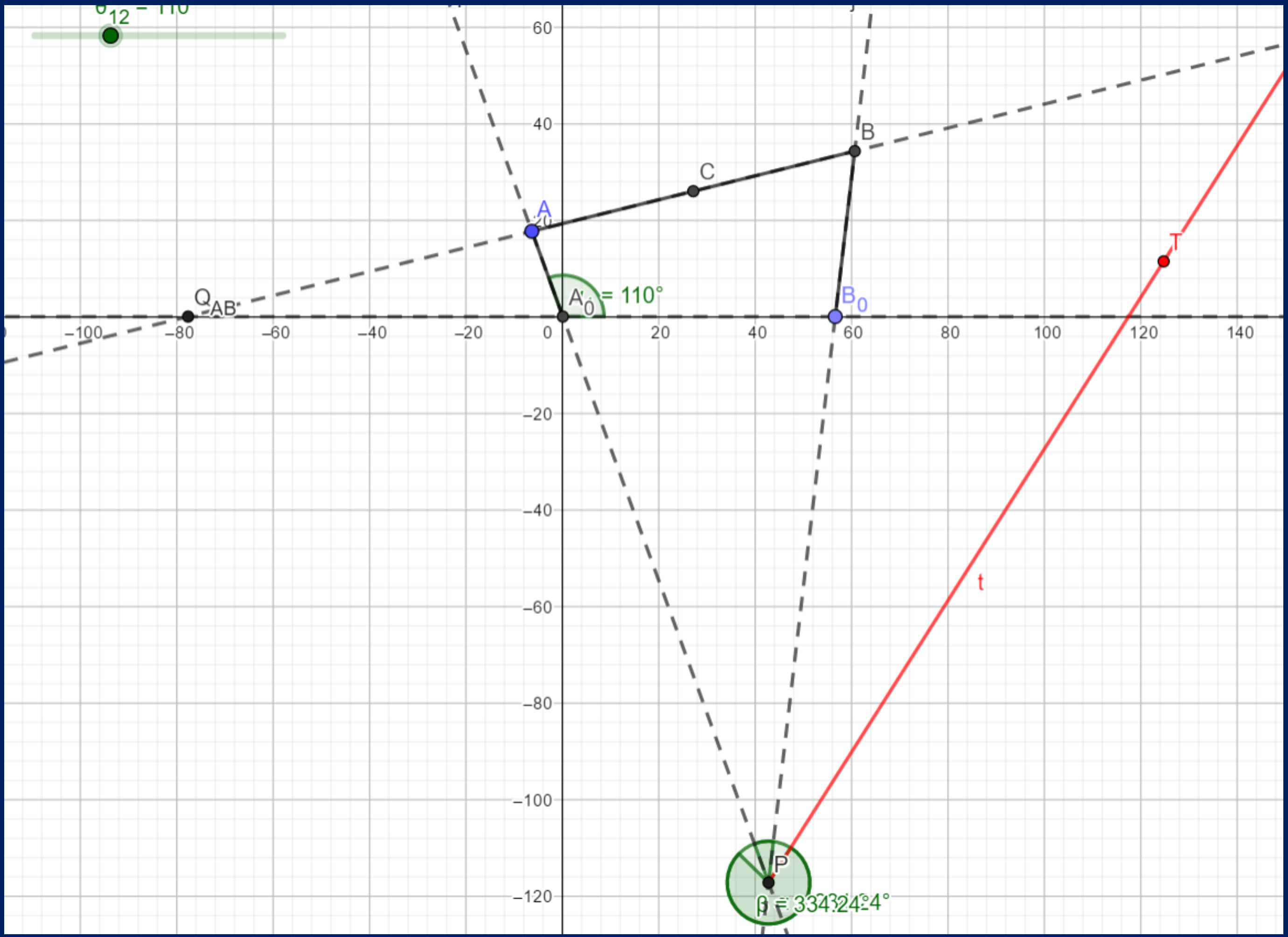
Q_{AB} is at the intersection of A_0B_0 and AB.



Bobillier's Theorem

Dwell Mechanisms:

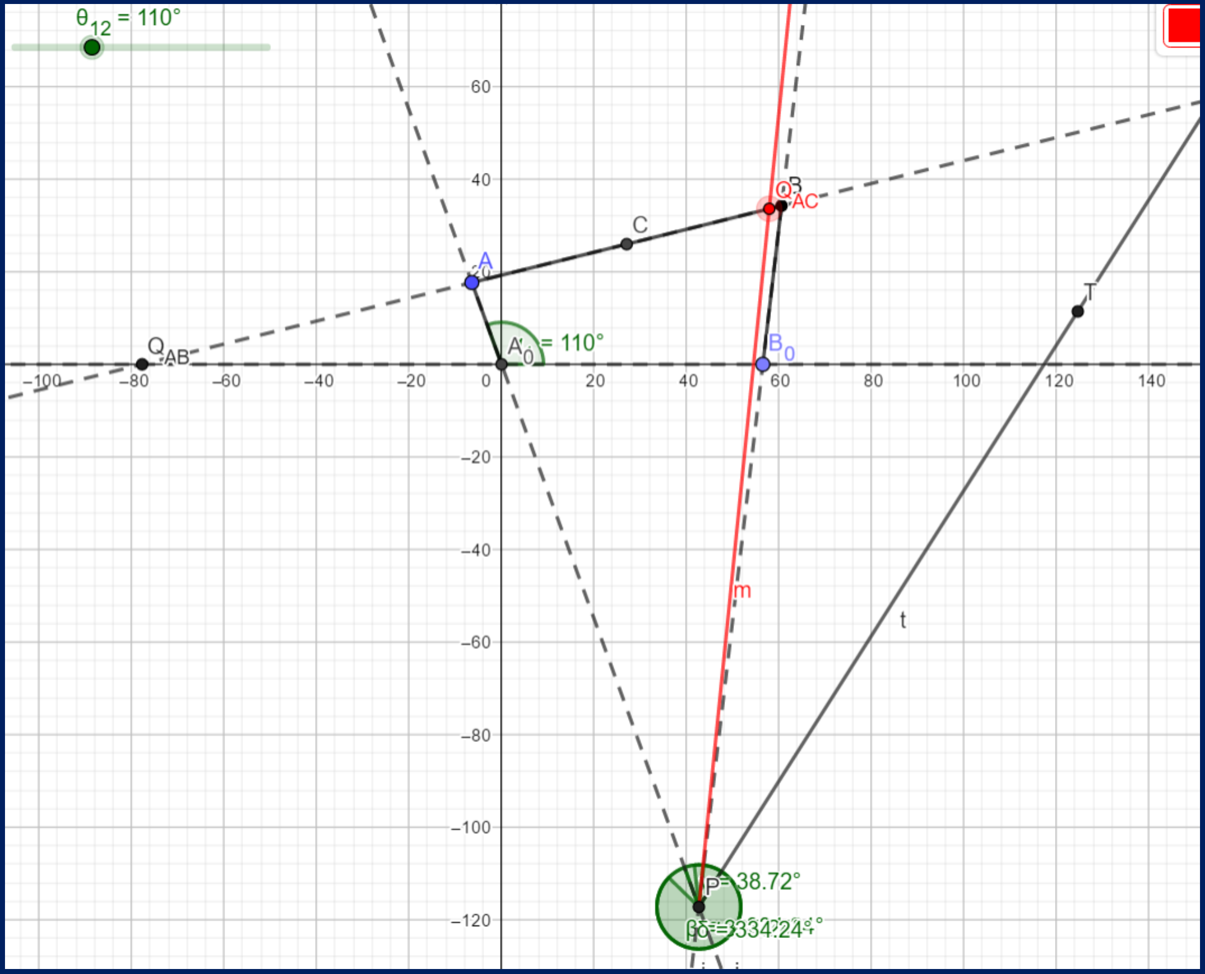
Bobiller's theorem states $\sphericalangle Q_{AB}PA = \sphericalangle BPT = \alpha$.



Bobillier's Theorem

Dwell Mechanisms:

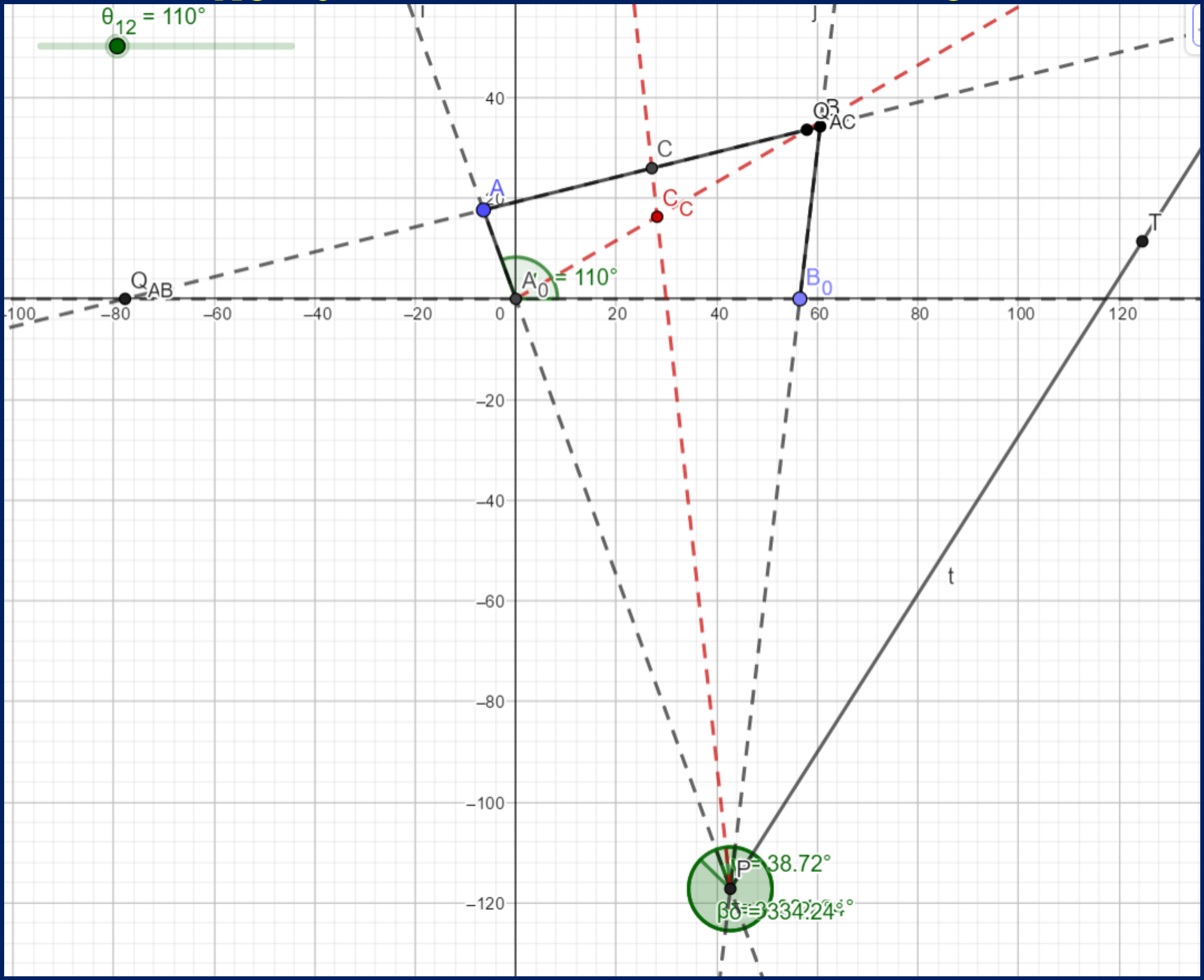
Bobiller's theorem states $\sphericalangle TPC = \sphericalangle APQ_{AC} = \beta$. Q_{AC} is the intersection of PQ_{AC} and AC.



Bobillier's Theorem

Dwell Mechanisms:

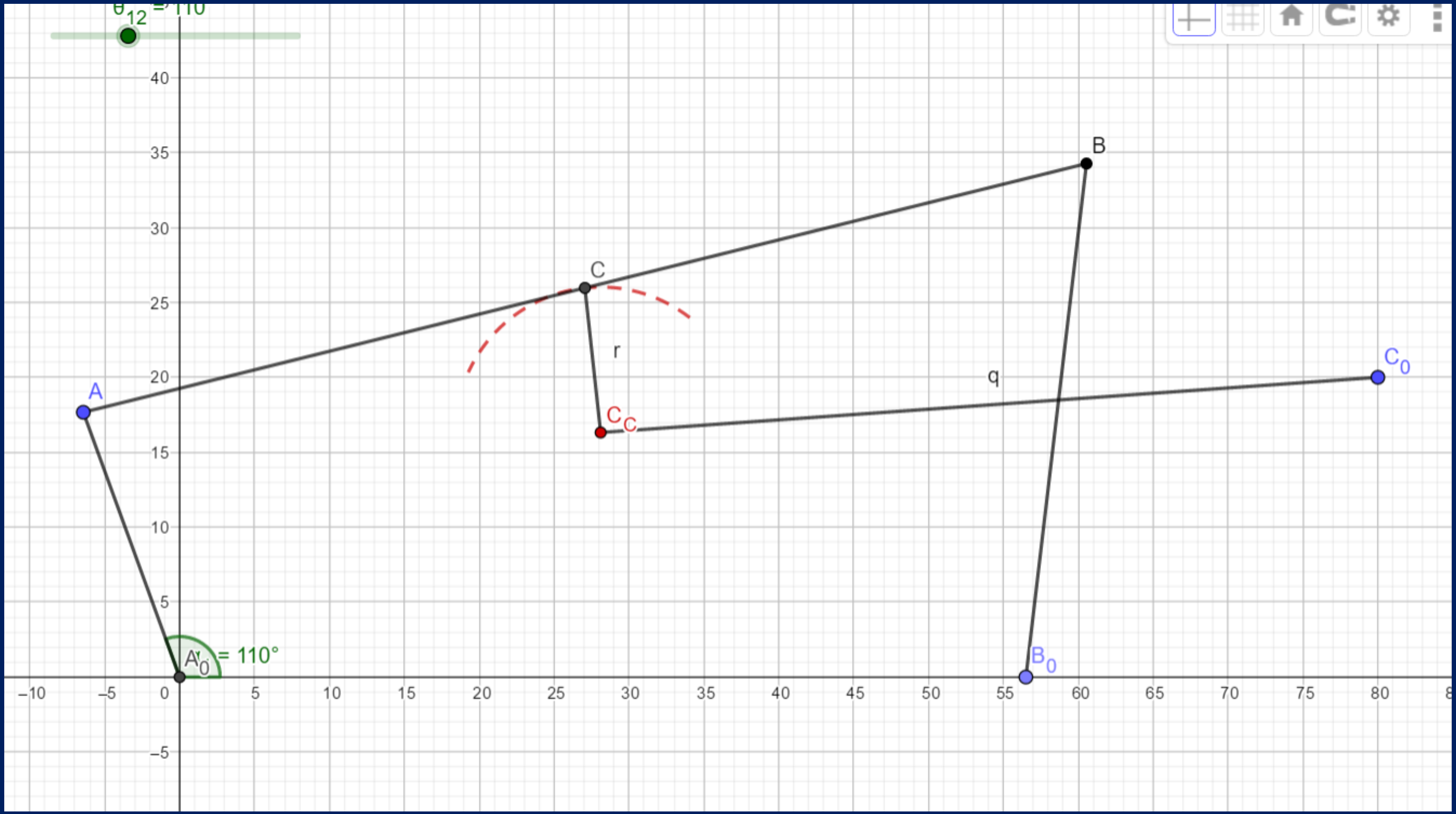
Intersection of $Q_{AC}A_0$ with ray PC yields C_C.



Bobillier's Theorem

Dwell Mechanisms:

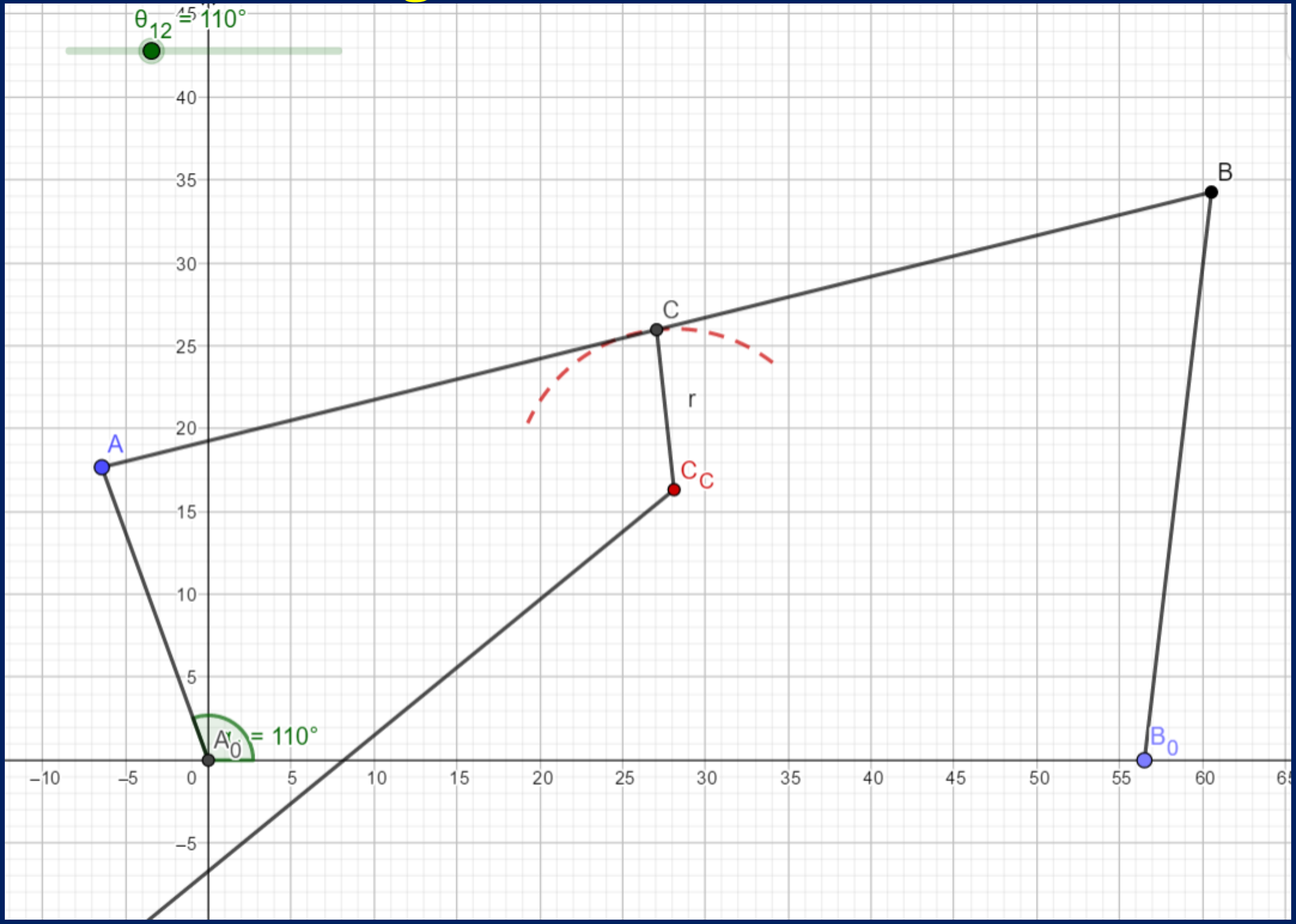
Dwell with oscillating arm.



Bobillier's Theorem

Dwell Mechanisms:

Dwell with translating arm.



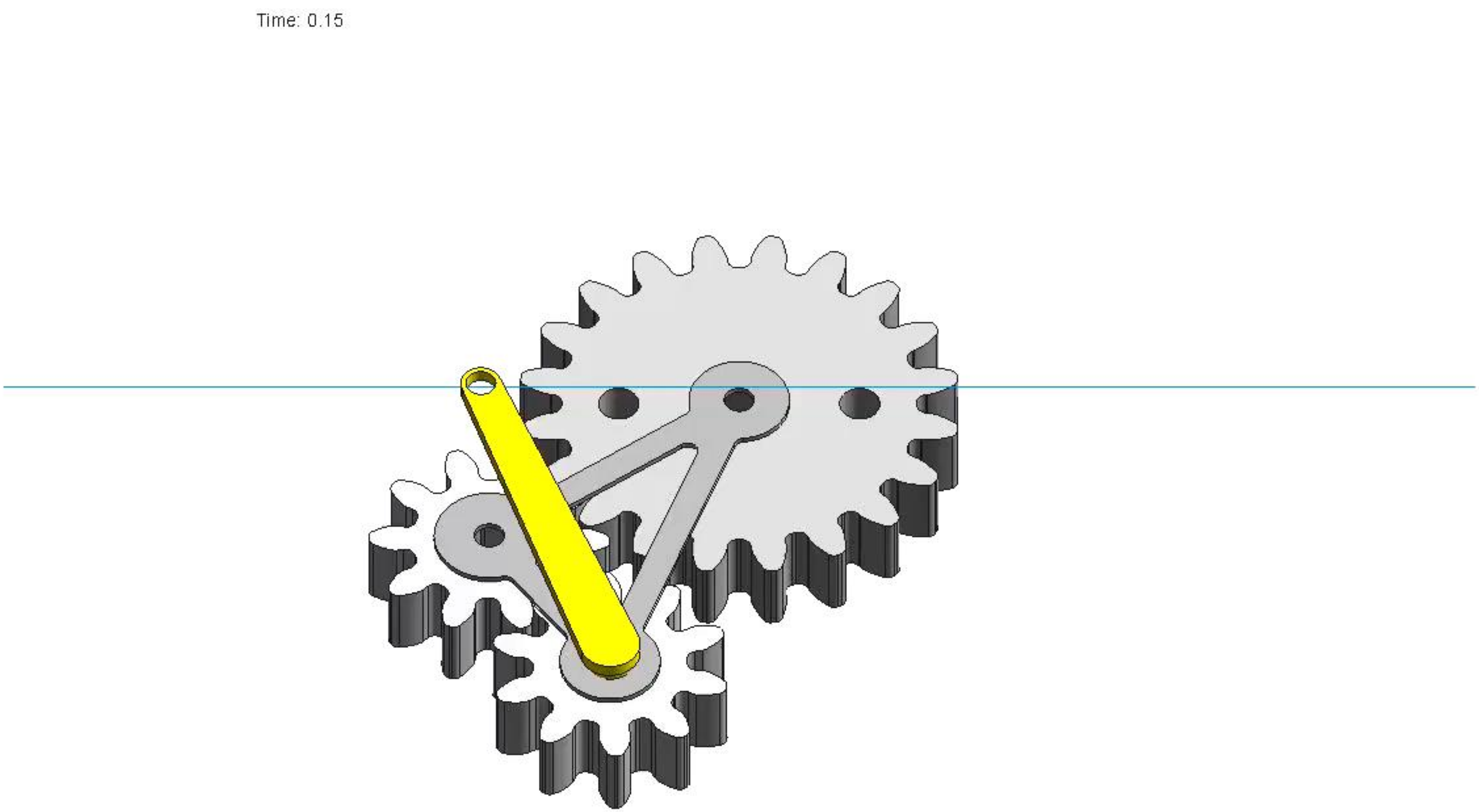
Bobillier’s Theorem

Straight Line Motion Mechanisms:

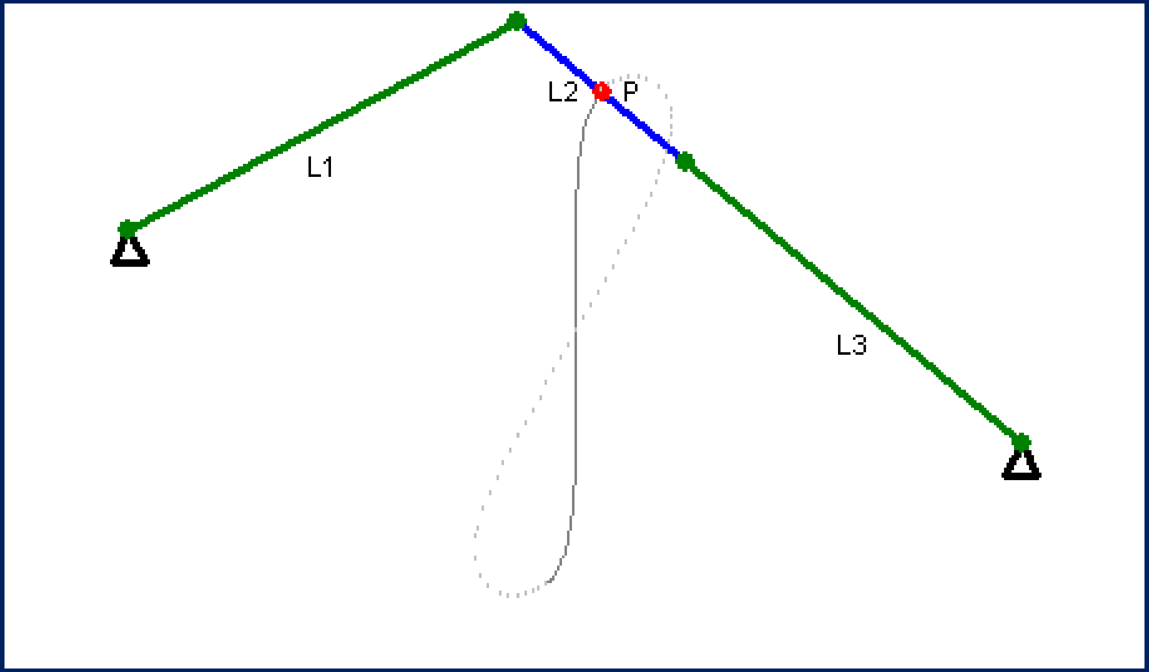
There are two major types of straight line motion mechanisms:

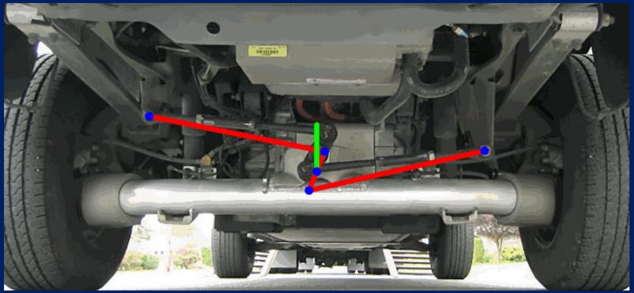
- Exact straight line motion mechanisms where the coupler curve or a portion of it is an exact straight line.
- Approximate straight line motion mechanisms where the coupler curve or a portion of it is very close to a straight line.

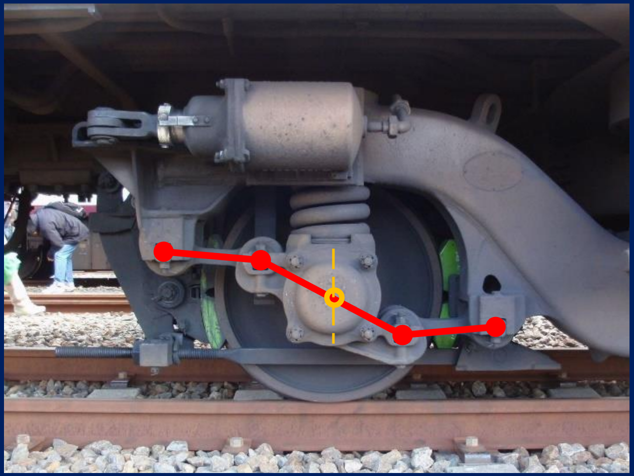
Exact Straight Line: Cardan Motion



Watt's Straight Line Motion Mechanism







https://en.wikipedia.org/wiki/Watt%27s_linkage

Bobillier's Theorem

Straight Line Motion Mechanisms:

There are two major types of straight line motion mechanisms:

- Exact straight line motion mechanisms where the coupler curve or a portion of it is an exact straight line.

Cardanic motion where the moving centrode of radius r_0 rolls inside a cylinder of radius $2r_0$ a point on the moving centrode describes an exact straight line.

- Approximate straight line motion mechanisms where the coupler curve or a portion of it is very close to a straight line.

The moving centrode rolling on the fixed centrode may be approximated up to a certain order the Cardanic motion so a point on the moving centrode approximates a straight line up to the same order at the design point.

First Order: Point and tangent the same (*two infinitesimally separated positions*).

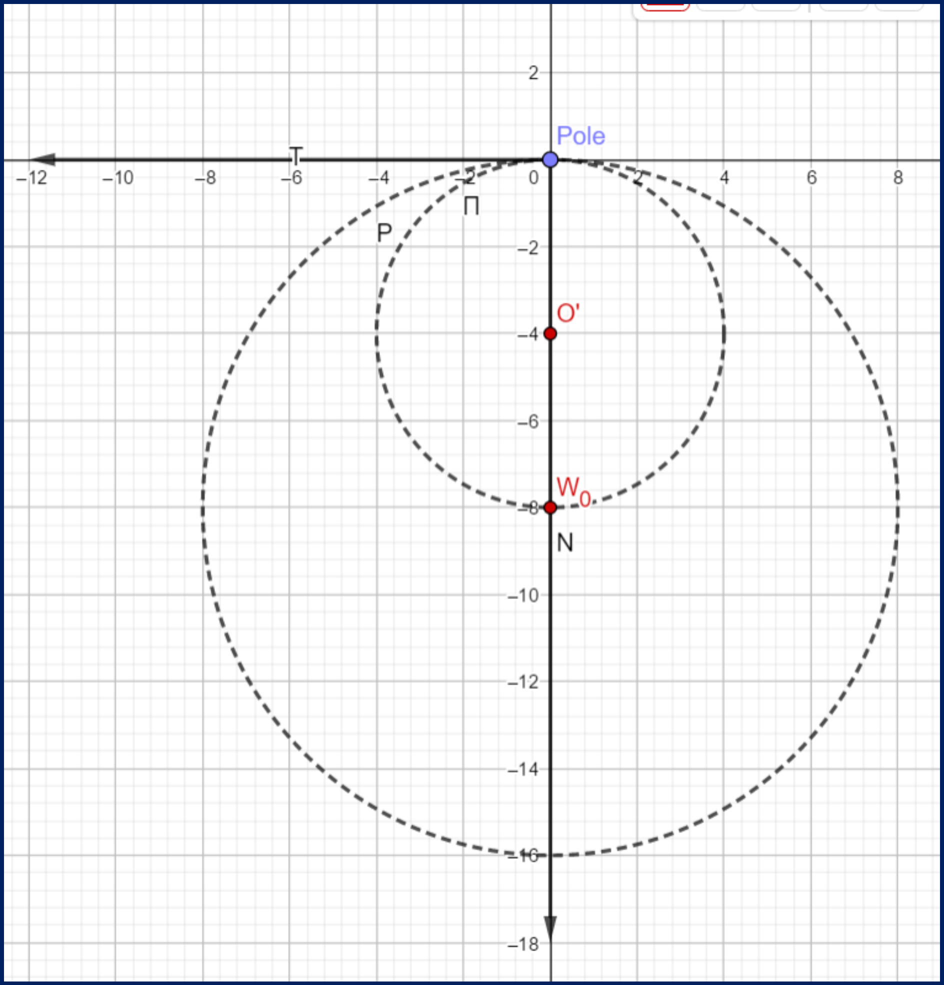
Second Order: Point, tangent and curvature the same (*three infinitesimally separated positions*).

Third Order: Point, tangent, curvature and rate of change of curvature the same (*four infinitesimally separated positions*).

Bobillier's Theorem

Straight Line Motion Mechanisms–Symmetric Four Bar:

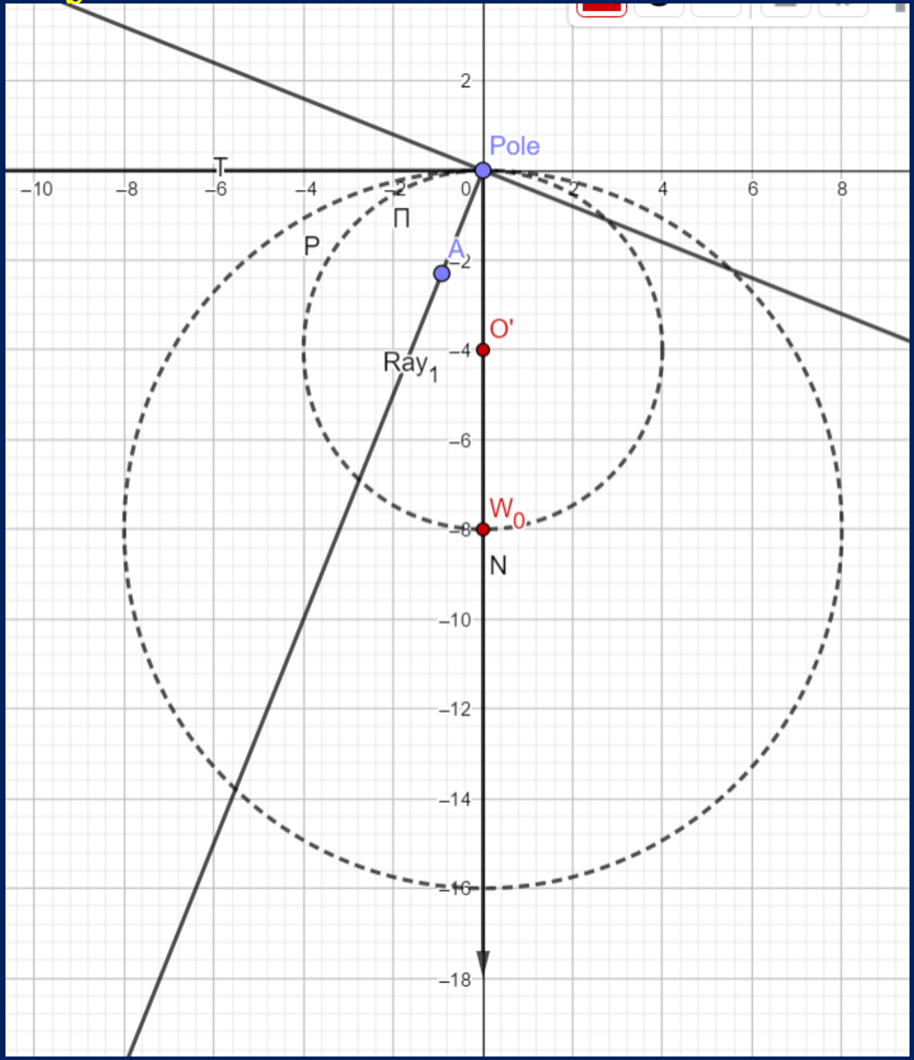
Use Cardanic motion centrodes, P the fixed centrode and Π the moving centrode.



Bobillier's Theorem

Straight Line Motion Mechanisms–Symmetric Four Bar:

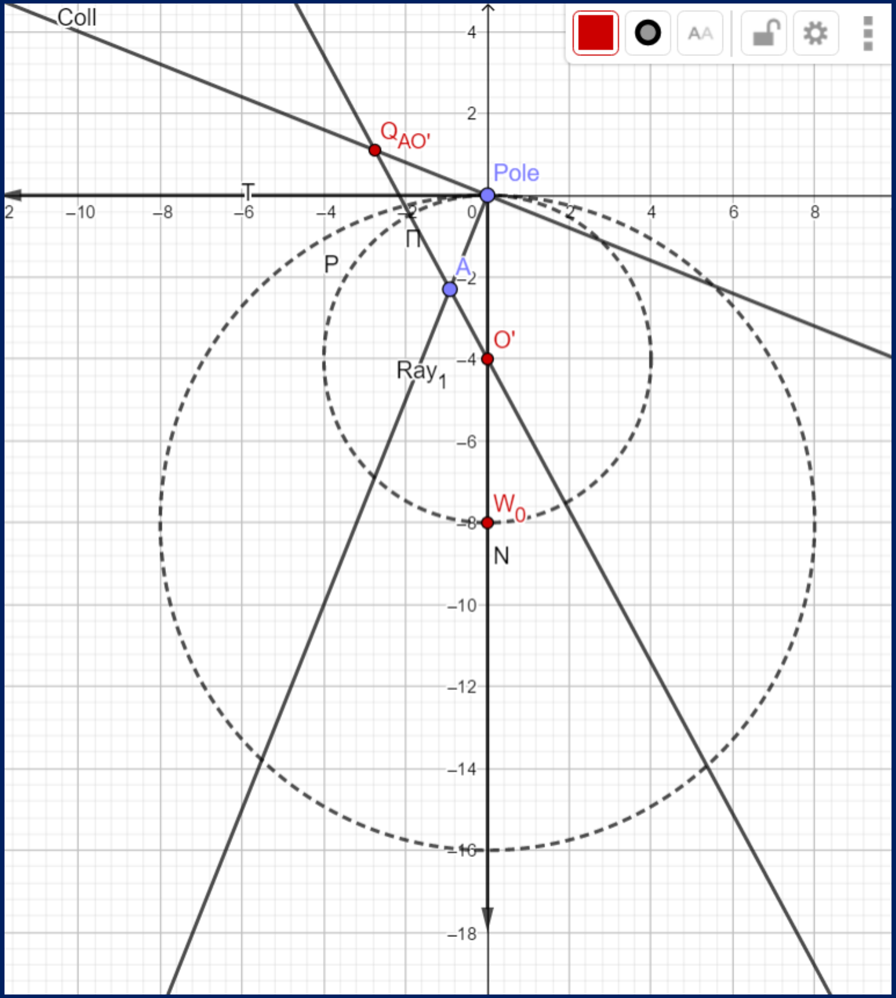
Select A arbitrarily. Like collineation axis is ⊥ AP since PT is ⊥ PN.



Bobillier's Theorem

Straight Line Motion Mechanisms–Symmetric Four Bar:

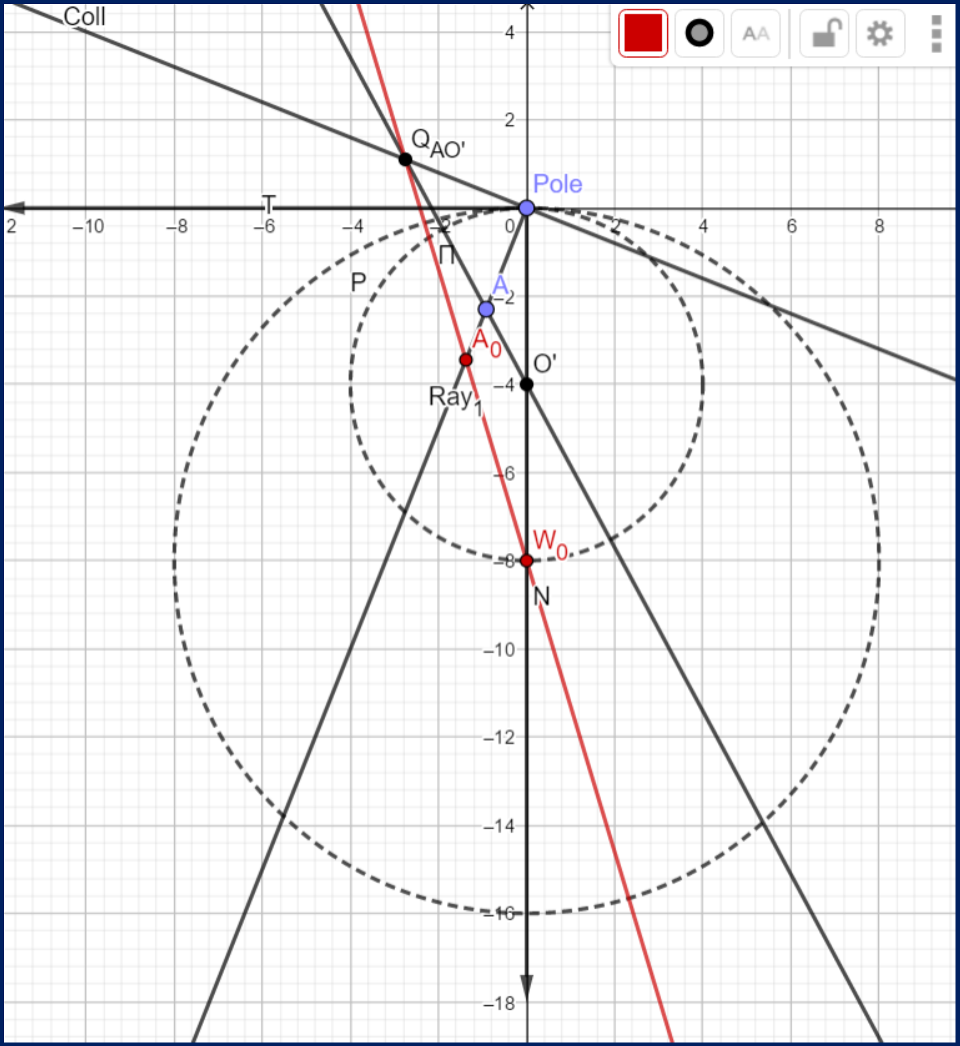
Draw AO' (O' is center of Π) intersection with collineation axis is $Q_{AO'}$.



Bobillier's Theorem

Straight Line Motion Mechanisms–Symmetric Four Bar:

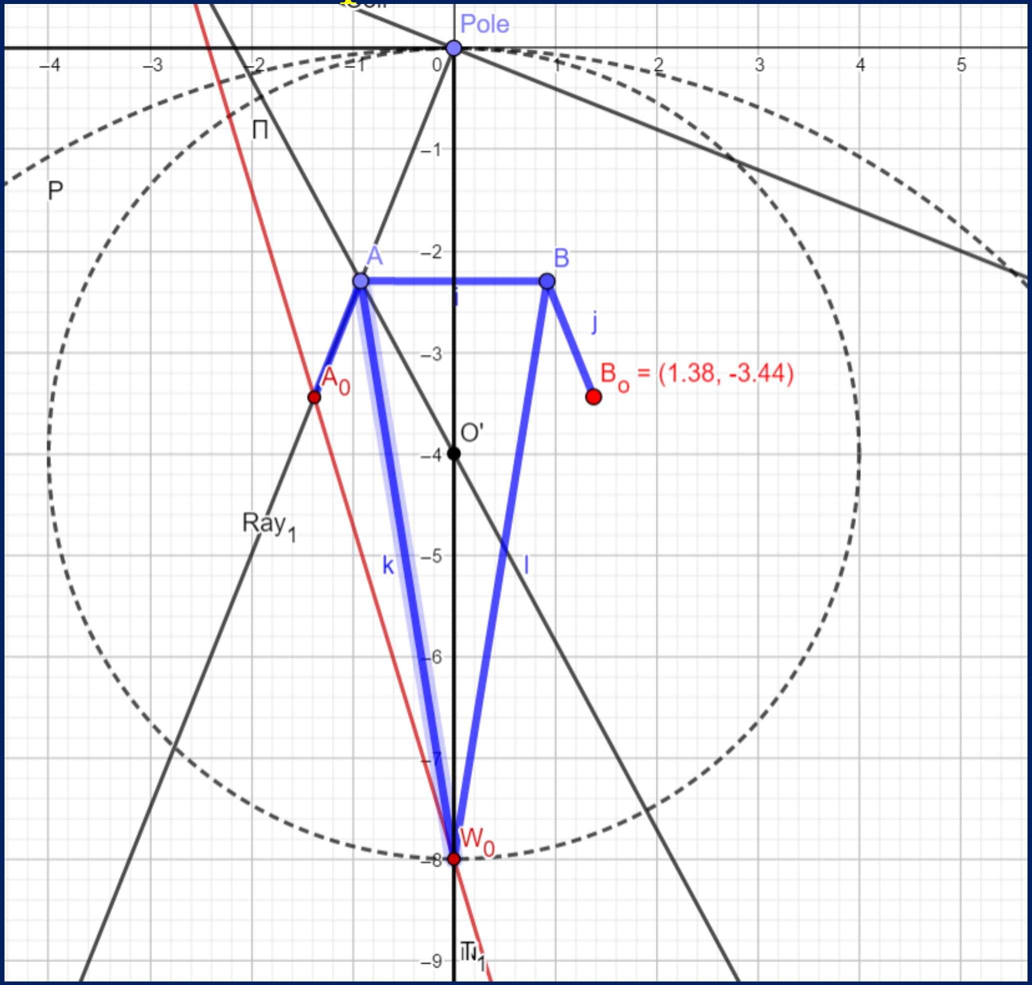
Intersection of $Q_{AO'}$, W_0 with Ray_1 yields A_0.



Bobillier's Theorem

Straight Line Motion Mechanisms–Symmetric Four Bar:

You may repeat the same procedure for B or you may select symmetric points about pole normal.



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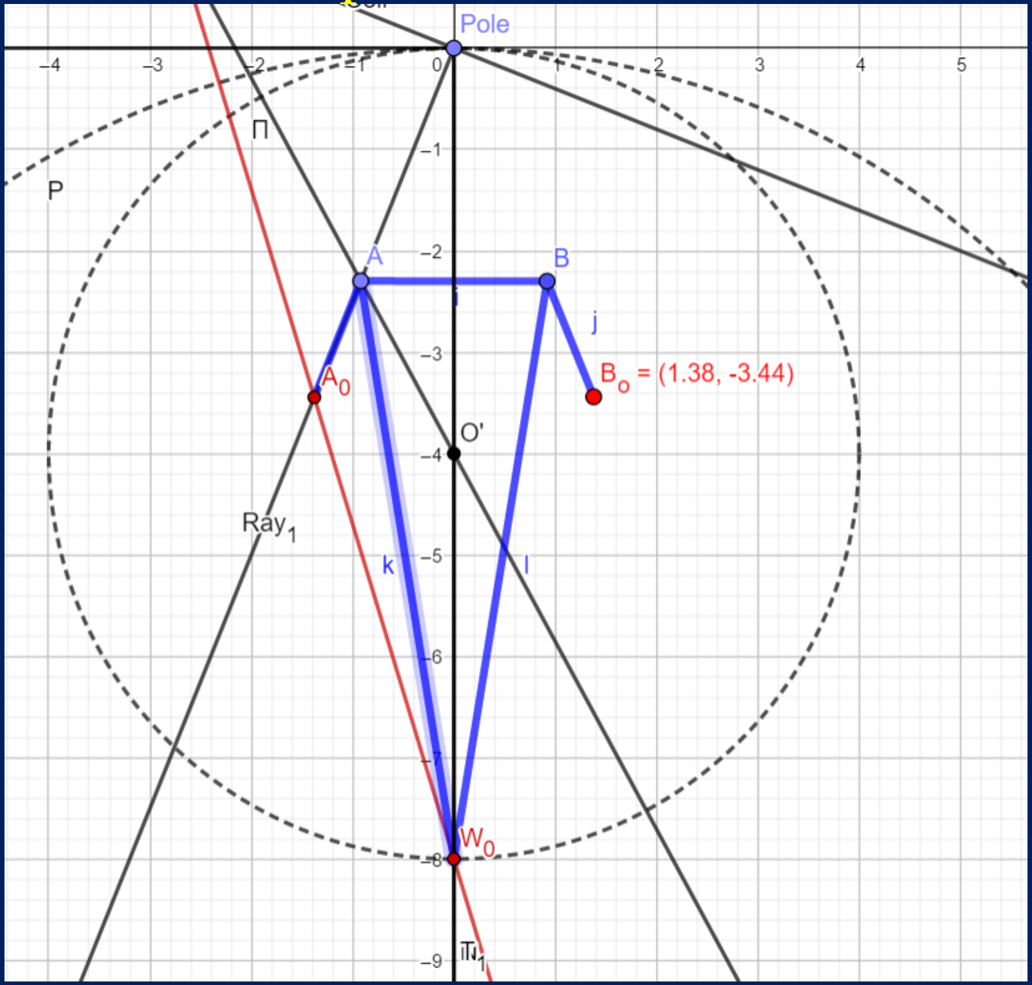


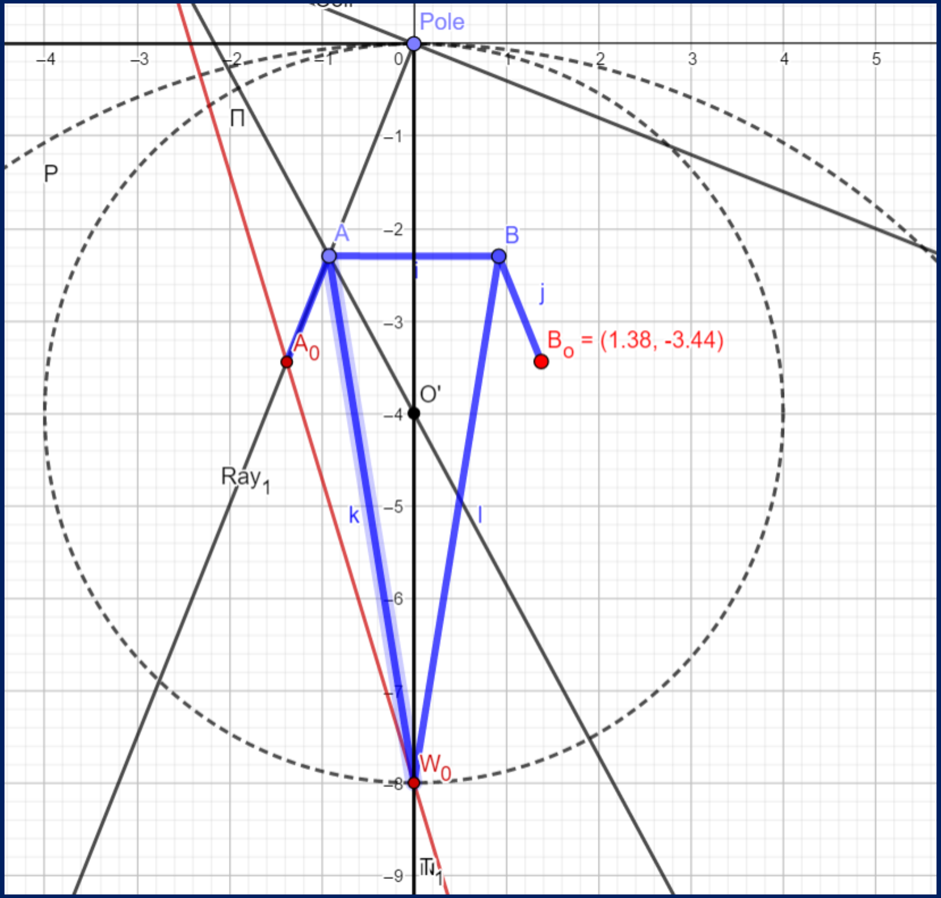
Bobillier's Theorem

Straight Line Motion Mechanisms–Symmetric Four Bar:

Bobillier's Theorem

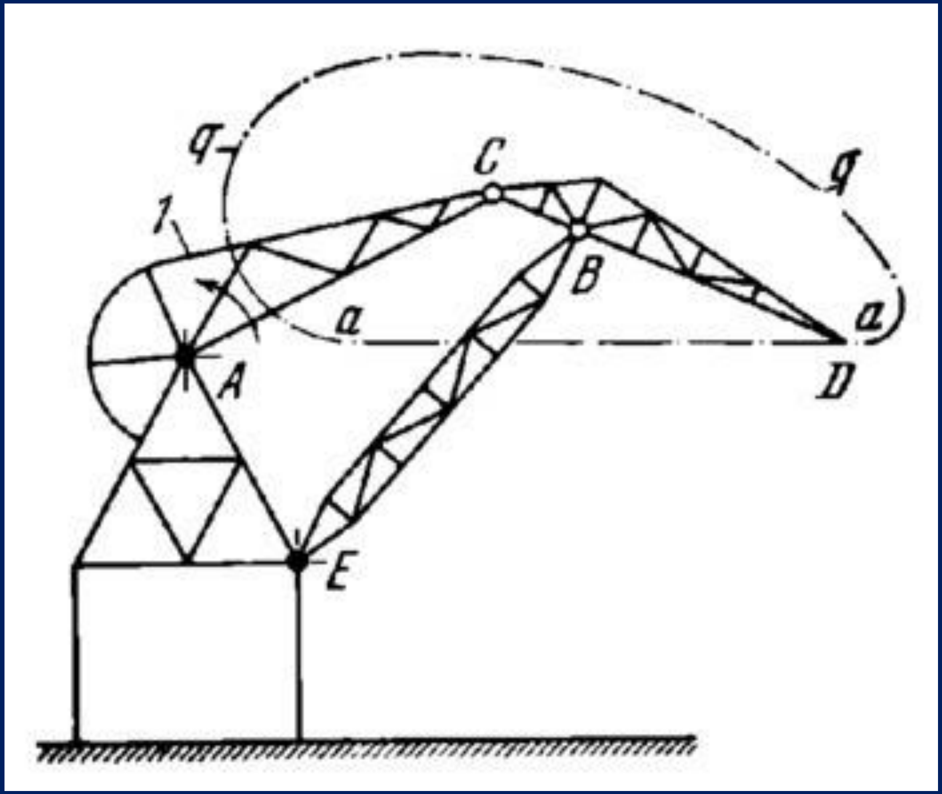
Straight Line Motion Mechanisms–Symmetric Four Bar:

Actually any point on inflection circle (which is also the moving centrode) describes a straight line. The reason for selecting the inflection pole W_0 will become clear soon.



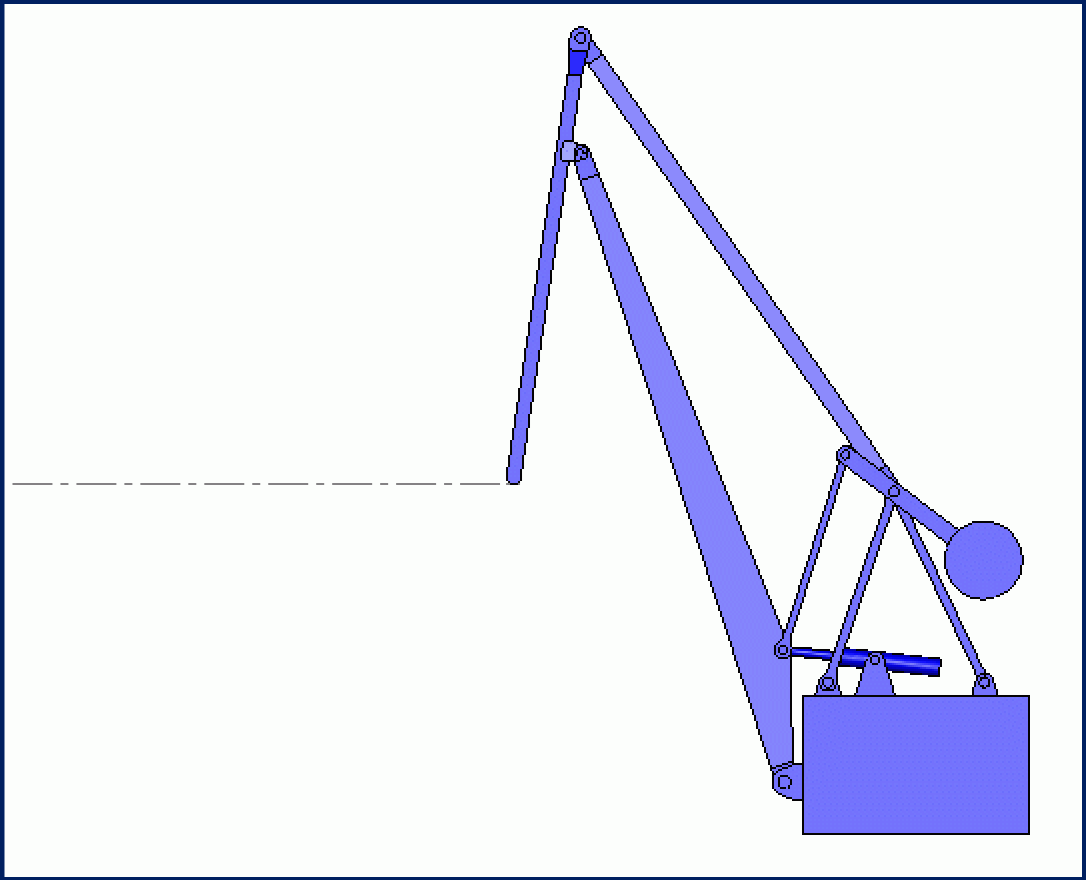
Bobillier's Theorem

Straight Line Motion Mechanisms–Level Luffing Crane:



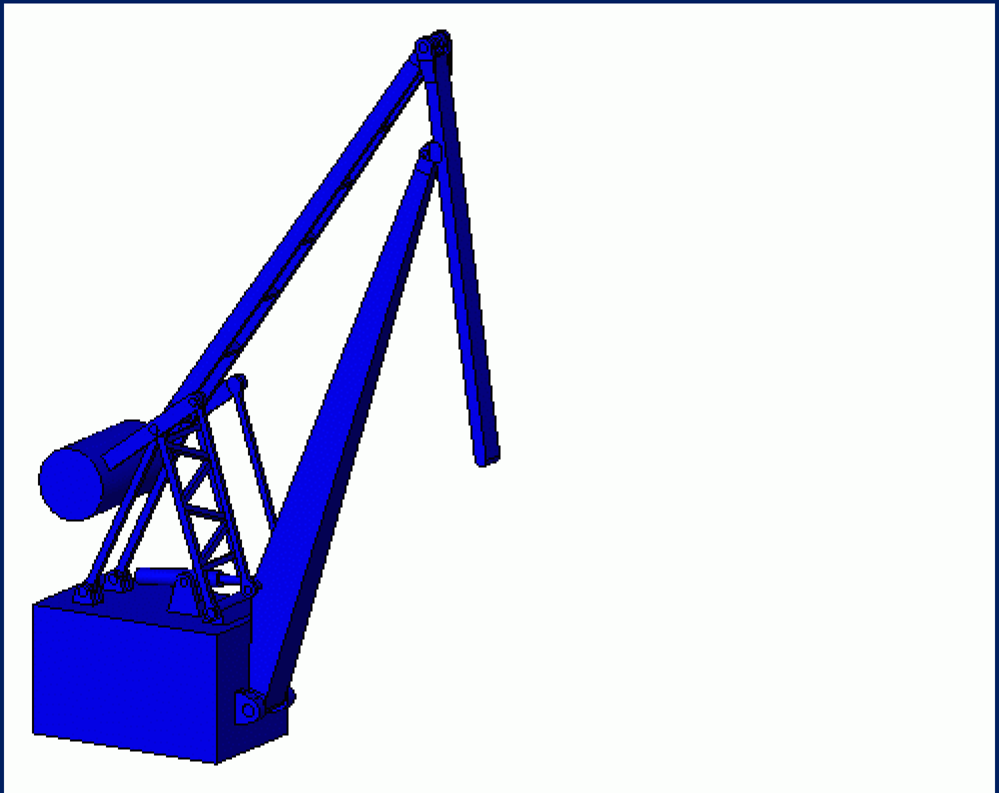
https://www.europeana.eu/portal/en/record/2020801/dmglib_handler_image_16783023.html

Four Bar Linkage (Level Luffing) Crane



https://commons.wikimedia.org/wiki/File:Crane_double-lever-jib-type_sideview_animated.gif

Four Bar Linkage (Level Luffing) Crane



https://upload.wikimedia.org/wikipedia/commons/4/48/Crane_double-lever-jib-type_3D_animated.gif

Four Bar Linkage (Level Luffing) Crane

http://www.jjhig.com/en/index.asp?16T33Mfo.html

Bobillier's Theorem

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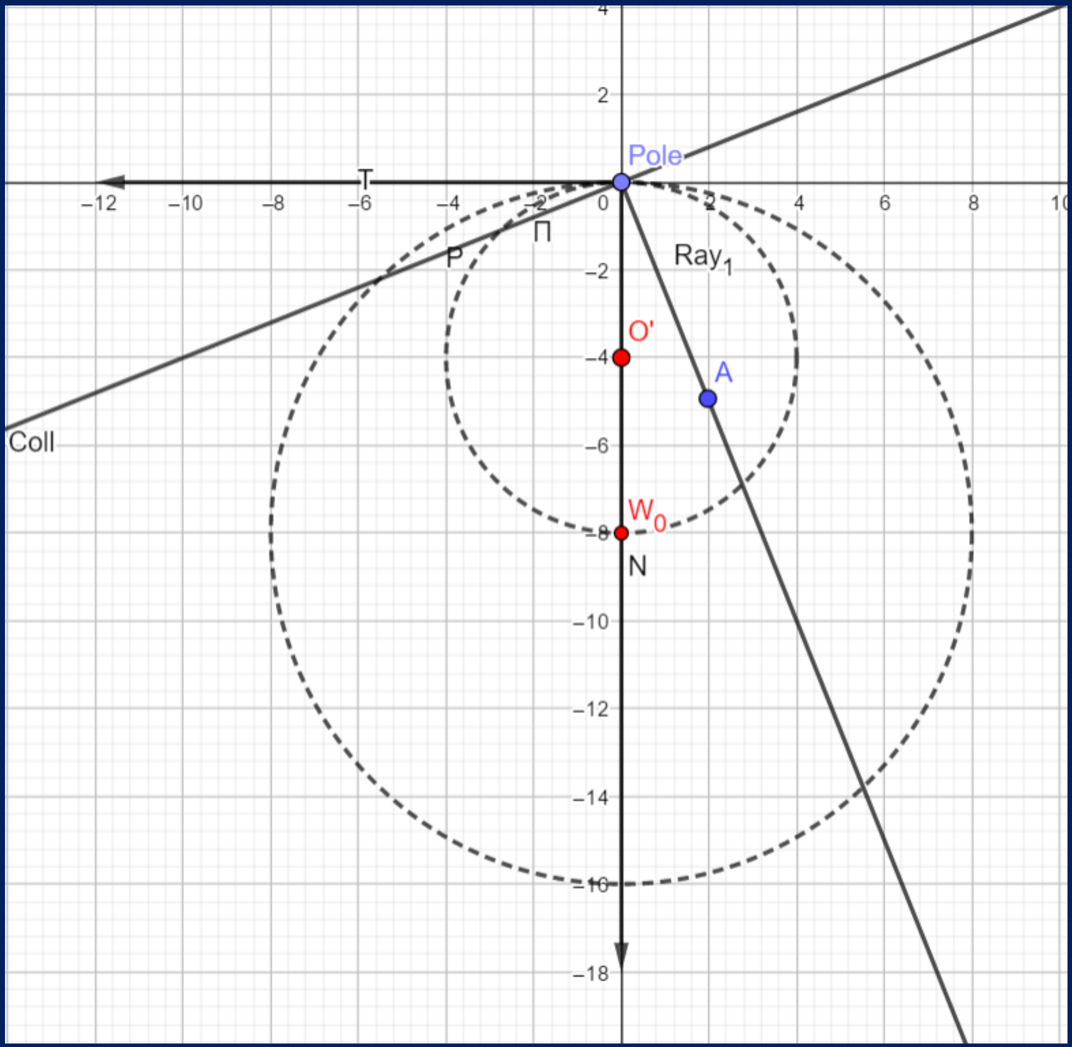
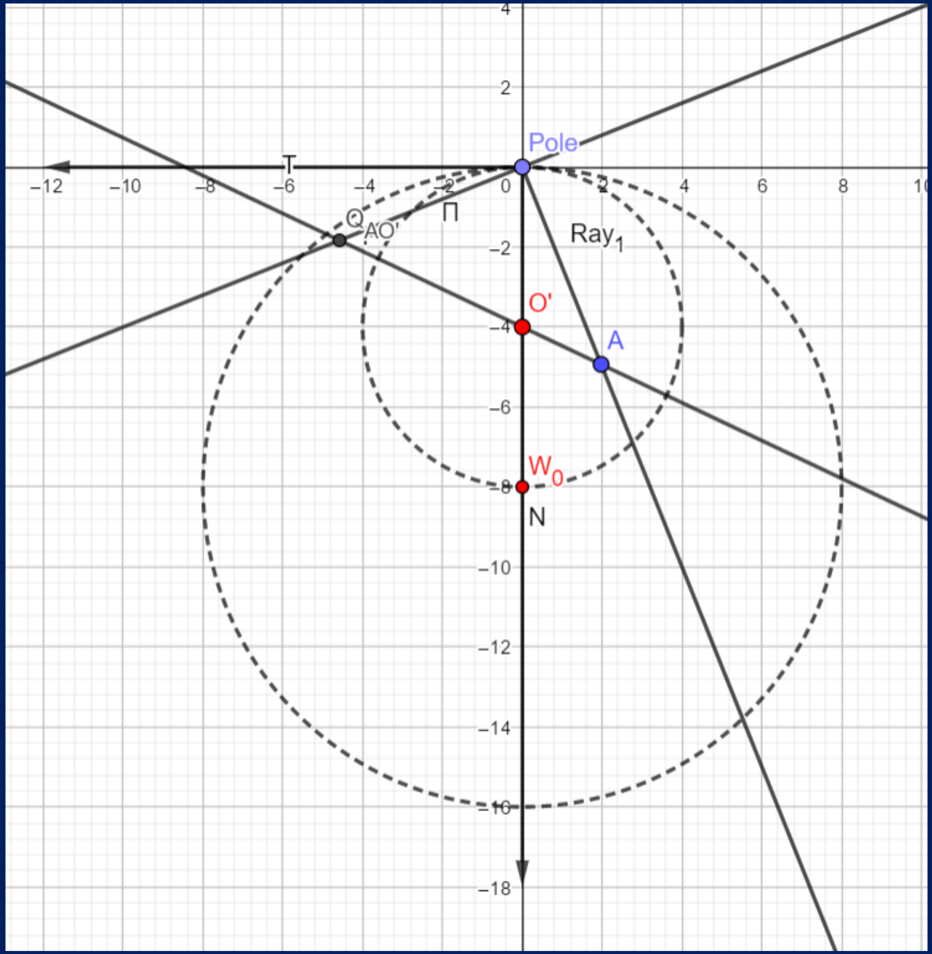
Select A arbitrarily. Like collineation axis is ⊥ AP since PT is ⊥ PN.



Bobillier's Theorem

Straight Line Motion Mechanisms–Level Luffing Crane:

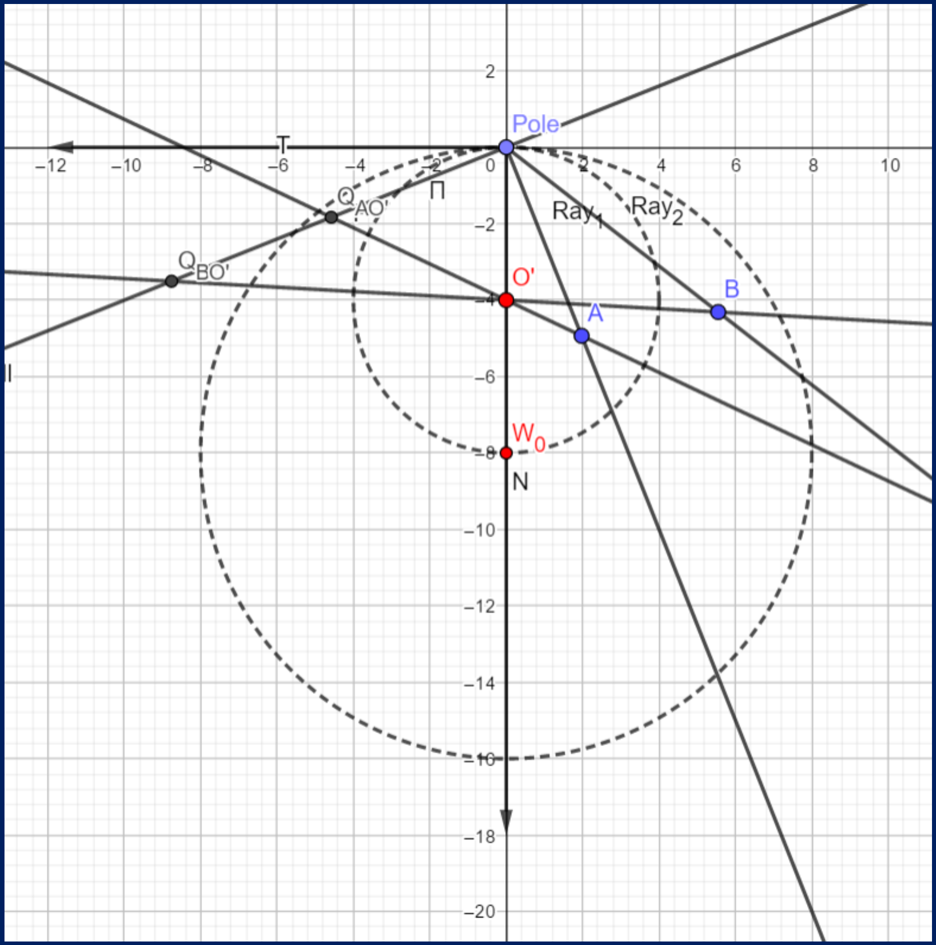
Draw AO' and BO' (B is selected arbitrarily too) intersection with collineation axis yield $Q_{AO'}$ and $Q_{BO'}$ respectively (1/2).



Bobillier's Theorem

Straight Line Motion Mechanisms–Level Luffing Crane:

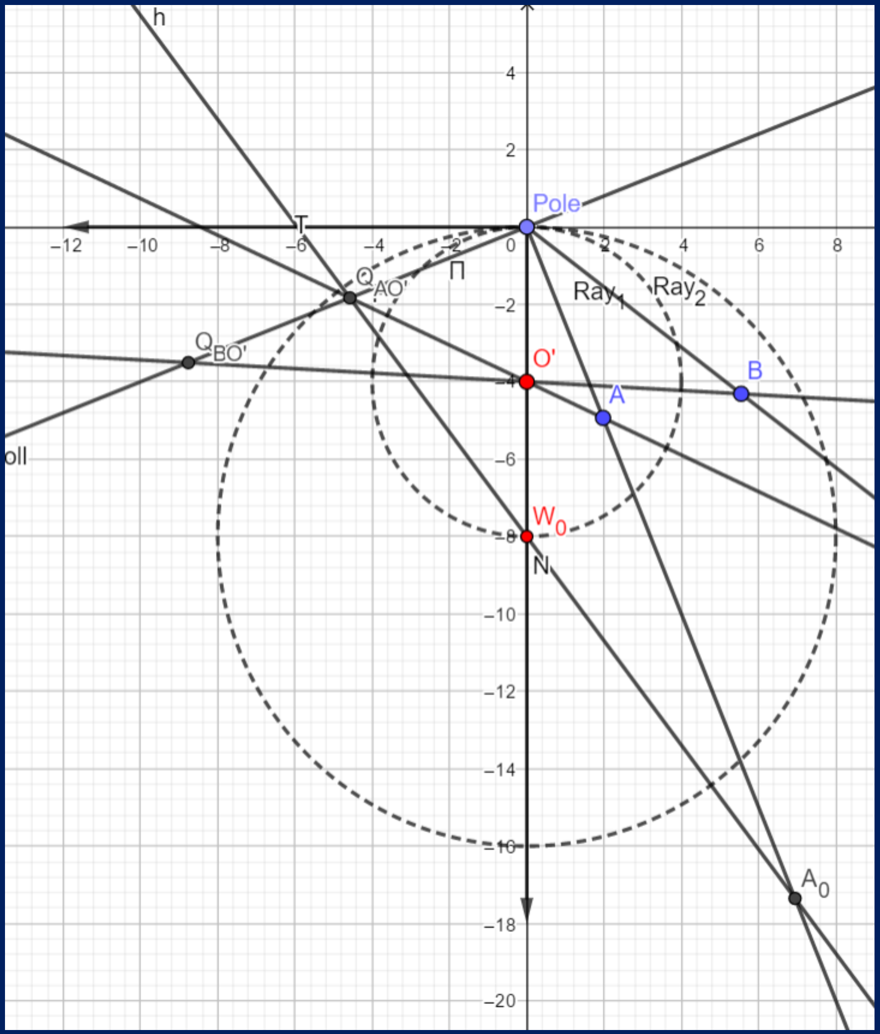
Draw AO' and BO' (B is selected arbitrarily too) intersection with collineation axis yield $Q_{AO'}$ and $Q_{BO'}$ respectively (2/2).



Bobillier's Theorem

Straight Line Motion Mechanisms–Level Luffing Crane:

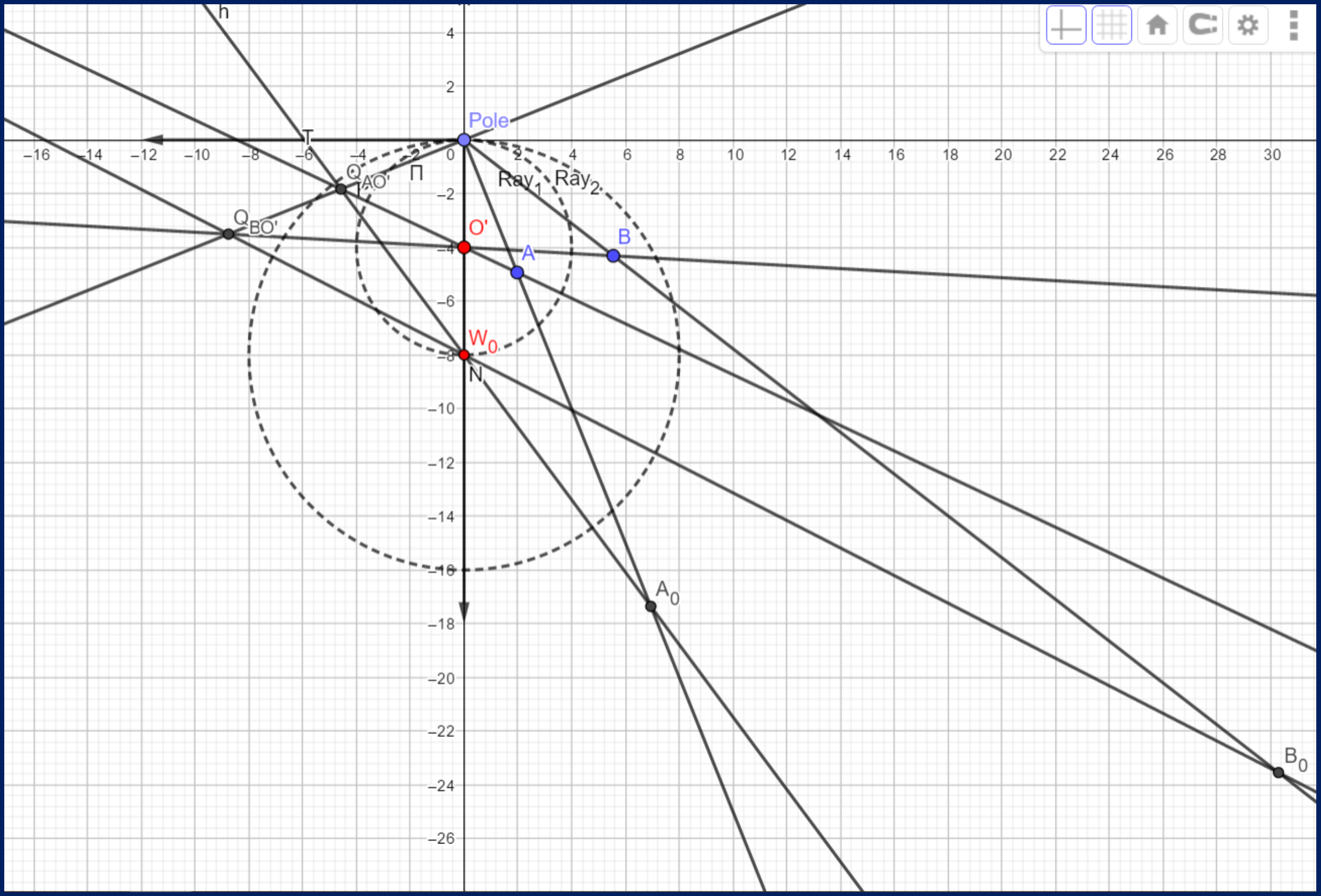
Intersection of $Q_{AO'}W_0$ (which is O as well) with Ray_1 yields A_0.



Bobillier's Theorem

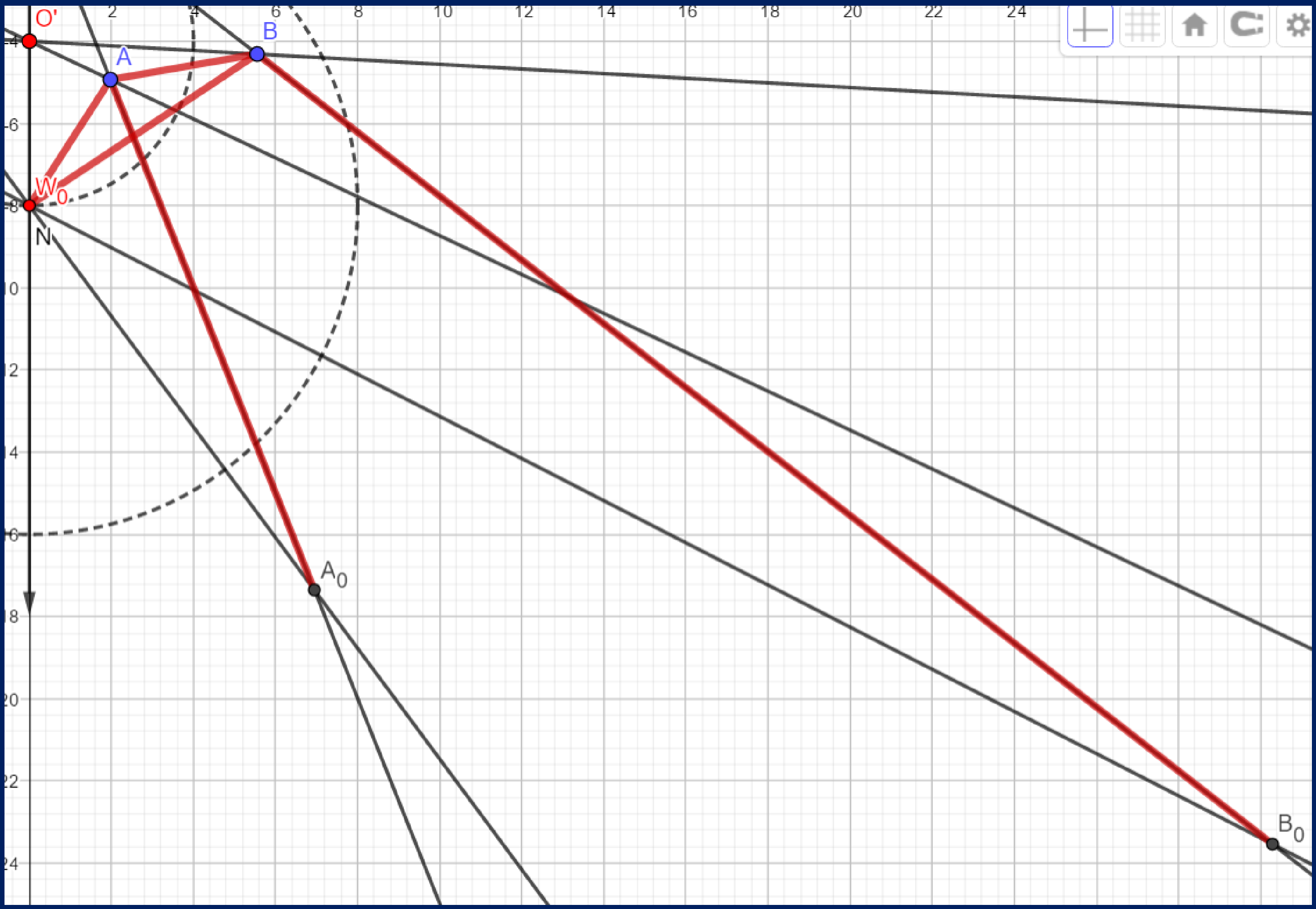
Straight Line Motion Mechanisms–Level Luffing Crane:

Intersection of $Q_{BO'}$, W_0 (which is O as well) with Ray_2 yields B_0.



Bobillier's Theorem

Straight Line Motion Mechanisms–Level Luffing Crane:



Bobillier's Theorem

Straight Line Motion Mechanisms–Level Luffing Crane:

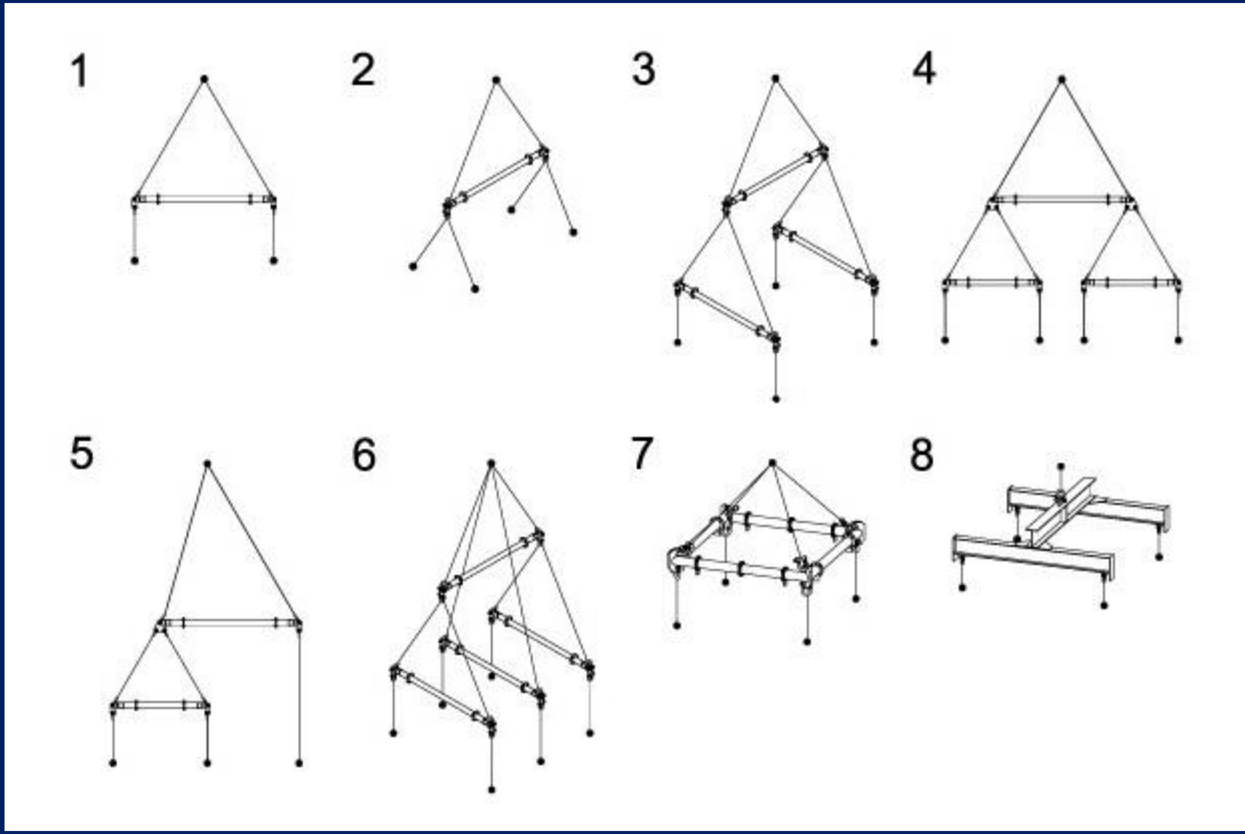
Level Luffing Crane Video Here (Ready!)

Bobillier's Theorem

Stability Analysis of Lifting Rigs & Spreader Frames:

They are used to lift huge components using cranes.

Rigs and Spreader Frames

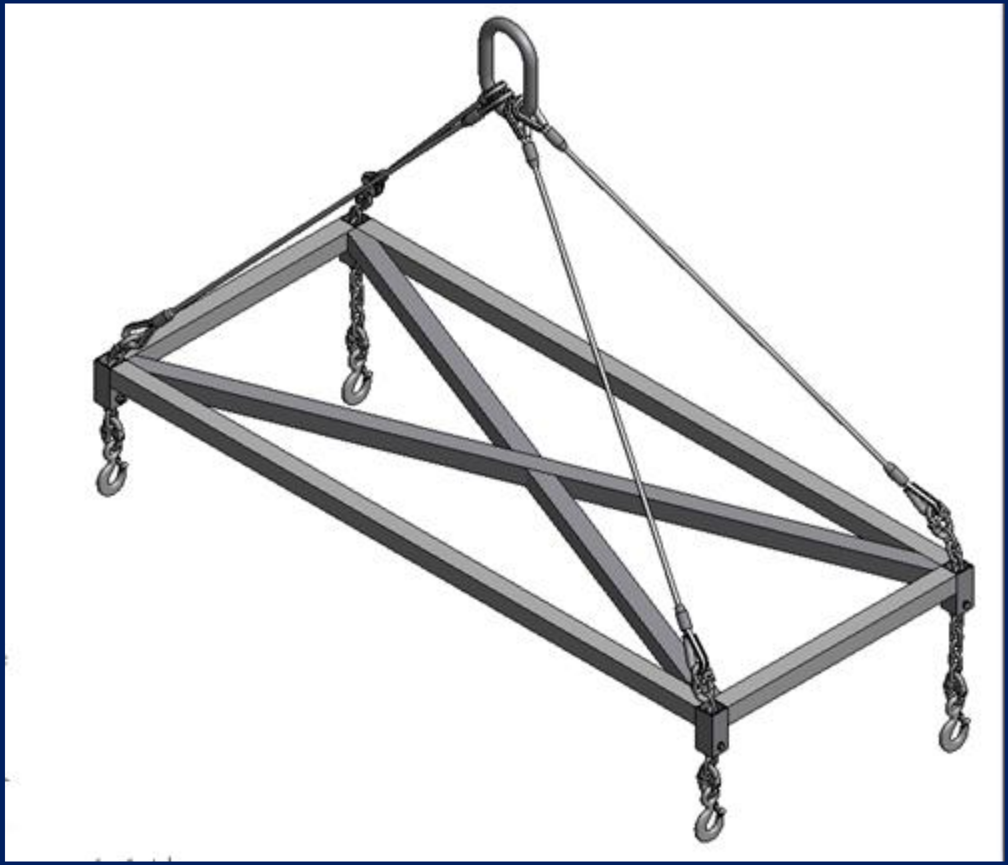


1. Single Spreader Beam: 2-Point Lift
2. Single Spreader Beam: 4-Point Lift
3. 3 Spreader Beams – "1-over-2" 4-Point Lift
4. 3 Spreader Beams – "1-over-2" in-line: 4-Point Lift
5. 2 Spreader Beams – "1-over-1" 3-Point Lift
6. Multiple Spreader Beams: Multi-Point Lift
7. Spreader Frames
8. Lifting Frames

https://www.modulift.com/rig-design/custom-design-solutions/rig-design

Rigs and Spreader Frames

Caravan Lifting Rig



https://www.lakeandair.com/Caravan-Lifting-Rig-p/1005247.htm

Rigs and Spreader Frames

Tein Otter Lifting Rig for Seaplanes

https://www.lakeandair.com/Twin-Otter-Lifting-Rig-p/1005836.htm

Rigs and Spreader Frames
Adjustable Spreader

https://www.amazon.com/Caldwell-Group-32C-10-4-Adjustable-Spreader/dp/B01KOURV3E

Rigs and Spreader Frames

https://www.lakeandair.com/Caravan-Lifting-Rig-p/1005247.htm

Rigs and Spreader Frames

http://amzoneinternational.com/Products/SpreaderBeam/

Rigs and Spreader Frames

https://www.maritimeprofessional.com/news/designed-heavy-dockside-lifts-318712

Rigs and Spreader Frames

http://chart-ur-bar.com/spreader-systems.php

Rigs and Spreader Frames

https://belowthehookrigging.com/spreader-bars/

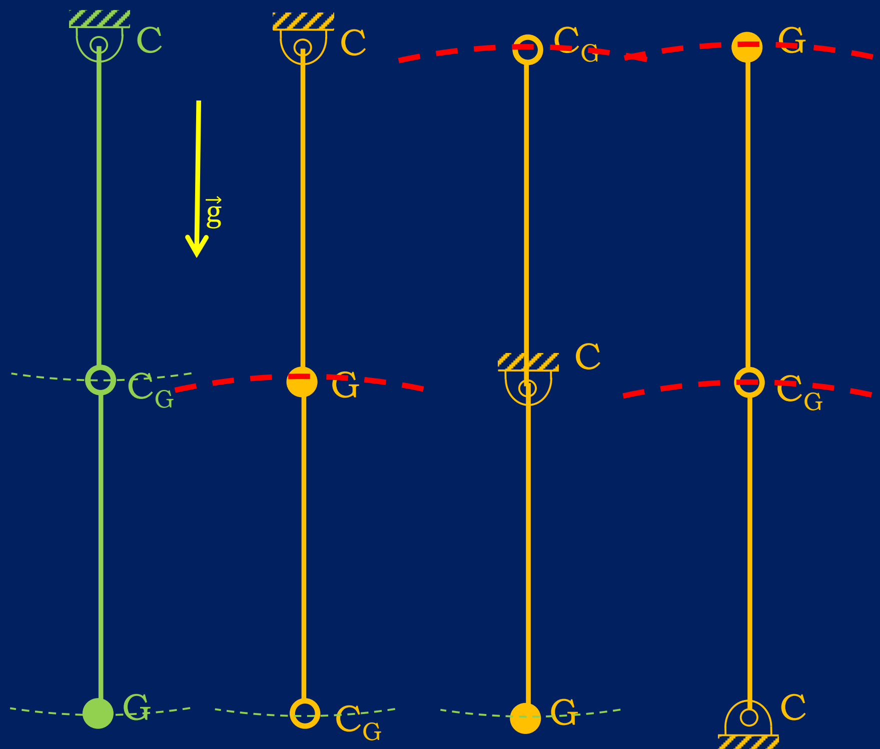
Rigs and Spreader Frames

https://es.123rf.com/photo_42336308_elevaci%C3%B3n-de-palets-con-elementos-de-encofrado-por-la-gr%C3%BAa-en-la-obra.html

Bobillier's Theorem

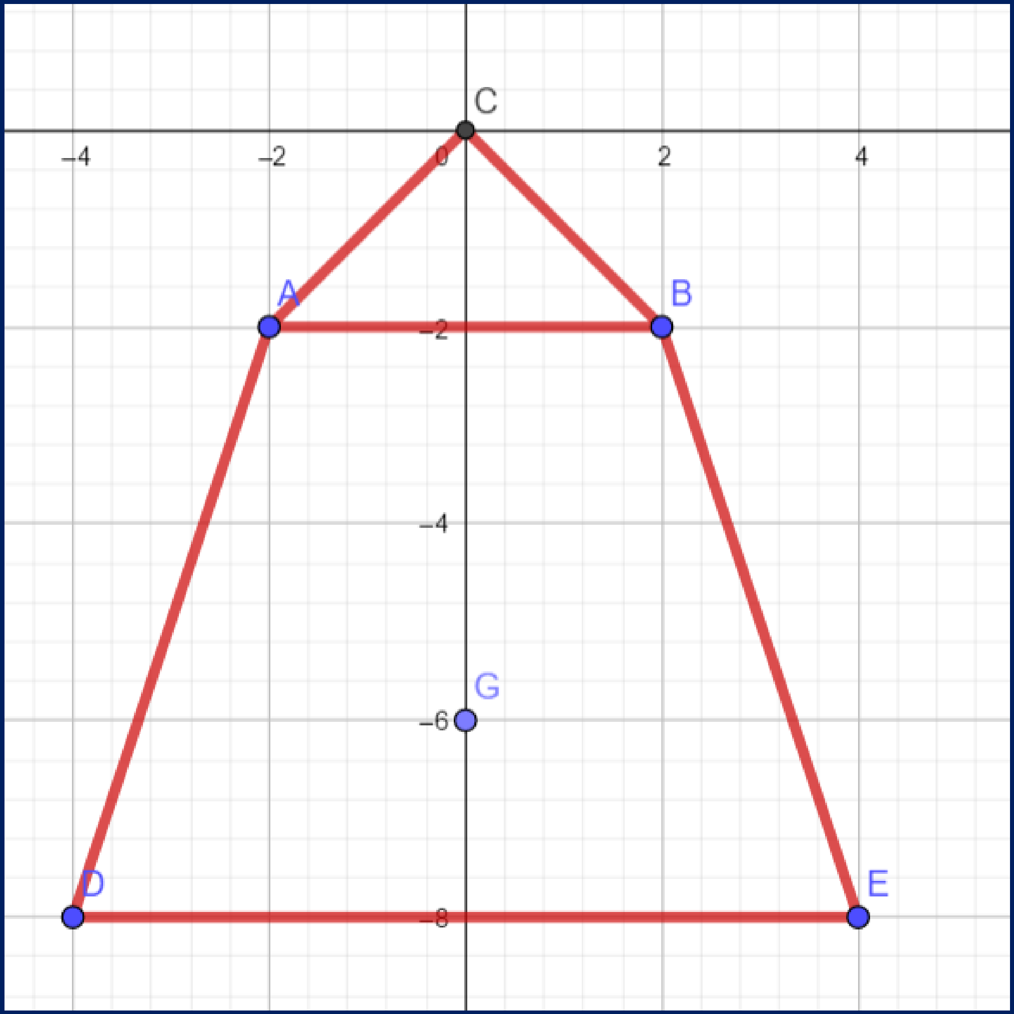
Stability Analysis of Lifting Rigs & Spreader Frames:

There are four possible *equivalent linkages* of the rigs and frames around the infinitesimal neighborhood of the equilibrium position. Only one of those is in stable, other three are in unstable equilibrium. Here C shows the hook of the crane, G is the center of gravity of the load together with rigs or frames and C_G is the center of curvature of the center of gravity, G.



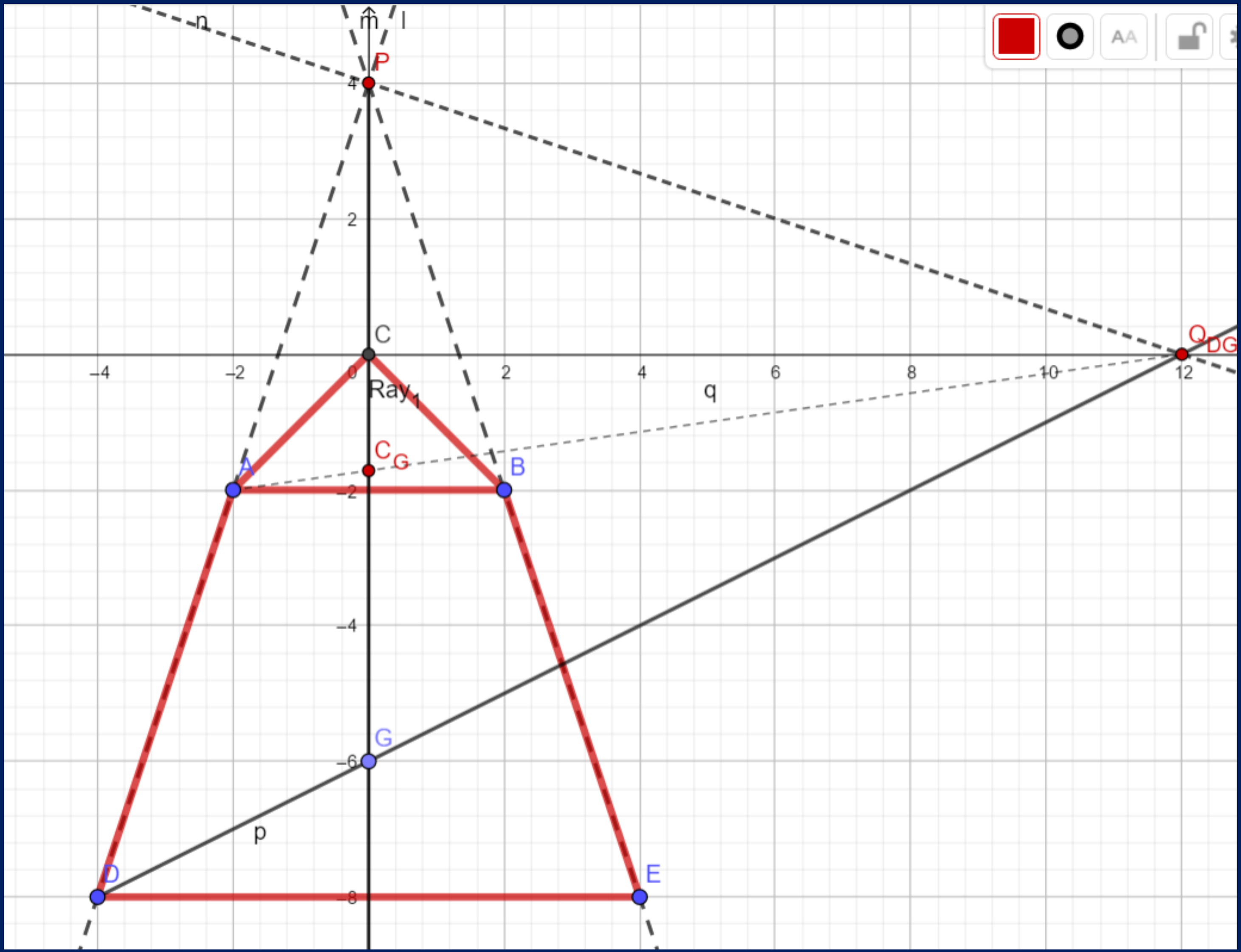
Bobillier's Theorem

Umbrella Tent:



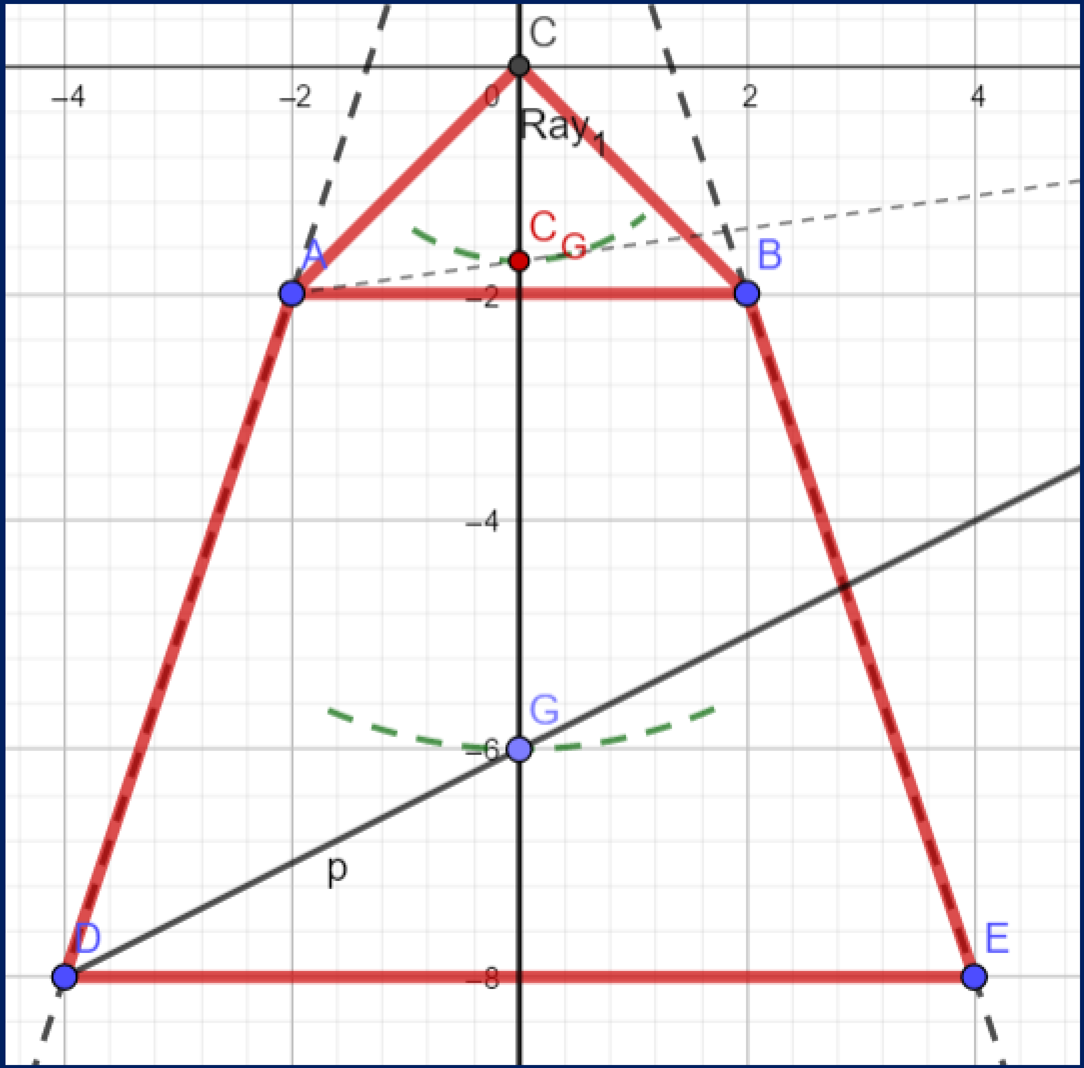
Bobillier's Theorem

Umbrella Tent:



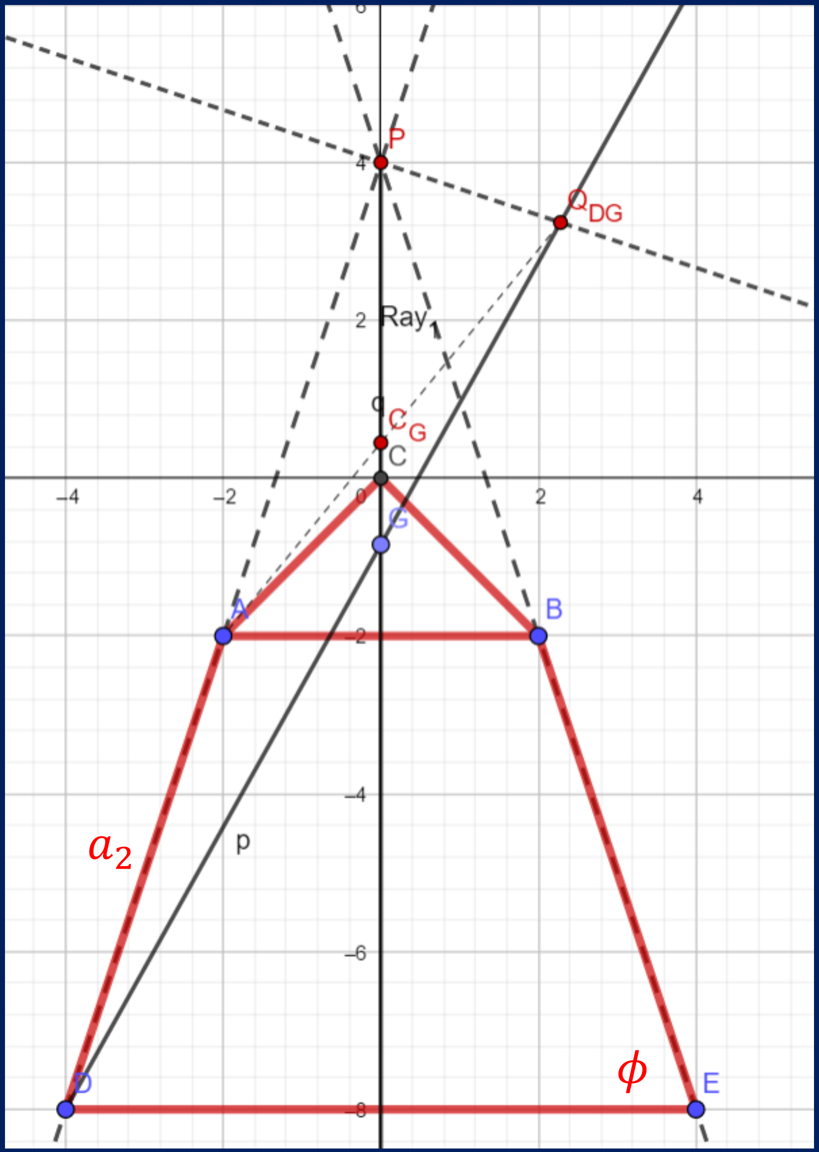
Bobillier's Theorem

Umbrella Tent:



Bobillier's Theorem

Umbrella Tent: Unstable!



Bobillier's Theorem

Umbrella Tent:

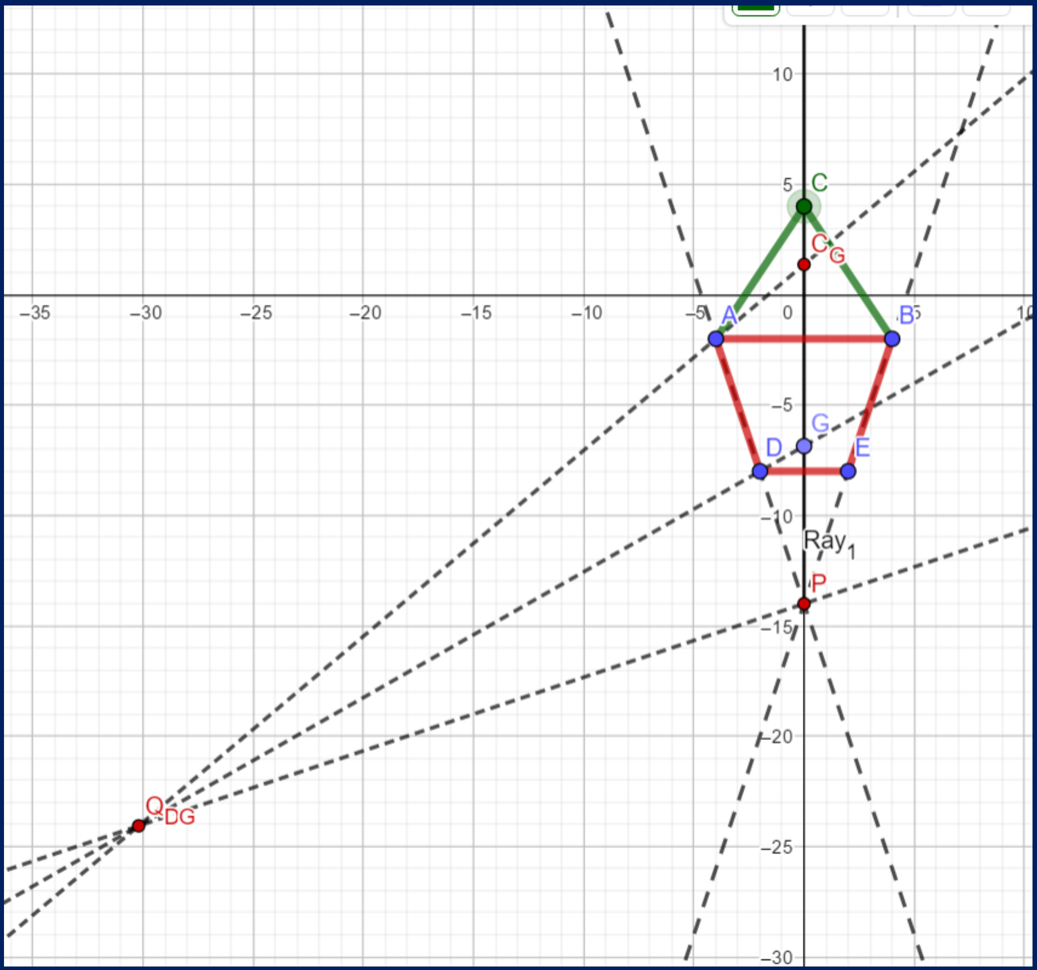
Center of curvature of mass center, CG, can be determined using Bobillier's construction and it can be shown by using Euler-Savary equation that CG is between C and G therefore stable for

$$\frac{1}{|PC|} > \left[\left(\frac{1}{|PA|} - \frac{1}{|PD|}\right) sin\phi + \frac{1}{|PG|}\right]$$

$$\frac{1}{|PA|} - \frac{1}{|PD|} = a_2$$

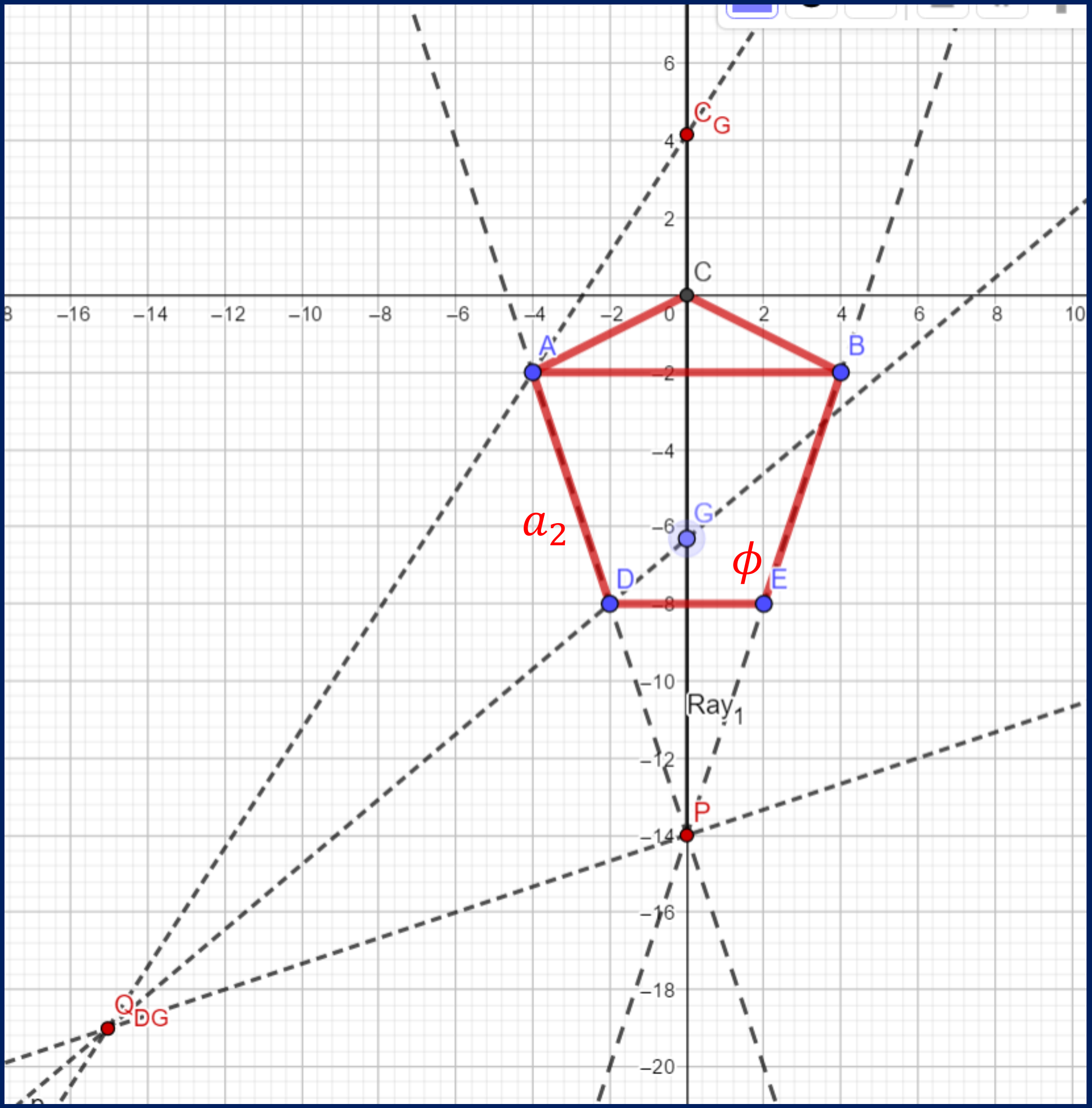
Bobillier's Theorem

Bird Cage:



Bobillier's Theorem

Bird Cage: Unstable!



Bobillier's Theorem

Bird Cage:

Center of curvature of mass center, CG, can be determined using Bobillier's construction and it can be shown by using Euler-Savary equation that CG is between C and G therefore stable for

$$\frac{1}{|PC|} < \left[\left(\frac{1}{|PA|} - \frac{1}{|PD|}\right) sin\phi + \frac{1}{|PG|}\right]$$

$$\frac{1}{|PA|} - \frac{1}{|PD|} = a_2$$

Generating Curves & Envelopes

For some applications (e.g. cams) rather than determining the centrodes it might be easier to find two curves, one fixed and other moving, *rolling and sliding* on each other. Still the centrodes exist but the equivalent linkages may also be derived from the generating curves and envelopes.

Generating Curves & Envelopes

https://www.youtube.com/watch?v=d3DpgF1-xdI

Generating Curves & Envelopes

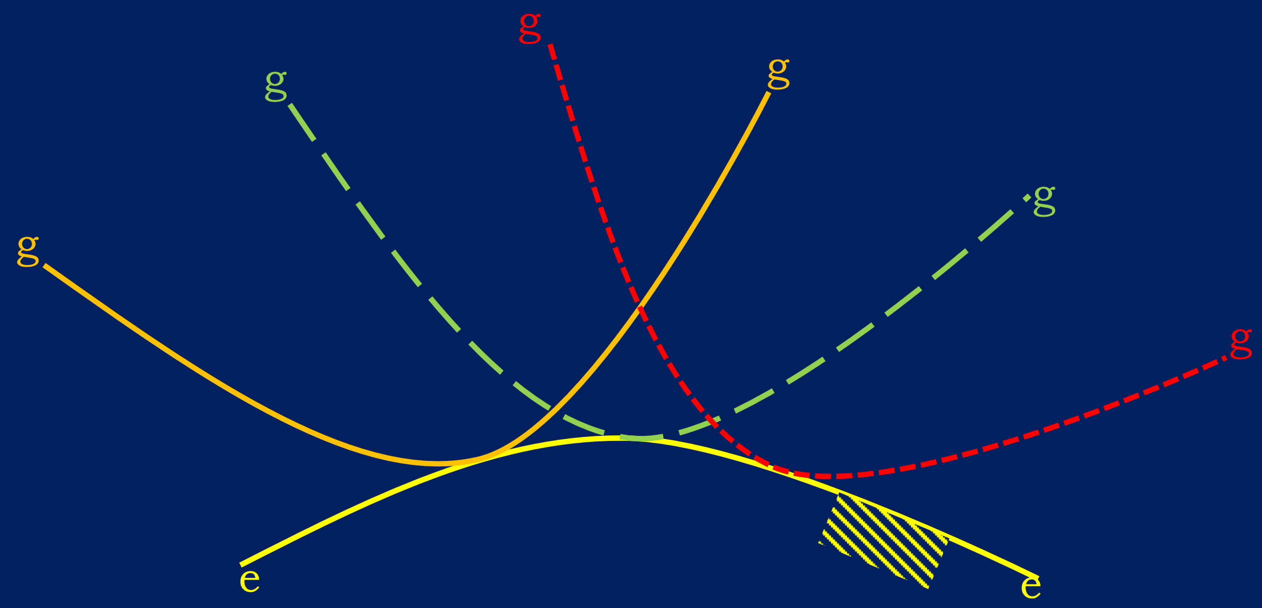
https://www.youtube.com/watch?v=9DhcAiV5U34

Generating Curves & Envelopes

https://www.youtube.com/watch?v=UtdSJZn62H8

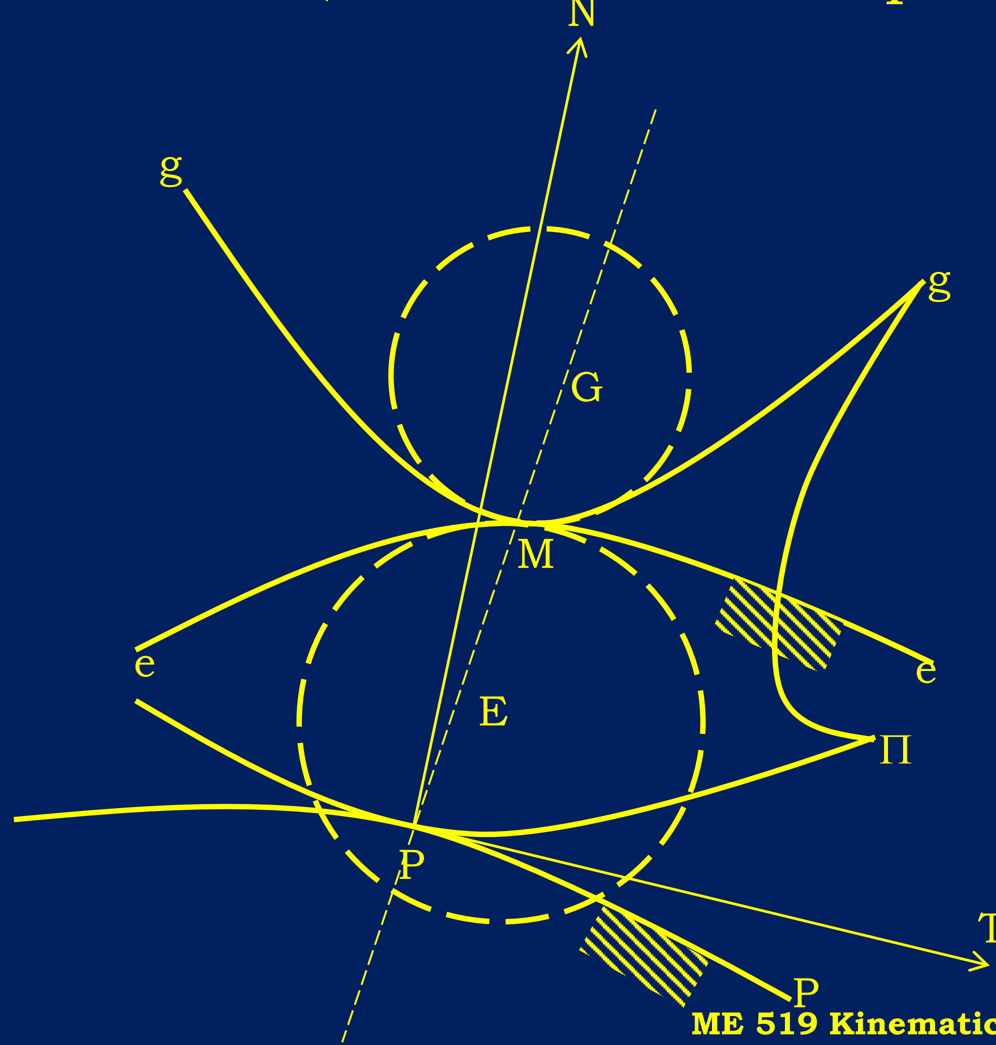
Generating Curves & Envelopes

g-g is the moving generating curve and e-e is the fixed envelope.



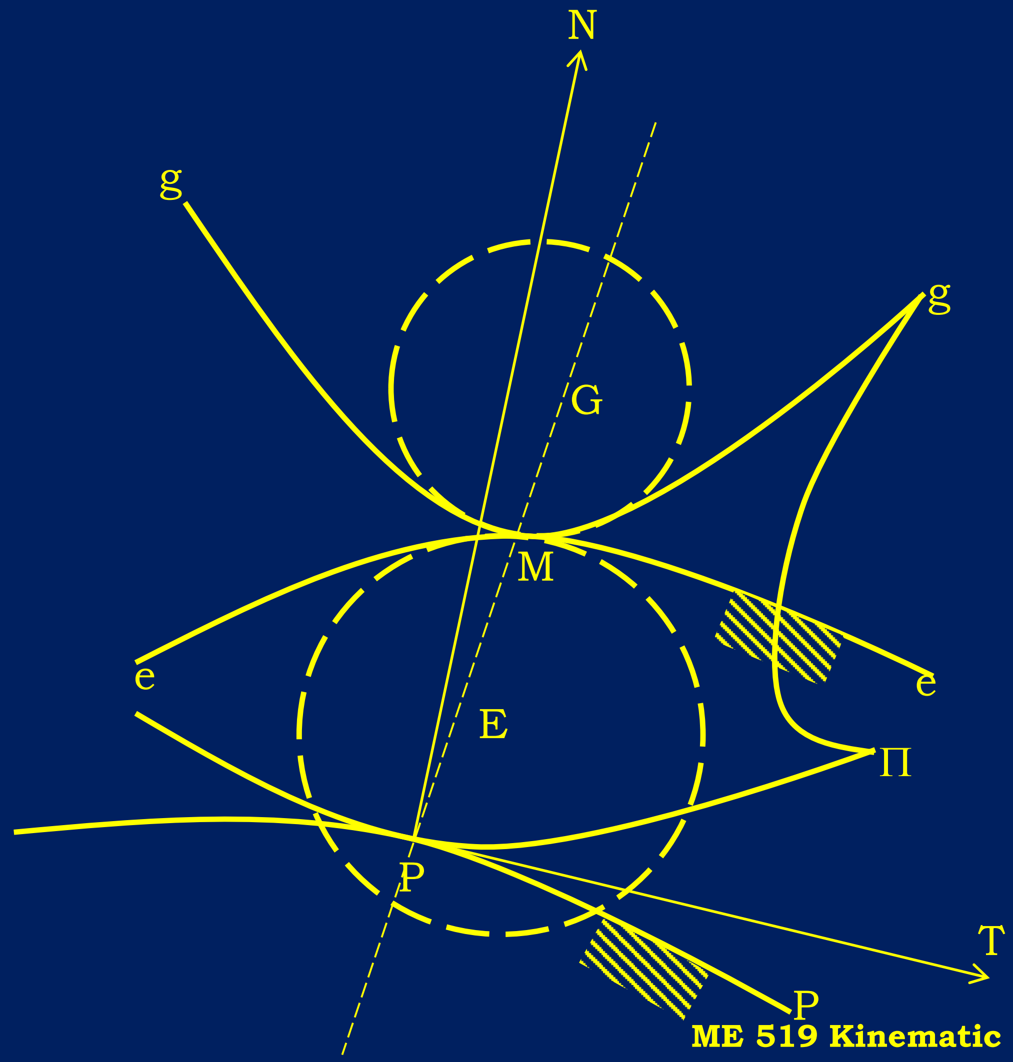
Generating Curves & Envelopes

While g-g is rolling and sliding on e-e, Π rolls on P without slipping. G is the center of curvature of g-g and E is the center of curvature of e-e, M is the contact point.



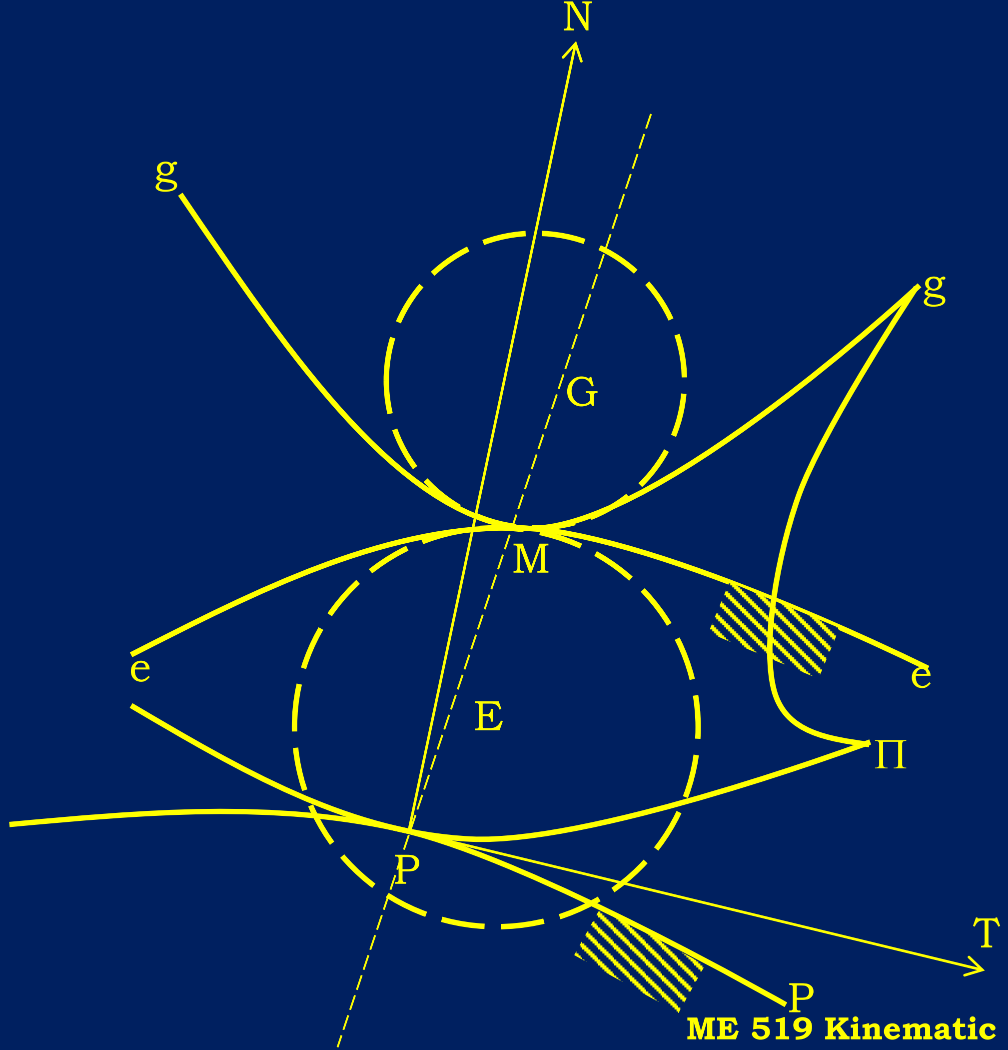
Generating Curves & Envelopes

Even if there is sliding at M still G momentarily traces a circle centered at E. Therefore E and G are conjugate points.



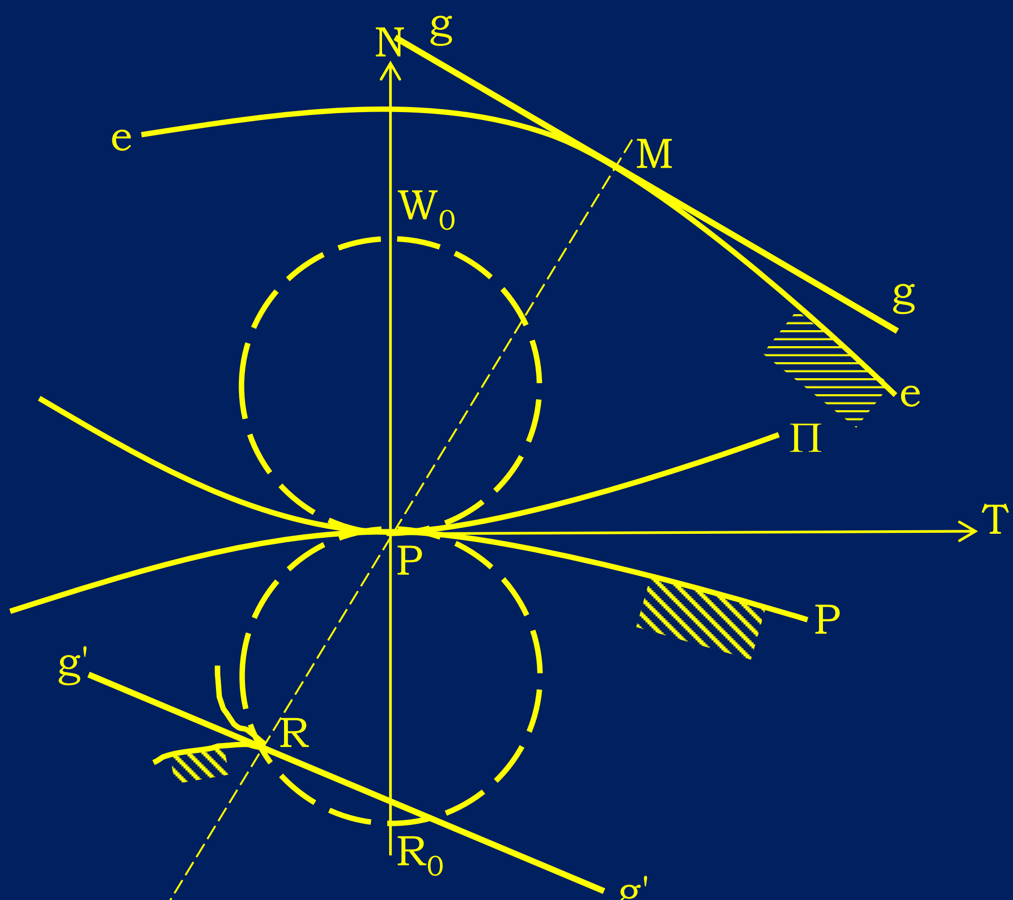
Generating Curves & Envelopes

Path of G is perpendicular to GM but it should also be perpendicular to PG (pole ray). Therefore P, E, M and G have to be collinear.



Generating Curves & Envelopes

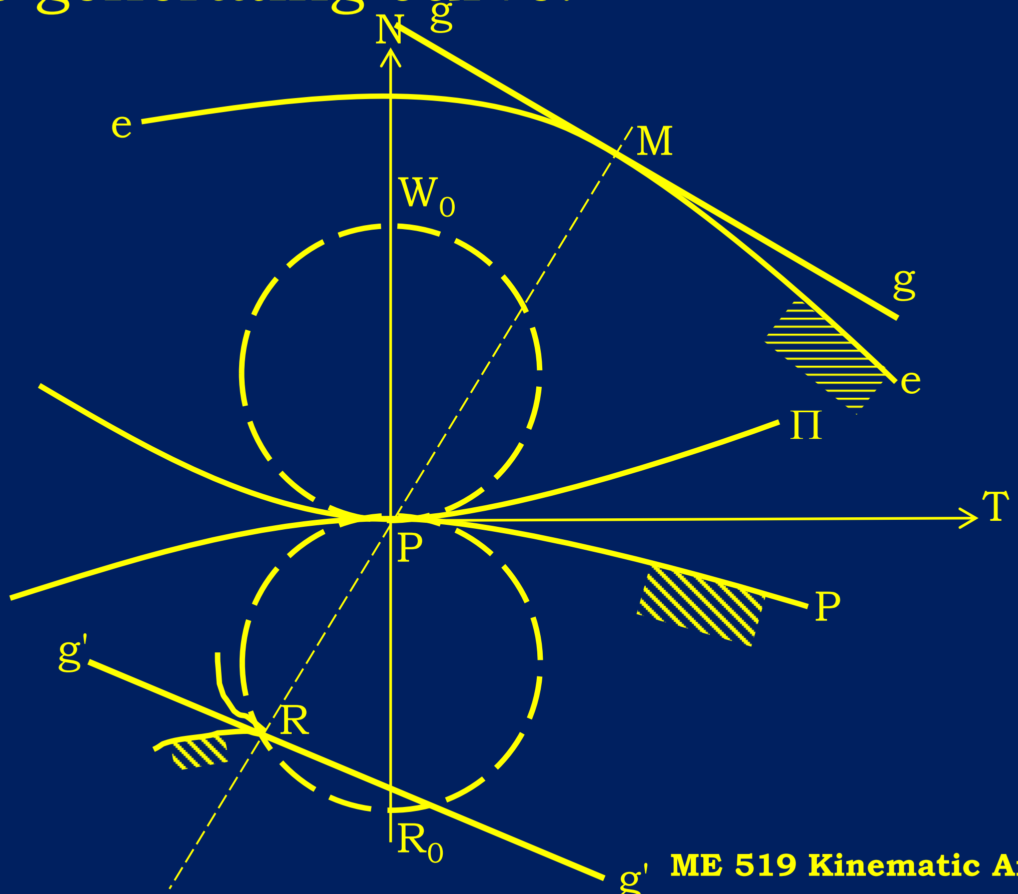
Aranhold's First Theorem: The return circle is the locus of centers of curvatures of all envelopes whose generating curves are straight lines.



Generating Curves & Envelopes

Aranhold's First Theorem, Proof:

For a straight generating curve g-g the center of curvature is at infinity on the ray. The center of curvature of all points at infinity lie on the return circle. Hence R is the center of curvature of the generating curve.



Generating Curves & Envelopes

Aranhold's First Theorem, Another Proof:

- Inflection circle is the locus of points whose infinitesimally separated positions lie on a straight line connecting that point to the inflection pole, W_0.
- For inverted motion the inflection and return circles exchange their roles. The return circle in inverted motion is the locus of points whose three infinitesimally separated positions lie on a straight line RR_0.
- If the straight line generating curve coincides with RR_0 then the envelope becomes a cusp.

Corollary: If a straight line on the moving plane always passes through a fixed point, then that point is on the return circle.

Generating Curves & Envelopes

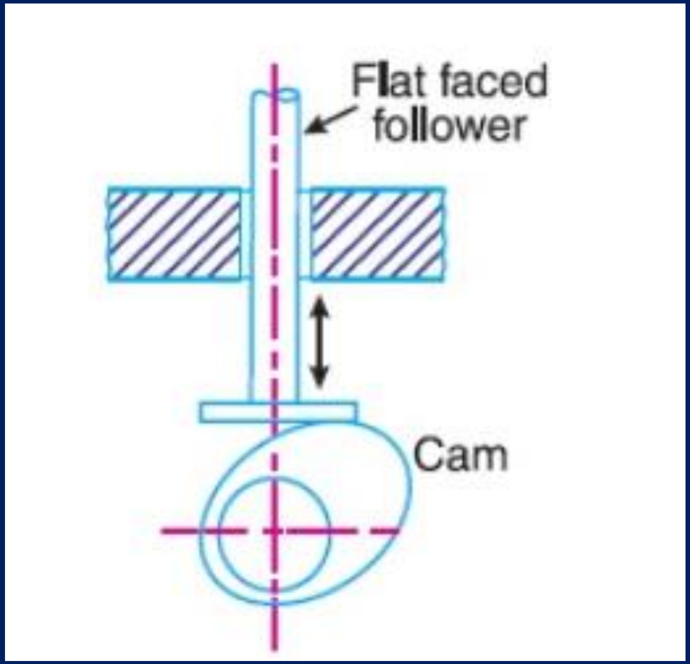
Aranhold's Second Theorem: The inflection circle is the locus of centers of curvatures of all generating curves whose envelopes are straight lines.

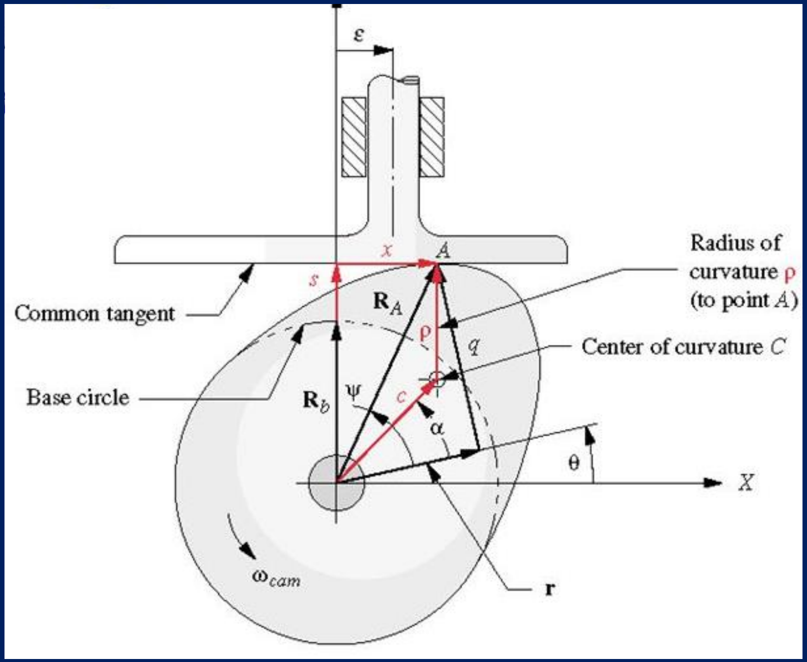
Using kinematic inversion and Aranhold's first theorem this can be proven.

Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

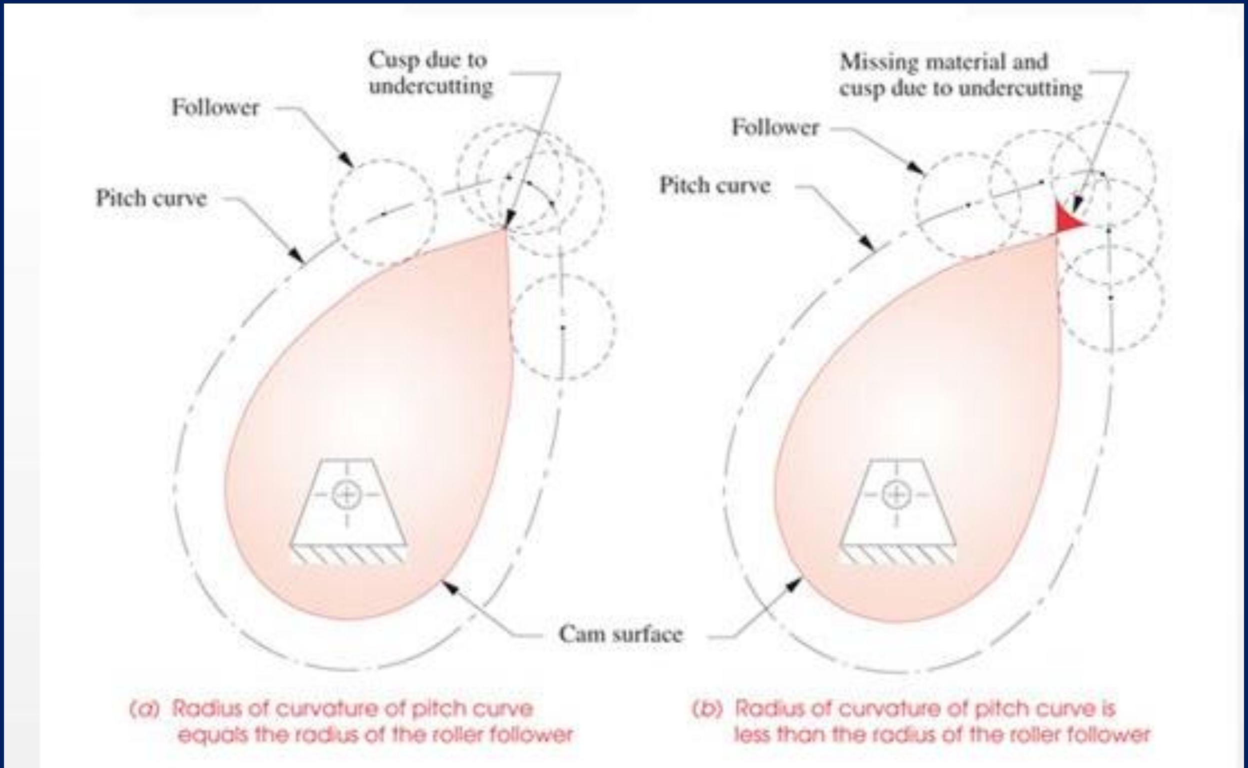
Recall from Machine Elements courses and Kinematic Synthesis of Mechanisms that undercutting ($\rho < 0$) is a situation where one cannot realize the desired follower motion with the cam. Further, to avoid high contact stresses radius of curvature of the cam surface has to be controlled.





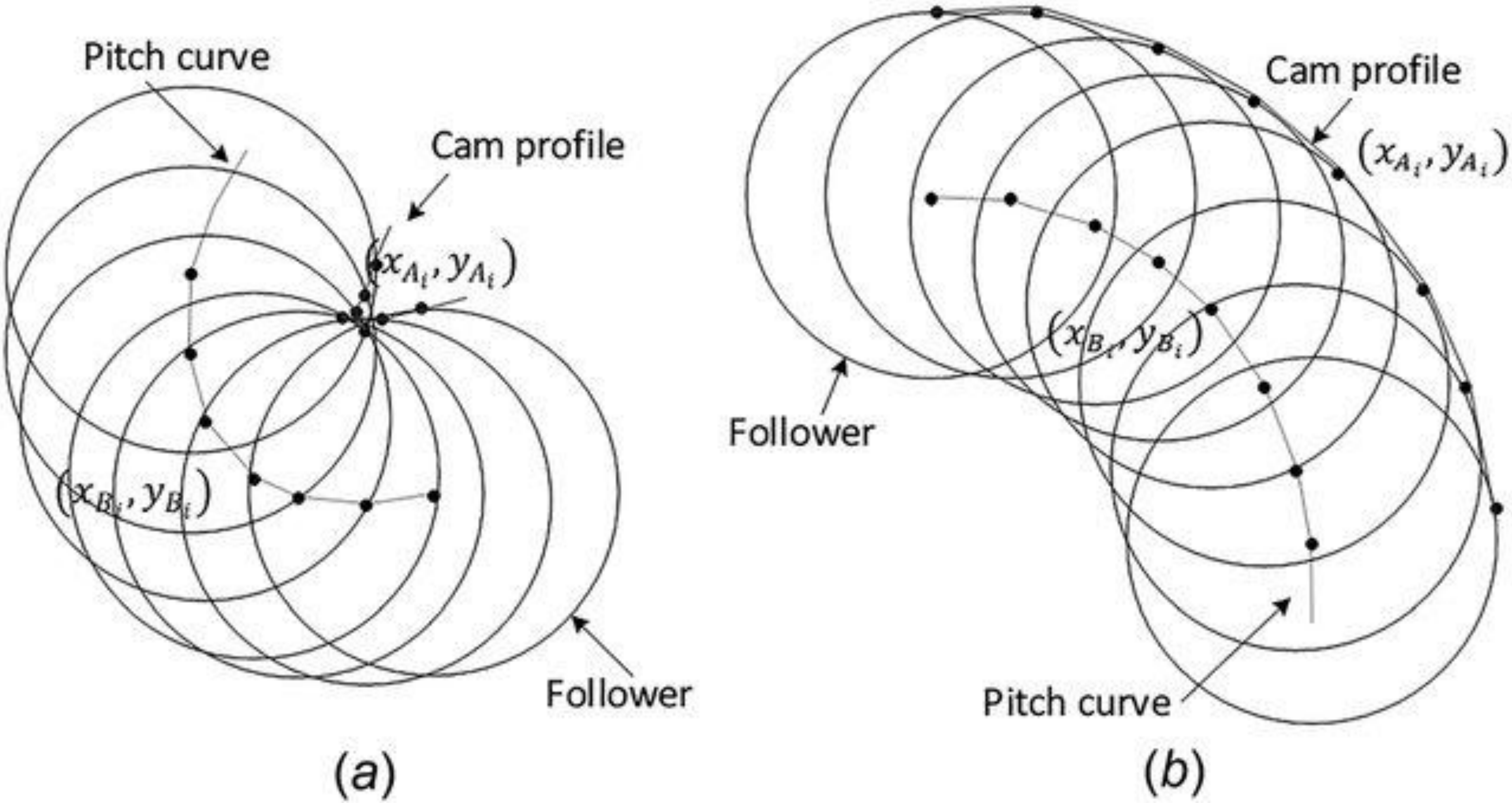
https://nirav56me.weebly.com/blog/cam-follower

Undercutting in Cams



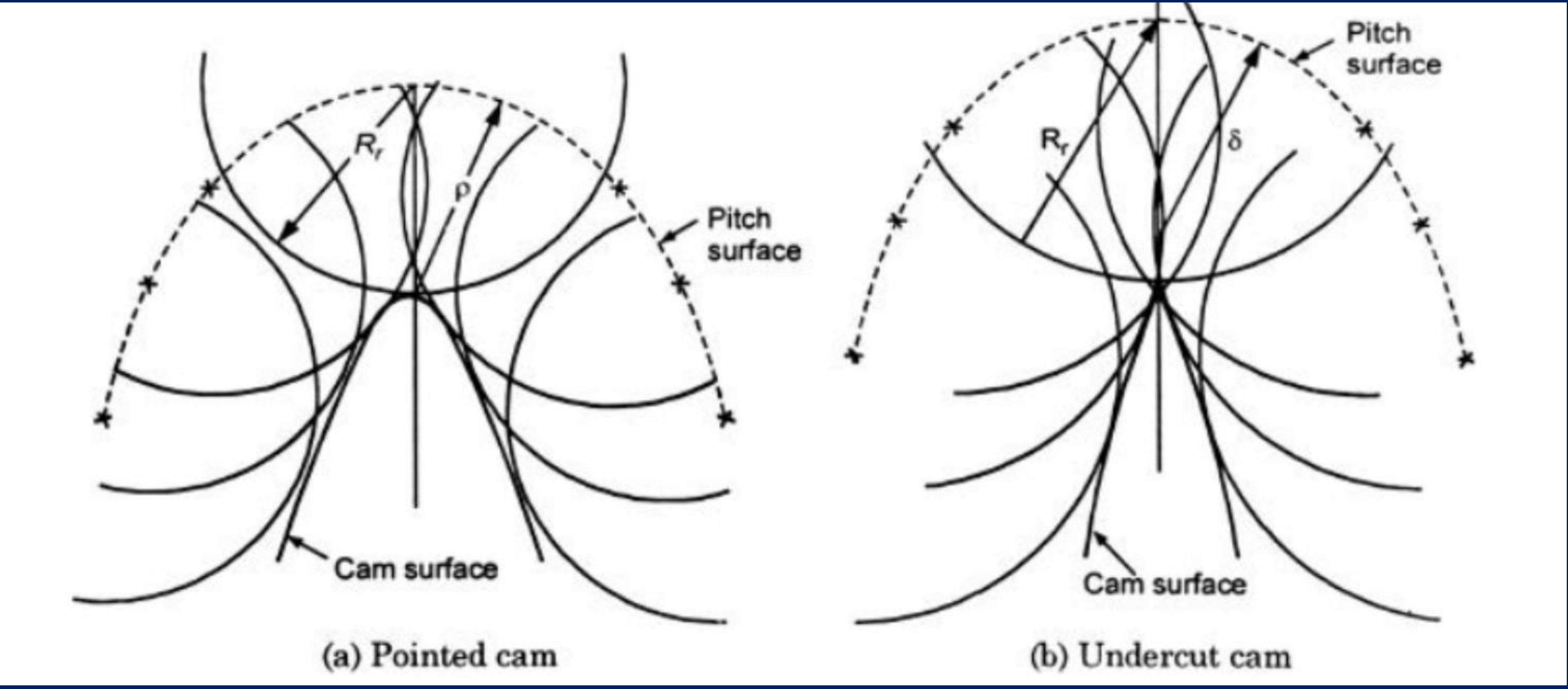
J. E. Shigley(?)

Undercutting in Cams



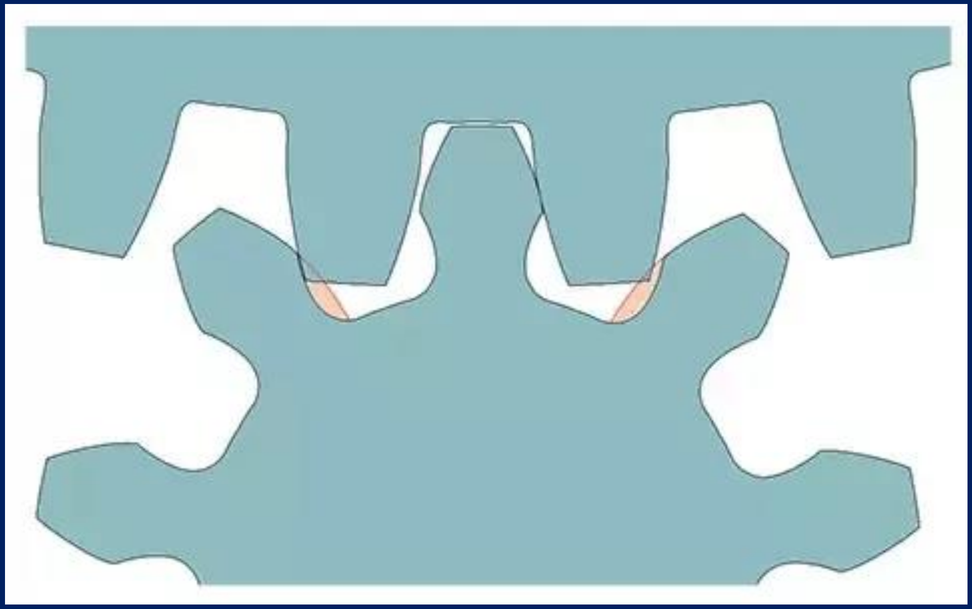
Cam Profile Generation for Cam-Spring Mechanism With Desired Torque, Gao Fei, Yannan Liu, Journal of Mechanisms and Robotics 10(4), DOI: 10.1115/1.4040270

Undercutting in Cams



https://www.slideshare.net/YatinSingh3/cams-and-followers

Undercutting in Gears

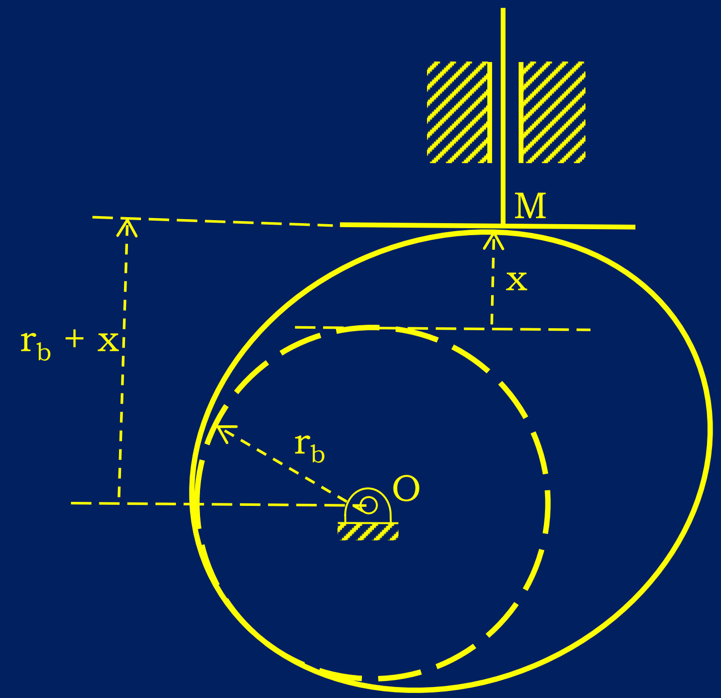


https://www.quora.com/What-is-undercutting-in-gear

Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

It is required to determine the radius of curvature of every contact point, M, as a function of r_b, x, $\dot{x}$ and $\ddot{x}$.

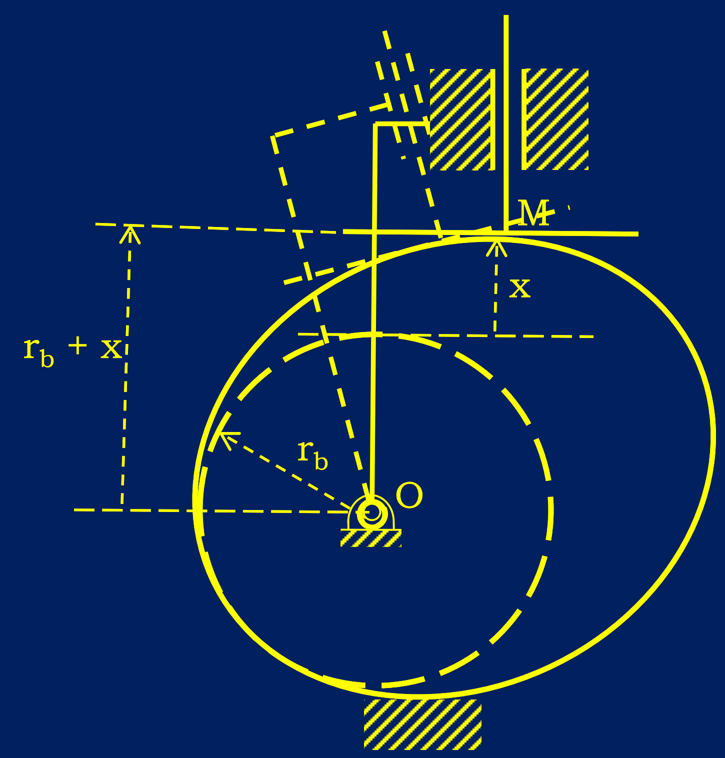


Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

By kinematic inversion fix the cam and let the follower rotate around it.

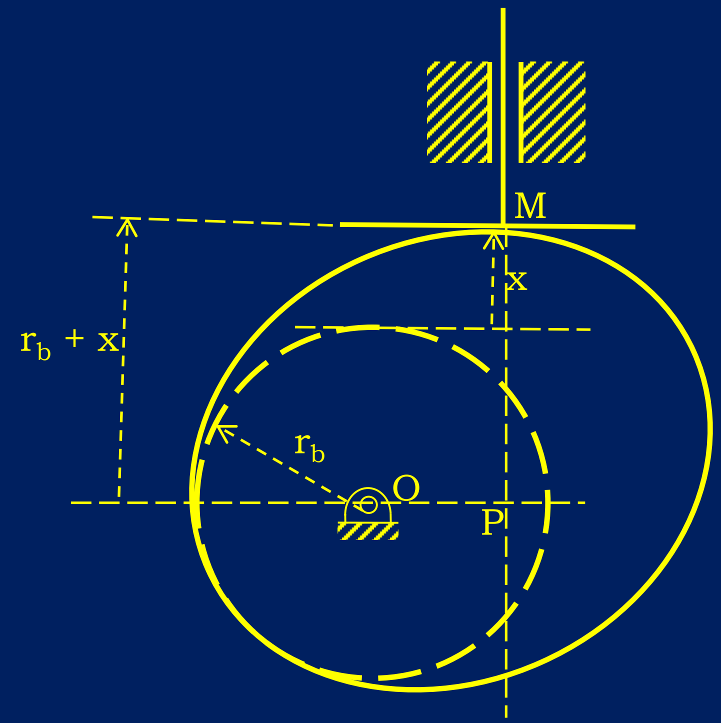
Now the follower is the generating curve and the cam is the envelope!



Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

The pole, P, is at the intersection of normal of the cam profile at M and a perpendicular drawn to the follower path from O.

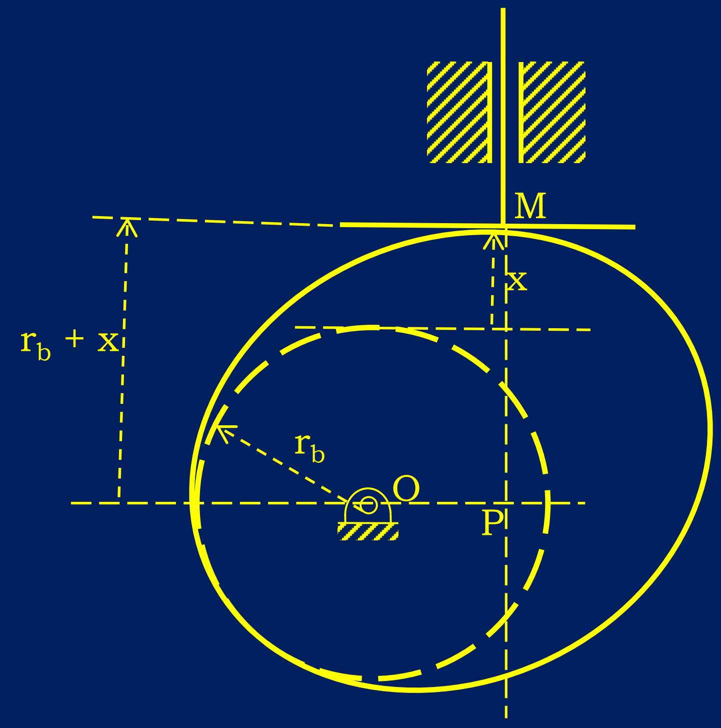


Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

According to Aranhold's first theorem the center of curvature, E, of the envelope, e-e, (i.e. the cam) lies on the return circle.

According to corollary to Aranhold's first theorem O lies on the return circle.

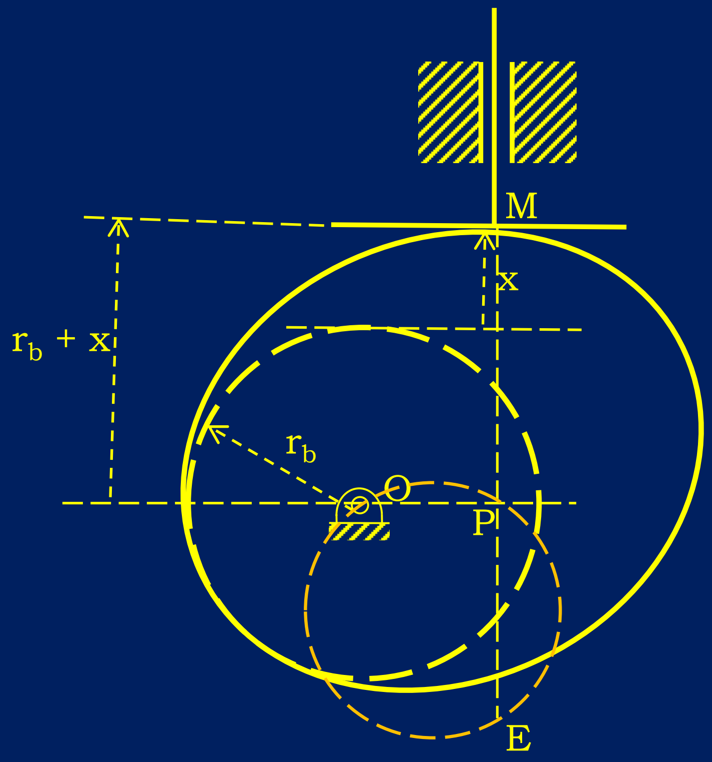


Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

According to corollary to Aranhold's first theorem O lies on the return circle.

Using P, E and O return circle is drawn.



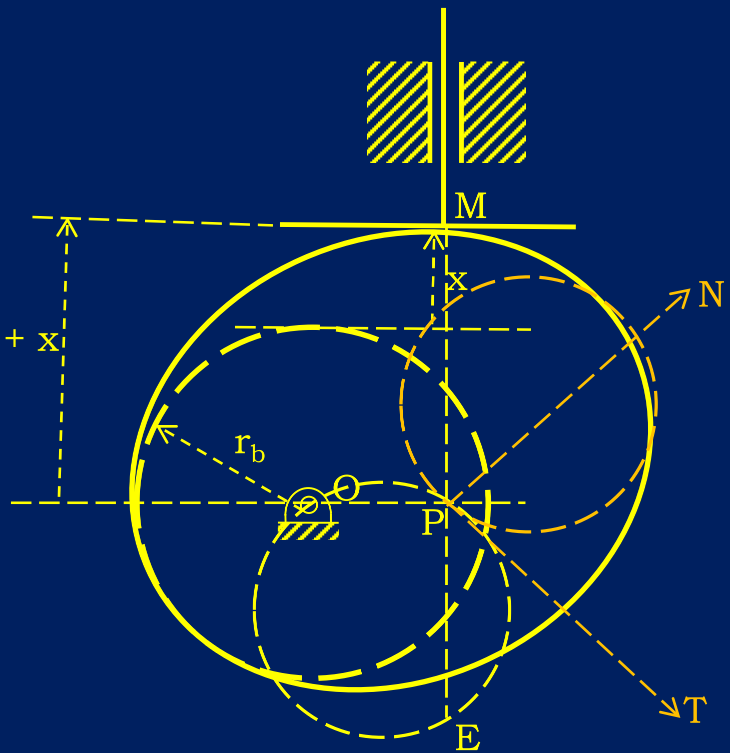
Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

Pole tangent is tangent to the return circle at P.

Pole normal is perpendicular to pole tangent.

Inflection circle is the mirror image of the return circle about pole tangent.



Generating Curves & Envelopes

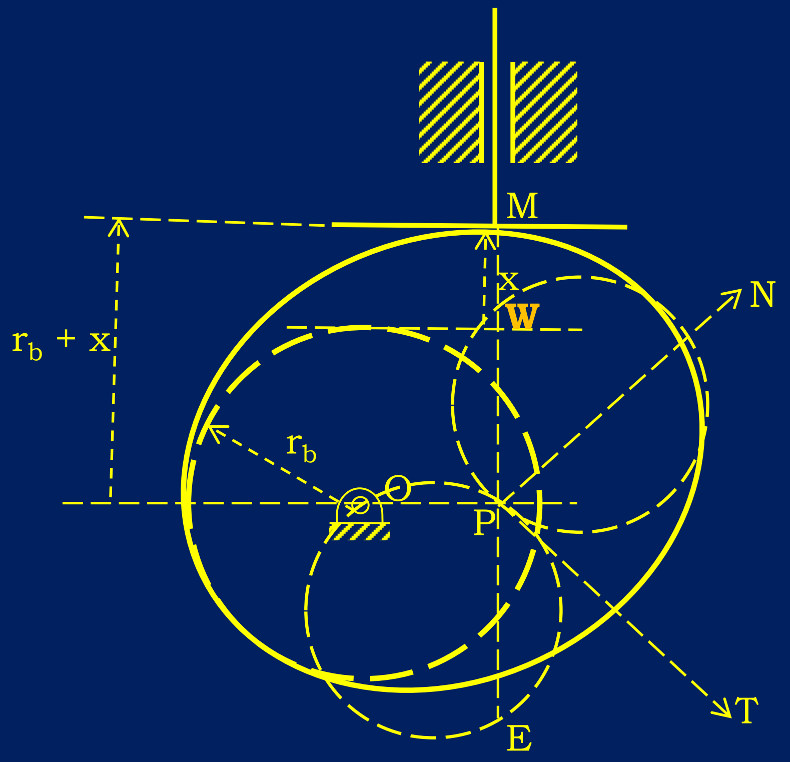
Example: Consider a disk cam with flat faced translating follower.

$|PE| = |PW|$ (inflection and return circles are mirror images of each other with respect to pole tangent).

$\rho = |EM| = |EP| + r_b + x$

$|EP| = |PW|$

$\rho = r_b + x + |PW|$



Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

For a cam rotating at constant speed:

$\vec{a}_P = \omega^2 \overrightarrow{PW}$

The vertical component of $\vec{a}_P$ is $\ddot{x}$.

$$\frac{\ddot{x}}{a_P} = \frac{|PW|}{|PW_0|}$$

$$|PW| = \frac{\ddot{x}}{a_P}|PW_0| = \frac{\ddot{x}|PW_0|}{\omega^2|PW_0|}$$

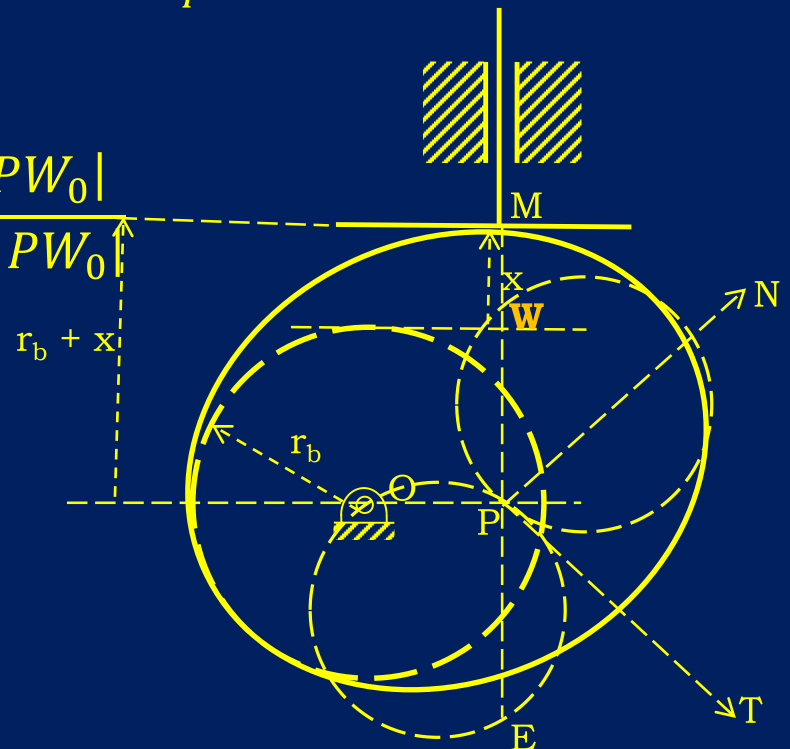
$$|PW| = \frac{\ddot{x}}{\omega^2}$$

Recall

$\rho = r_b + x + |PW|$

so

$$\rho = r_b + x + \frac{\ddot{x}}{\omega^2}$$

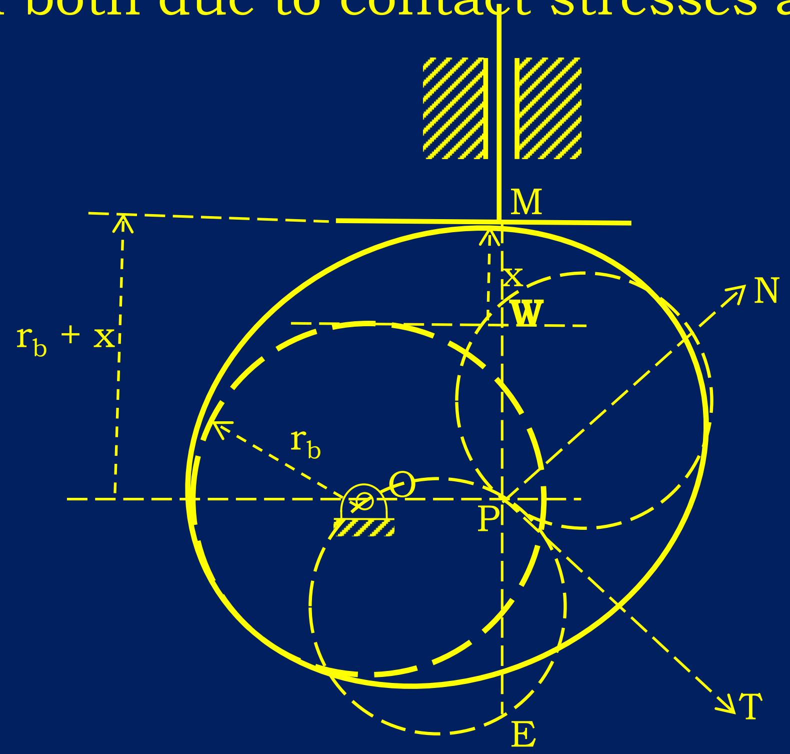


Generating Curves & Envelopes

Example: Consider a disk cam with flat faced translating follower.

$$\rho = r_b + x + \frac{\ddot{x}}{\omega^2}$$

This equation proves to be useful in determining the radius of curvature of the cam both due to contact stresses and undercutting.



Generating Curves & Envelopes

Example: Consider cycloidal motion rise curve of height H during β cam rotation.

Equation of normalized motion curve is:

$$\frac{x}{H} = \frac{1}{\pi}\left[\frac{\pi\theta}{\beta} - \frac{1}{2}sin\left(\frac{2\pi\theta}{\beta}\right)\right]$$

$$\frac{\dot{x}}{H} = \frac{1}{\pi}\left[\frac{\pi\dot{\theta}}{\beta} - \frac{2\pi\dot{\theta}}{2\beta}cos\left(\frac{2\pi\theta}{\beta}\right)\right] = \frac{\dot{\theta}}{\beta}\left[1 - cos\left(\frac{2\pi\theta}{\beta}\right)\right]$$

$$\frac{\ddot{x}}{H} = \frac{2\pi\dot{\theta}^2}{\beta^2}sin\left(\frac{2\pi\theta}{\beta}\right)$$

$$\rho = r_b + x + \frac{\ddot{x}}{\omega^2}$$

$$\rho = r_b + \frac{H}{\pi}\left[\frac{\pi\theta}{\beta} - \frac{1}{2}sin\left(\frac{2\pi\theta}{\beta}\right)\right] + \frac{H}{\omega^2}\frac{2\pi\dot{\theta}^2}{\beta^2}sin\left(\frac{2\pi\theta}{\beta}\right)$$

define

$$\rho' = \frac{\rho}{r_b}, H' = \frac{H}{r_b}$$

Generating Curves & Envelopes

Example: Consider cycloidal motion rise curve of height H during β cam rotation.

$$\rho' = 1 + \frac{H'\theta}{\beta} + \frac{H'}{2\pi}\left[\left(\frac{2\pi}{\beta}\right)^2 - 1\right] sin\left(\frac{2\pi\theta}{\beta}\right)$$

Differentiate this equation with respect to θ to obtain ρ_{min}:

i. For $\beta > 152.1^o$ $\rho_{min} = r_b$

ii. For $\beta < 152.1^o$

$$\rho'_{min} = 1 - \frac{H'}{2\pi}\left[tan\left(\frac{2\pi\theta_m}{\beta}\right) - \frac{2\pi\theta_m}{\beta}\right]$$

where

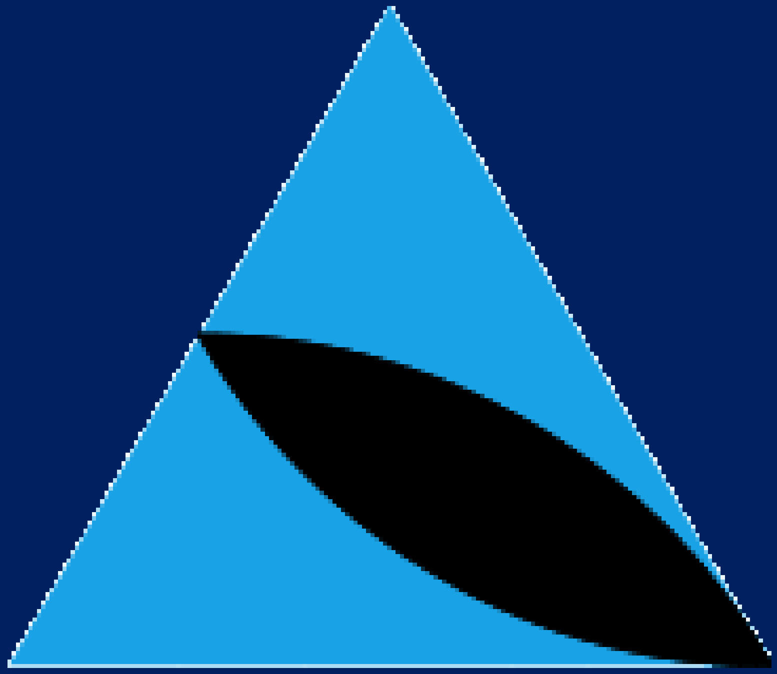
$$\theta_m = \frac{\beta}{2\pi} cos^{-1}\left[\frac{1}{1 - \left(\frac{2\pi}{\beta}\right)^2}\right], \frac{\beta}{2} < \theta_m < \frac{3\beta}{4}$$

- To realize motion $\rho' > 0$
- To control contact stresses ρ_{min} should be controlled.

Generating Curves & Envelopes

Example: Constant Breadth (Diameter) Cams

Biangle in a triangle

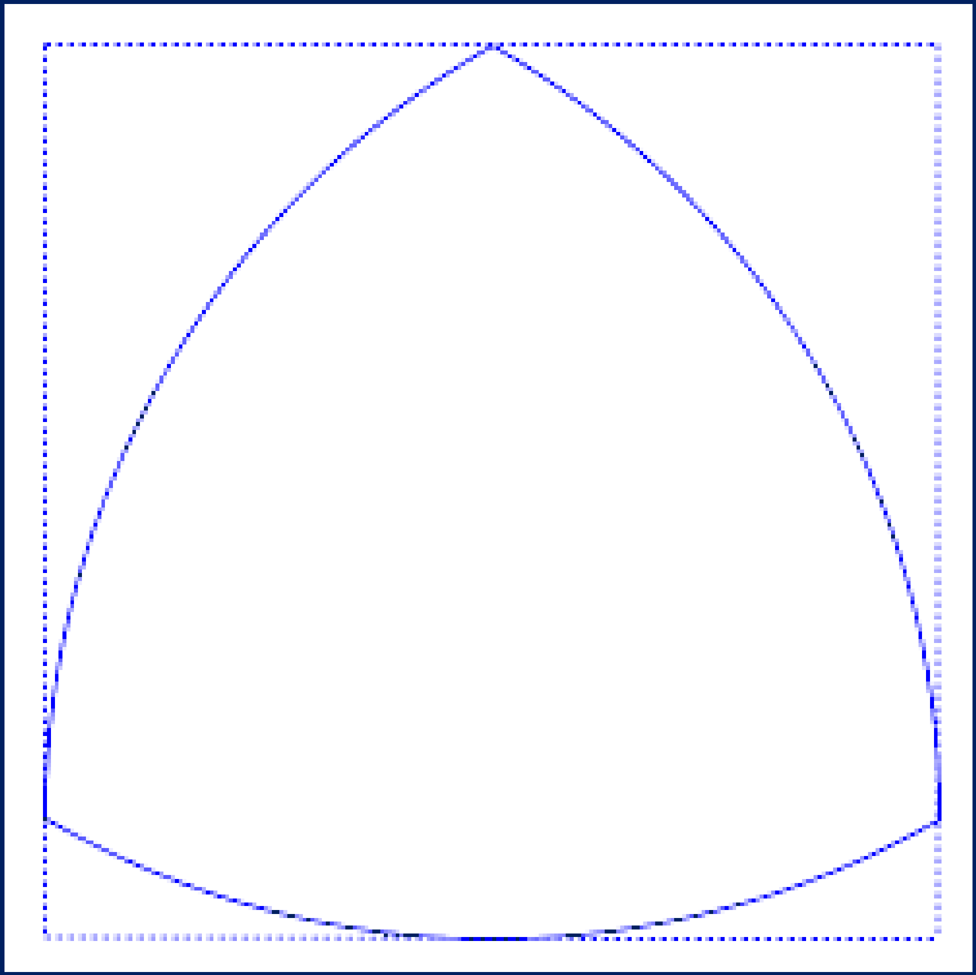


https://en.wikipedia.org/wiki/Curve_of_constant_width#/media/File:Lens_Rotating_in_Triangle.gif

Generating Curves & Envelopes

Example: Constant Breadth (Diameter) Cams

Reuleaux Triangle

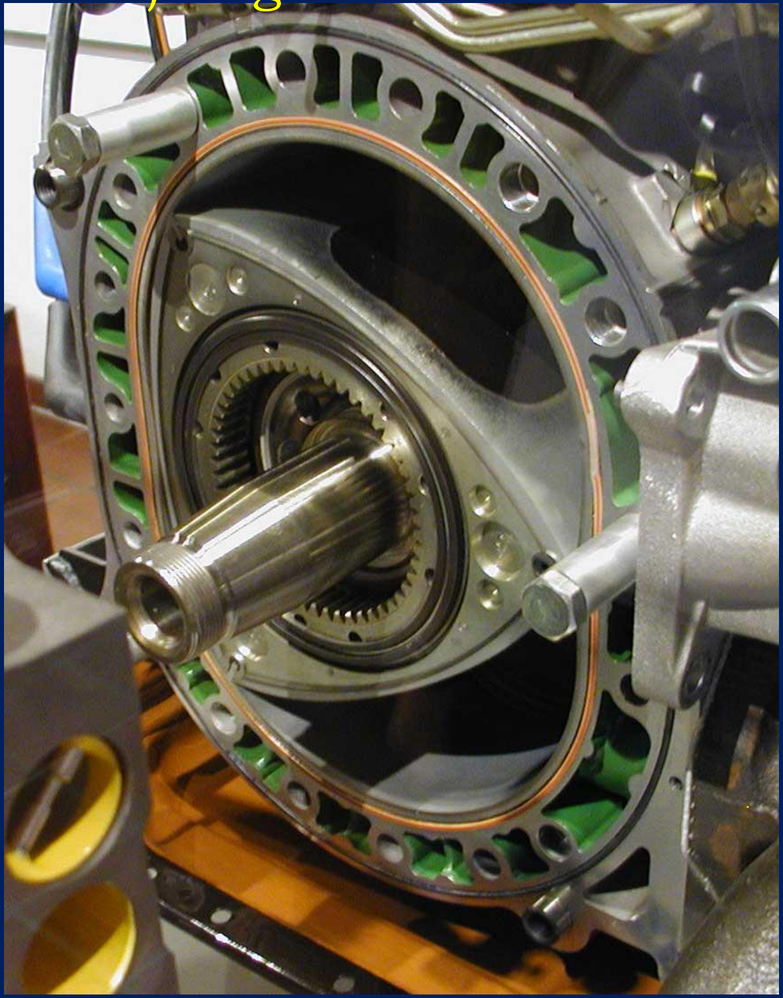


https://en.wikipedia.org/wiki/Curve_of_constant_width#/media/File:Reuleaux_triangle_Animation.gif

Generating Curves & Envelopes

Example: Constant Breadth (Diameter) Cams

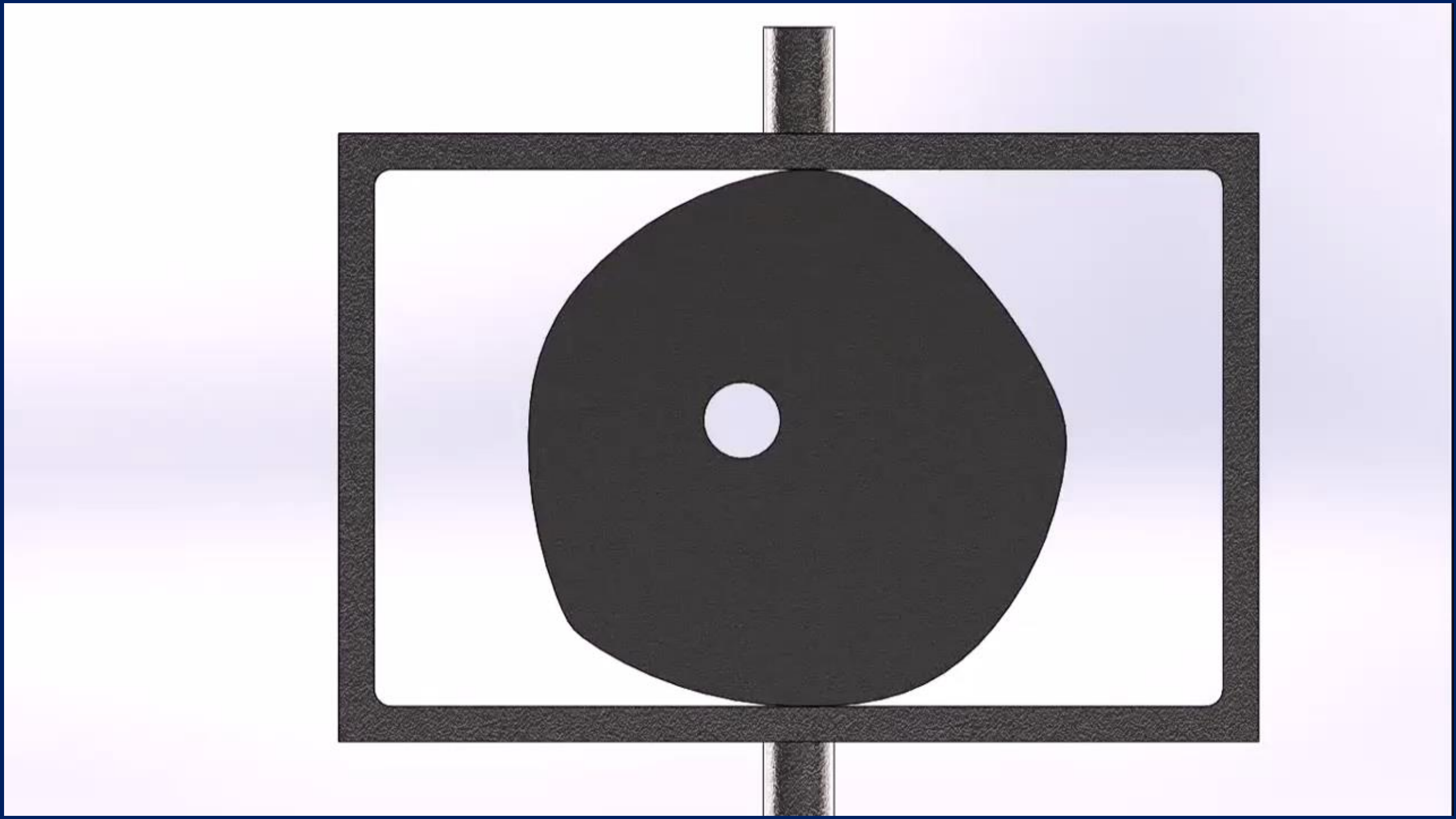
Wankel (Rotary/Pistonless) Engine



https://upload.wikimedia.org/wikipedia/commons/b/ba/Wankel-1.jpg

Generating Curves & Envelopes

Example: Constant Breadth (Diameter) Cams

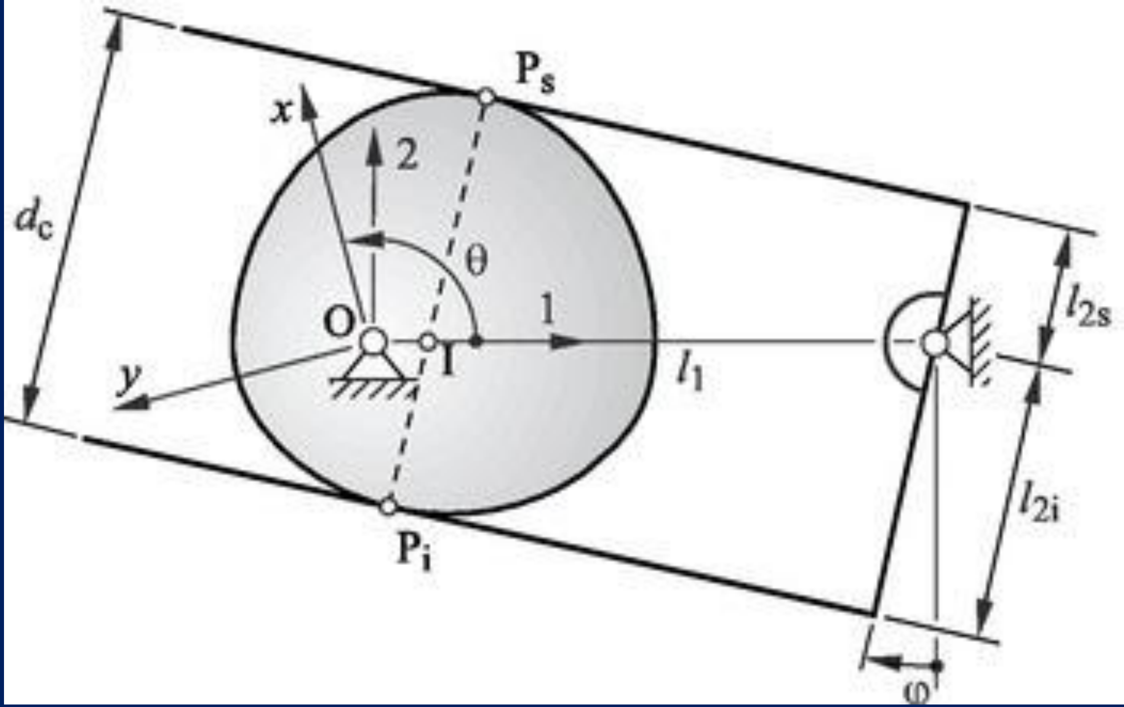


https://www.youtube.com/watch?v=ENnLMWD03HQ

Generating Curves & Envelopes

Example: Constant Breadth (Diameter) Cams

Unlike regular cams which are open kinematic pairs, constant breadth cams are form closed. They do not jump and good for high speed applications however their design and production are complex compared to force closed cams.



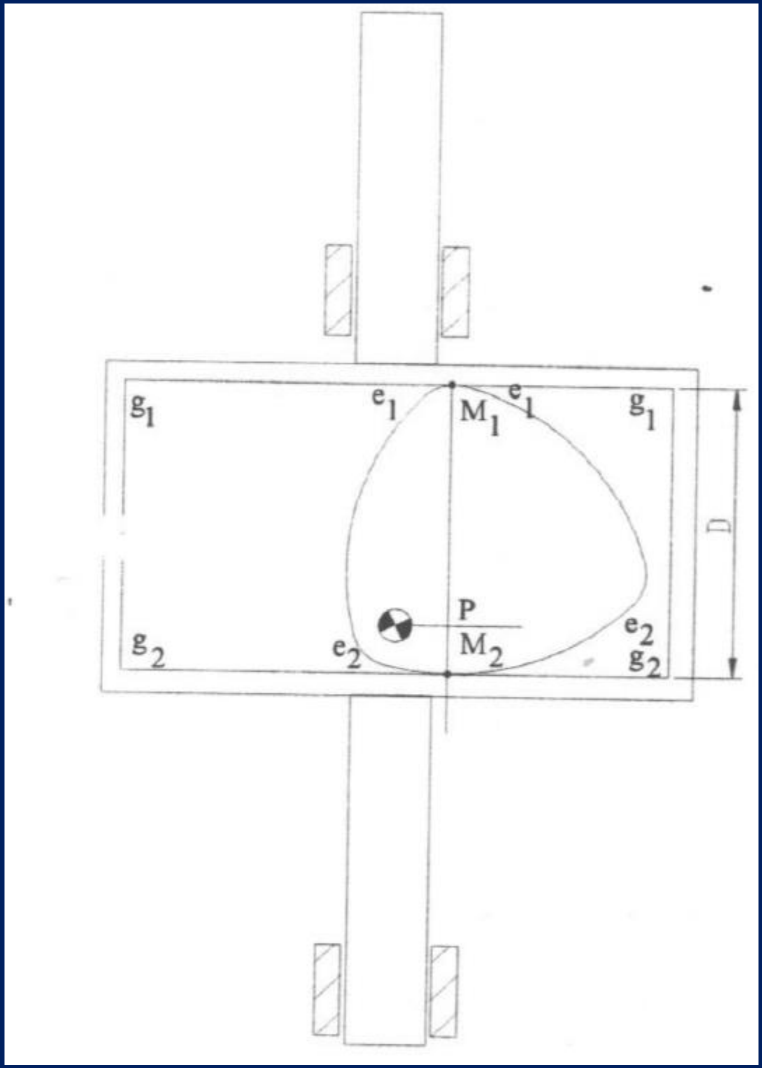
https://www.europeana.eu/portal/en/record/2020801/dmglib_handler_image_6868023.html

Generating Curves & Envelopes

Example: Constant Breadth (Diameter) Cams

In a constant breadth cam there are two generating curves and two envelopes (upper and lower). Let 1 denote upper and 2 lower.

For movability the contact points M_1, M_2 and P must be collinear all the times.



Eres Söylemez (unpublished ME 519 lecture notes)

Generating Curves & Envelopes

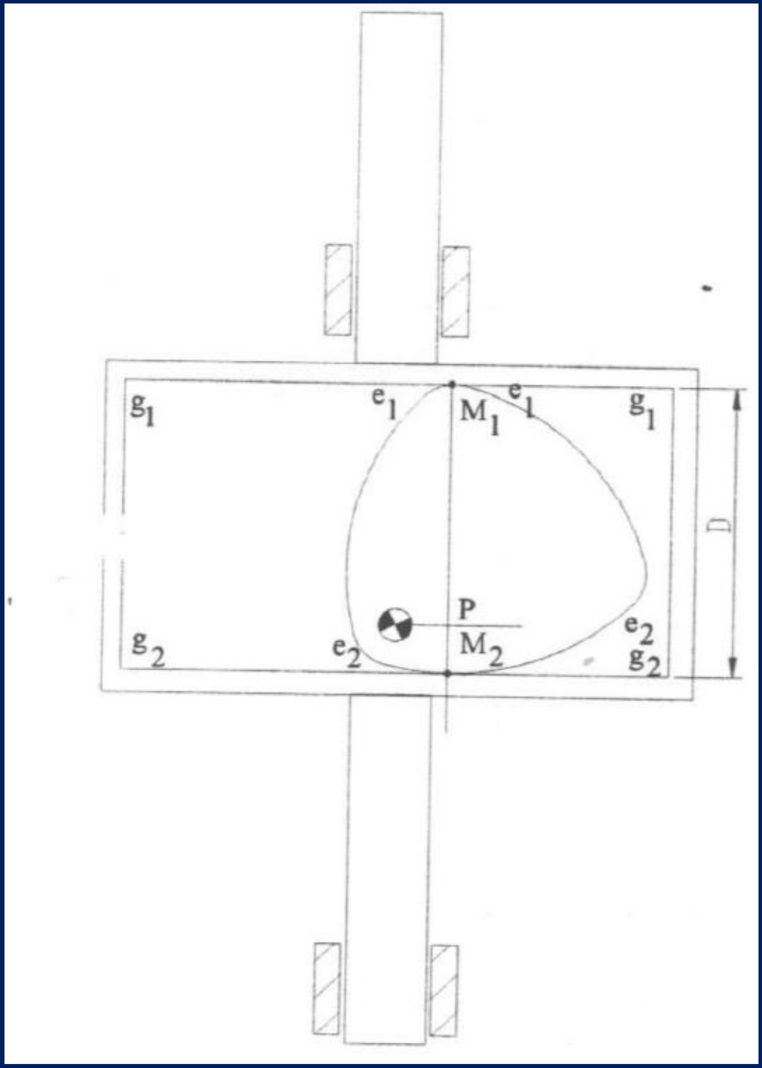
Brunell's First Theorem: The normal at the point of tangency between a constant breadth cam and its follower must be collinear.

$$\rho_1 = r_{b_1} + x + \frac{\ddot{x}}{\omega^2}$$

Aranhold's first theorem states that centers of curvature of M_1 and M_2 lay on the return circle (single because single follower!) which implies these centers of curvatures must be coincident.

Let D be the distance between the two followers:

$$\rho_1 + \rho_2 = D$$



Eres Söylemez (unpublished ME 519 lecture notes)

Generating Curves & Envelopes

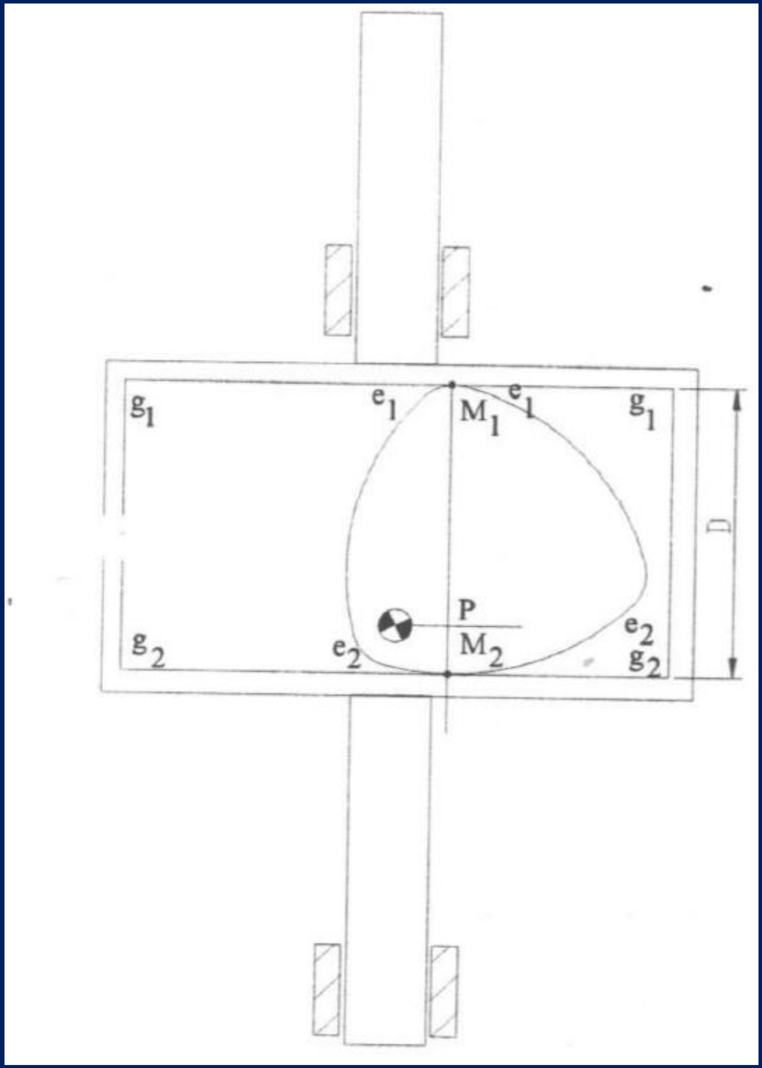
$$\rho_1 + \rho_2 = D$$

$$\rho_1 = r_{b_1} + x + \frac{\ddot{x}}{\omega^2}$$

$$\rho_2 = r_{b_2} - x - \frac{\ddot{x}}{\omega^2}$$

$$r_{b_1} + r_{b_2} = D$$

$$r_{b_2} = D - r_{b_1}$$

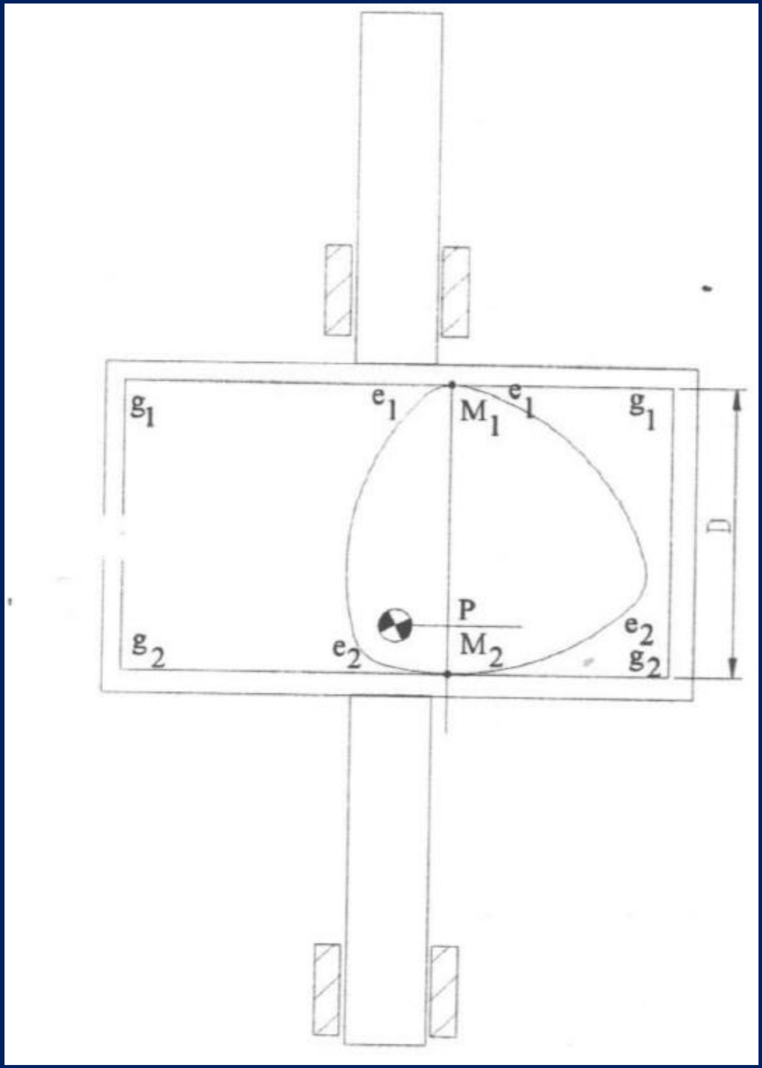


Eres Söylemez (unpublished ME 519 lecture notes)

Generating Curves & Envelopes

Brunell's Second Theorem: The sum of curvatures of points of constant breadth cam is equal to the through diameter D.

Any curve (not only circular arcs) satisfying Brunell's two theorems can be used for constant breadth cams.



Eres Söylemez (unpublished ME 519 lecture notes)

(Kinematically) Equivalent Linkages

The *unfamiliar* mechanisms may be replaced by a kinematically equivalent mechanism which is familiar. Higher kinematic pairs may be replaced by lower ones.

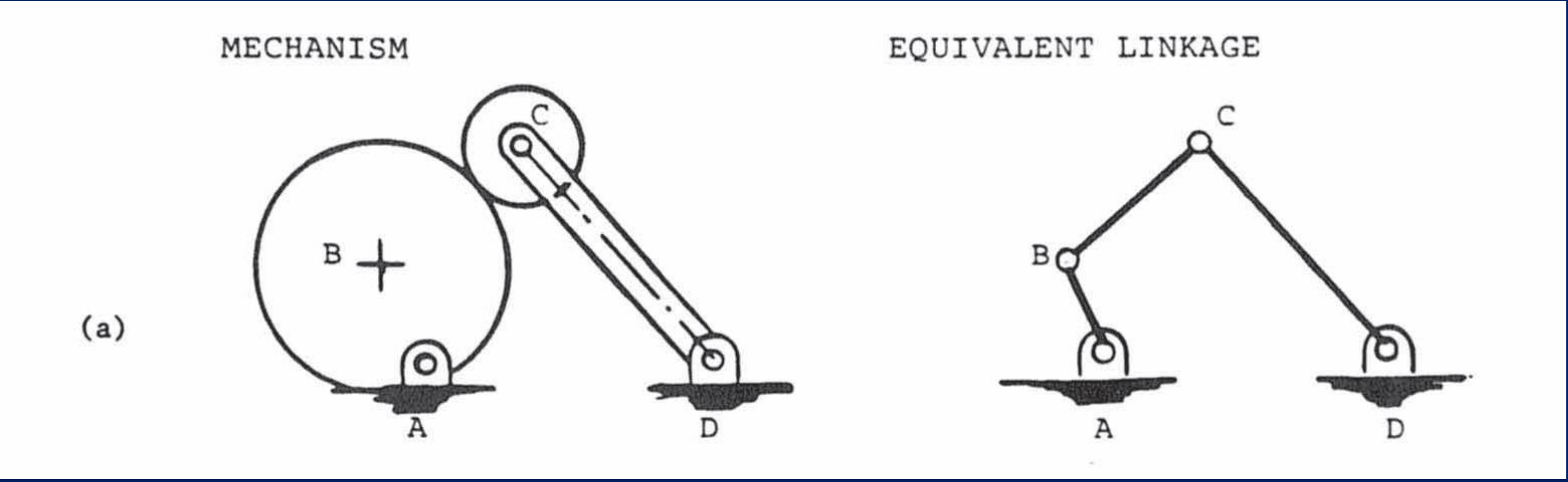
Kinematic equivalence is defined in accordance with the purpose of the kinematic analysis, that is equivalent mechanism should have *identical* kinematic behavior with the original mechanism for the purpose of the analysis.

Equivalent linkages may be used for:

1. Determination of velocities, accelerations (or sometimes higher order derivatives),
2. Determination of motion of a link of a mechanism up to a certain order,
3. Determination of path curvature.

Four-bar, slider crank and its inversions are the most familiar equivalent linkages.

(Kinematically) Equivalent Linkages



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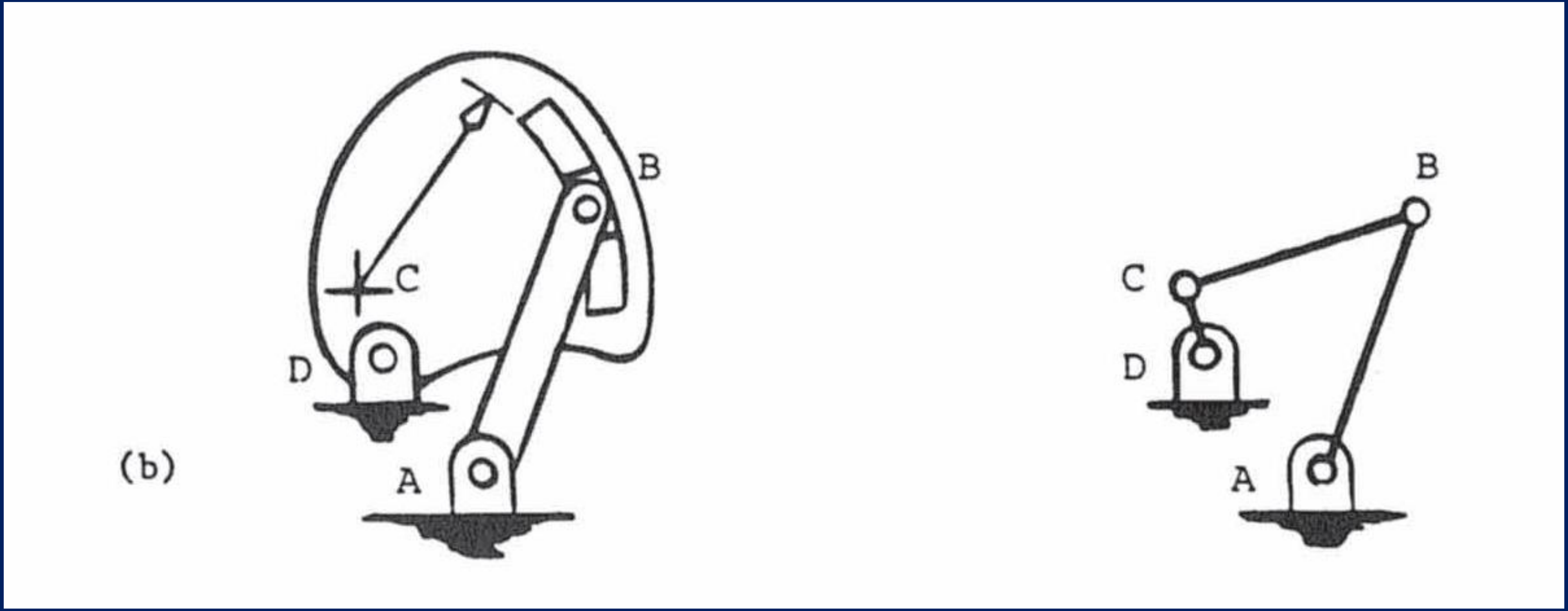
(Kinematically) Equivalent Linkages



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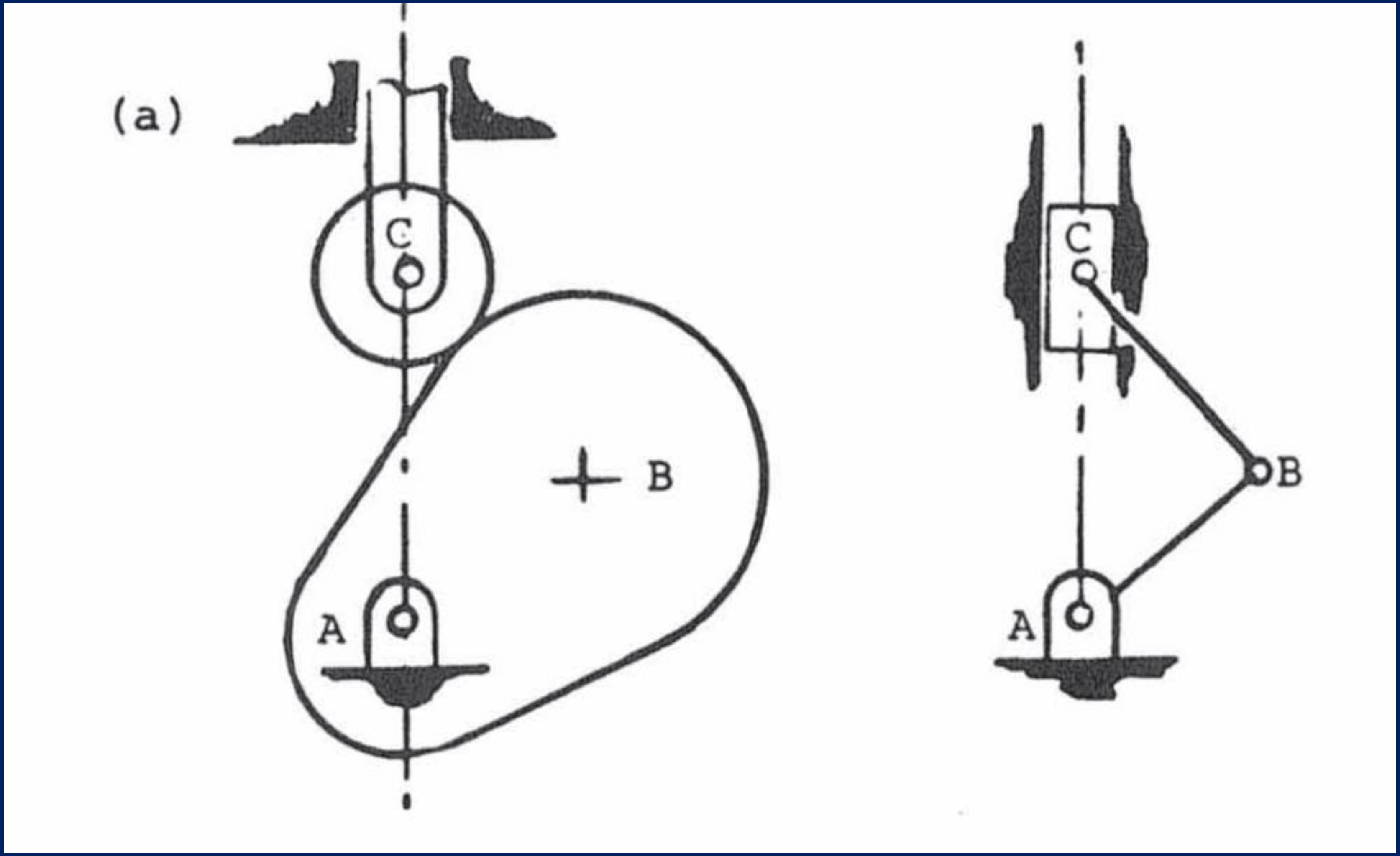
(Kinematically) Equivalent Linkages



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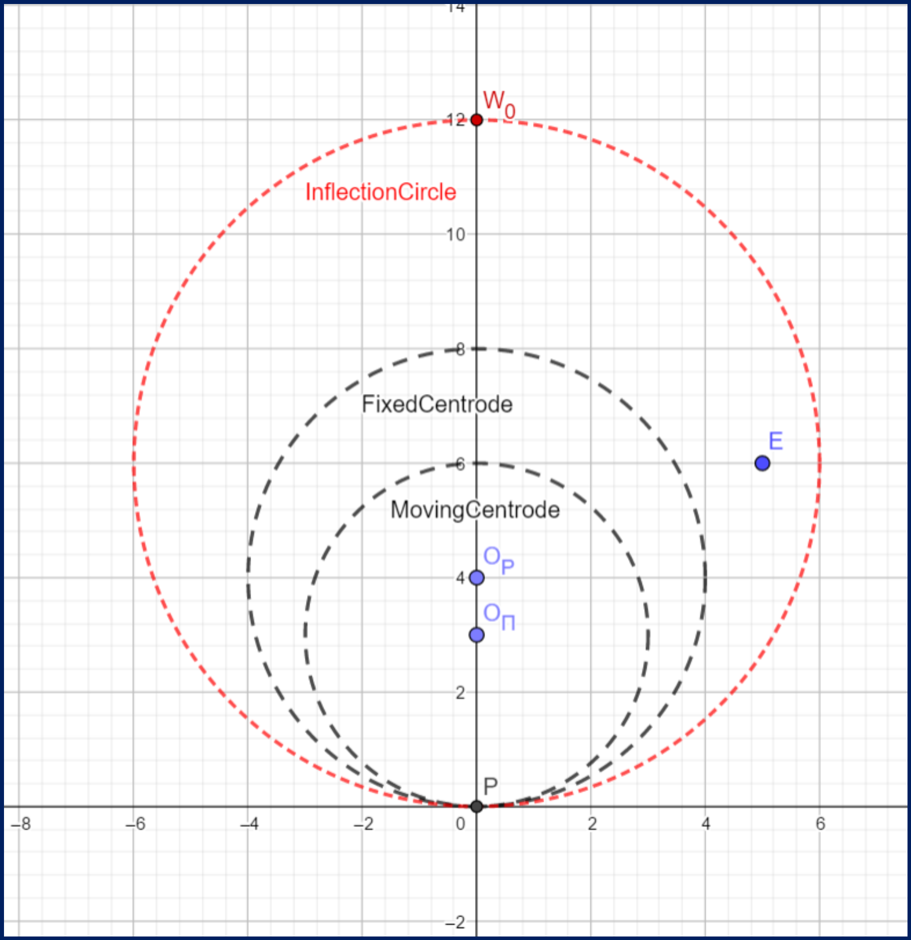
(Kinematically) Equivalent Linkages

Example: Consider point E on the moving plane. At the instant considered the radii of curvature of the fixed (O_P) and moving (O_Π) centrodes are known.

The inflection circle diameter is

$$\frac{1}{\delta} = \frac{1}{r_\Pi} - \frac{1}{r_P} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \delta = 12$$



(Kinematically) Equivalent Linkages

Example:

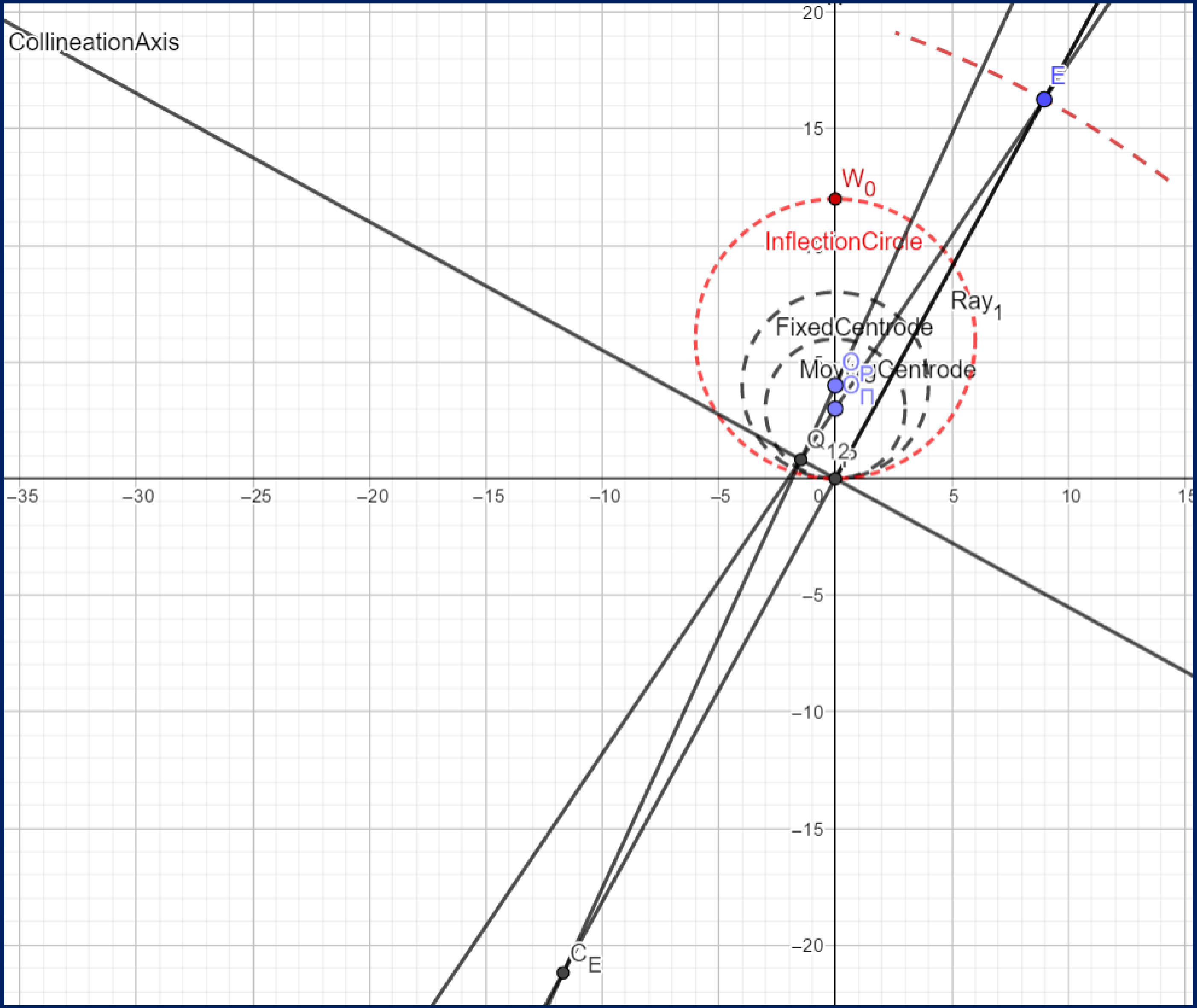
One may determine center of curvature of any point on the moving plane by Bobillier's theorem (Application Examples Case 2).

Select two arbitrary points, A and B on the moving plane and determine their centers of curvature, A_0 and B_0 which are unique.

By selecting different points, A and B, one may obtain infinitely many (*four free parameters, say* r_A, ψ_A, r_B *and* ψ_B) four bar mechanisms approximating the motion of point E (and therefore the moving plane) *in the infinitesimal neighborhood of the design position* to the second order (i.e. position, tangent and curvature or infinitesimally separated three positions or *position, velocity and acceleration*).

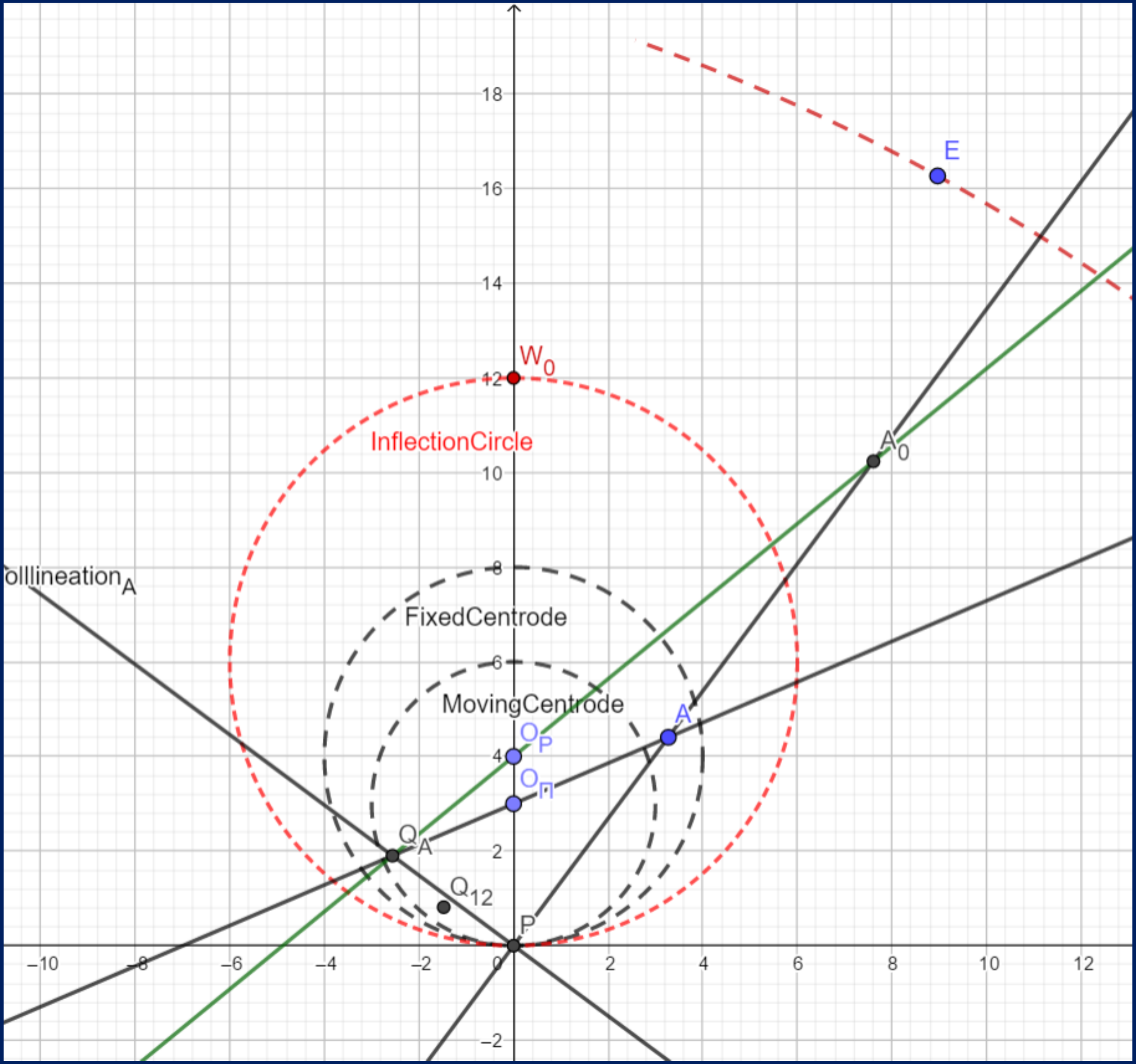
(Kinematically) Equivalent Linkages

Example:



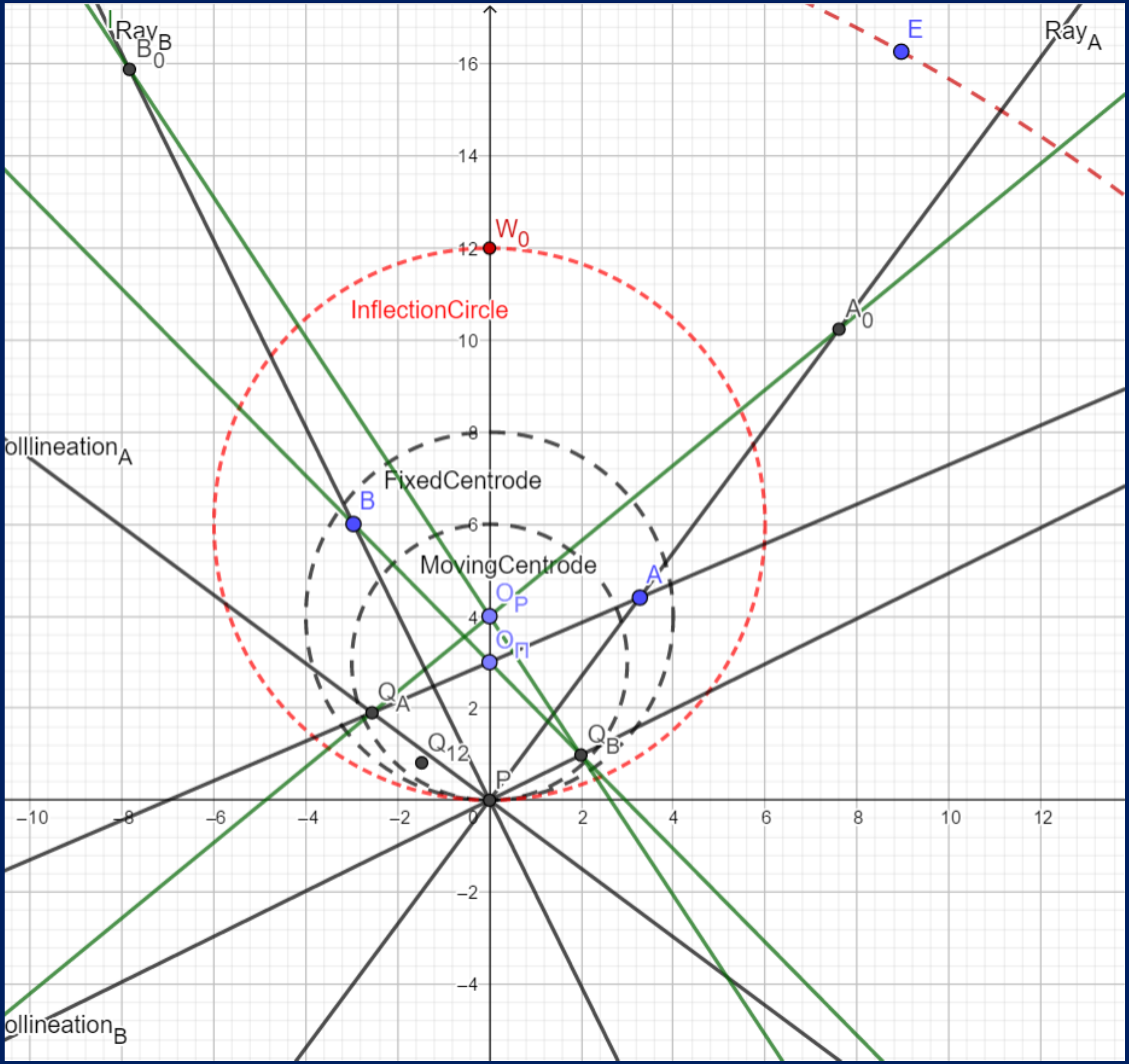
(Kinematically) Equivalent Linkages

Example:



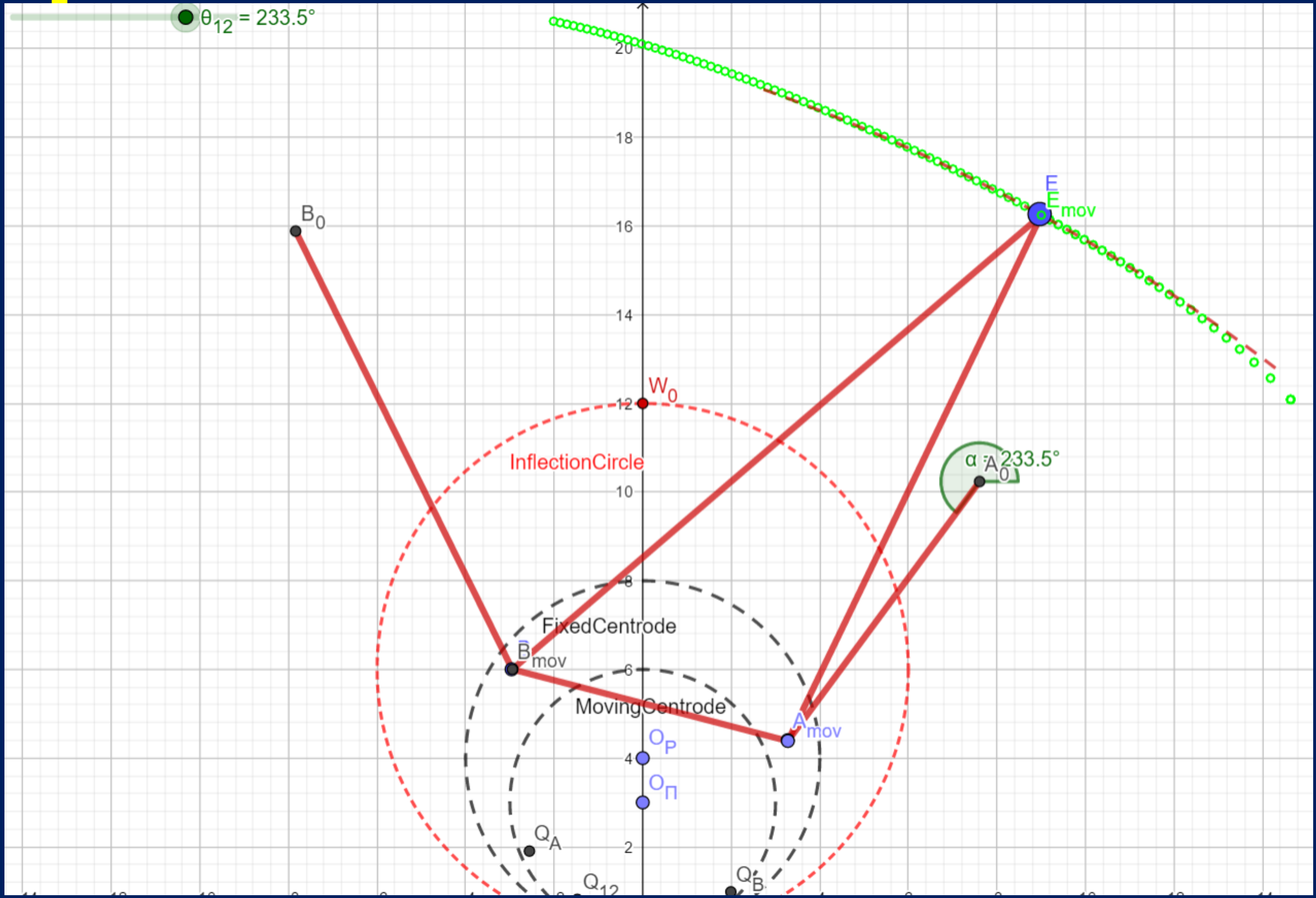
(Kinematically) Equivalent Linkages

Example:



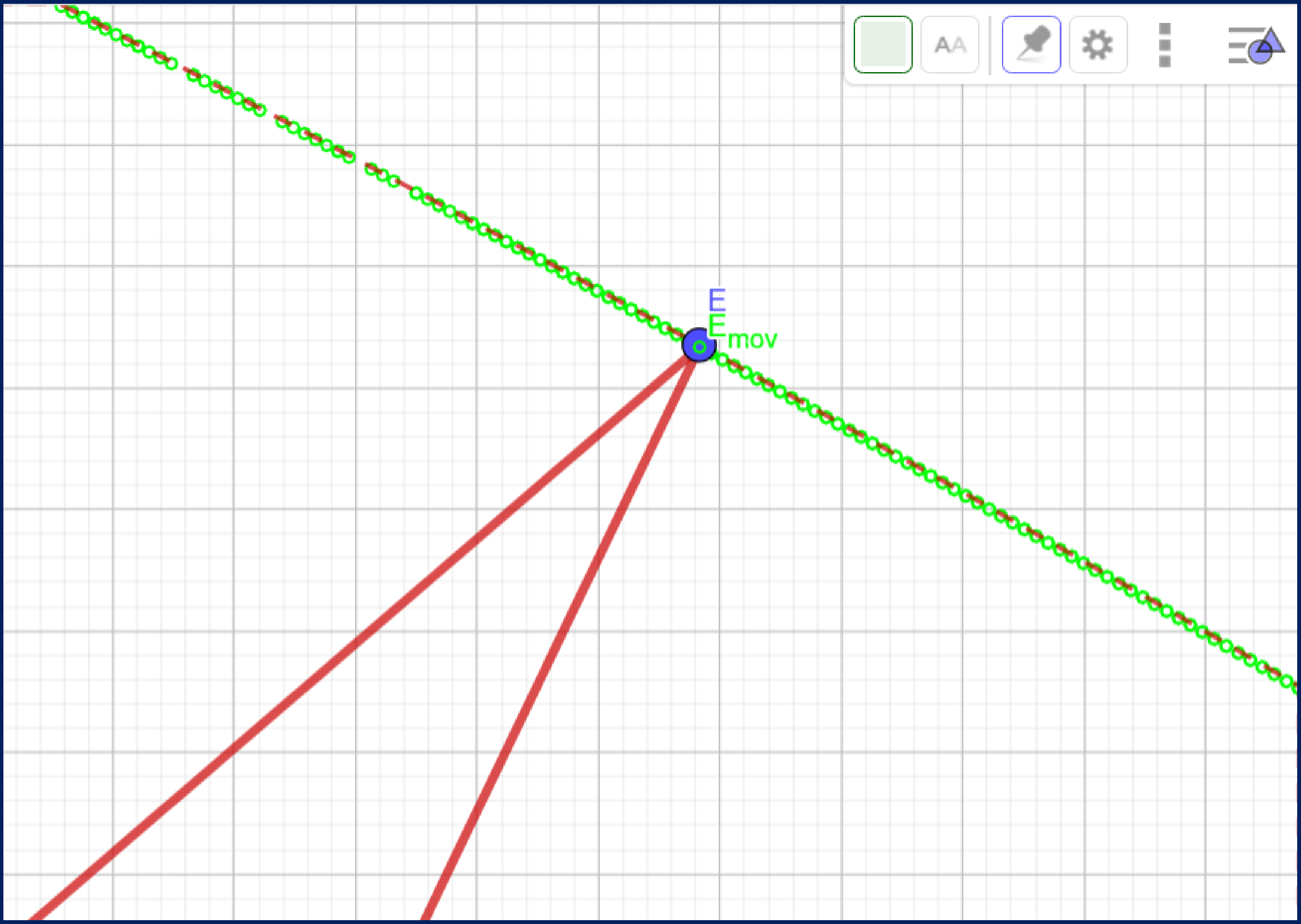
(Kinematically) Equivalent Linkages

Example:



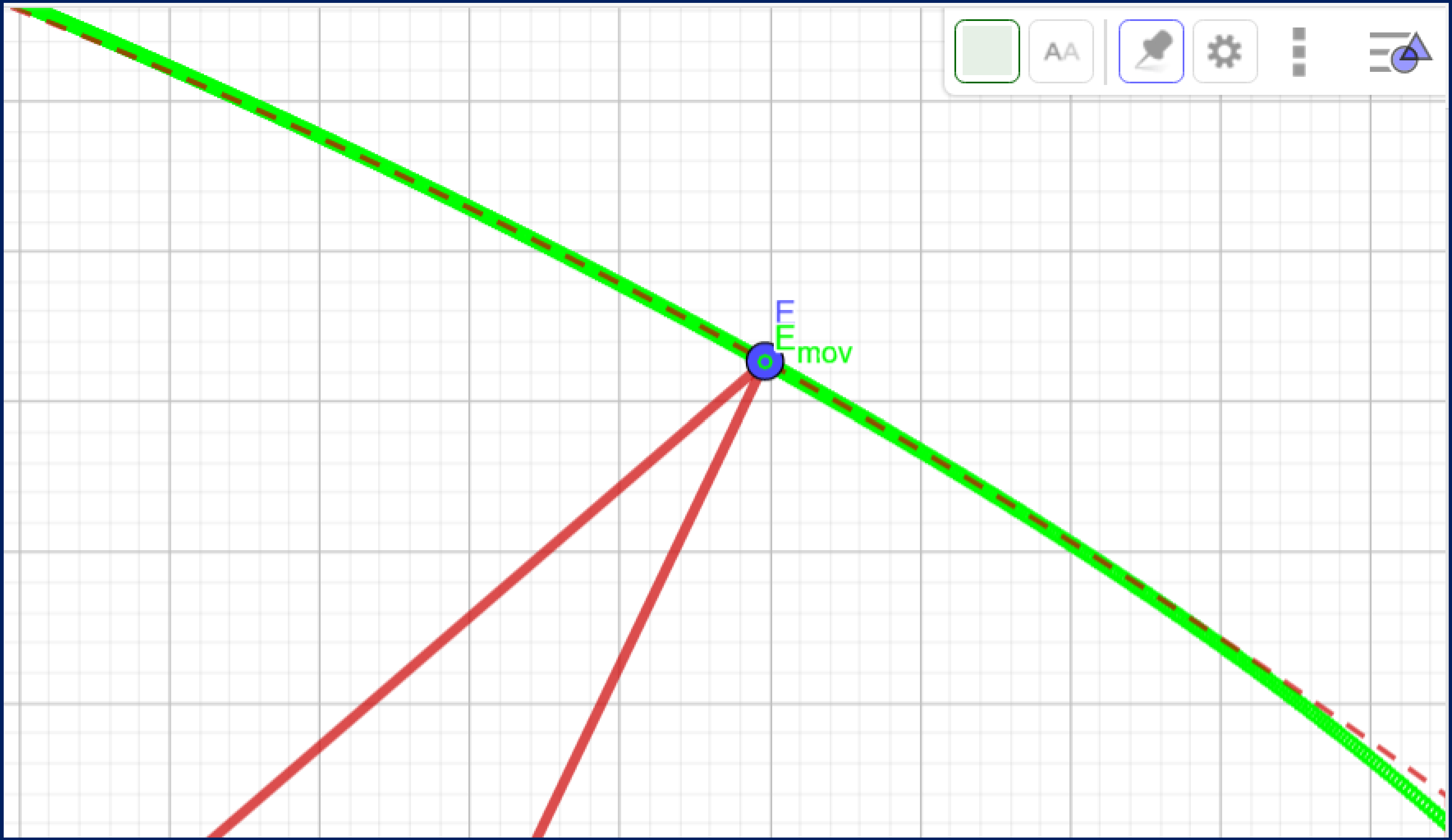
(Kinematically) Equivalent Linkages

Example:



(Kinematically) Equivalent Linkages

Example:



(Kinematically) Equivalent Linkages

Example:

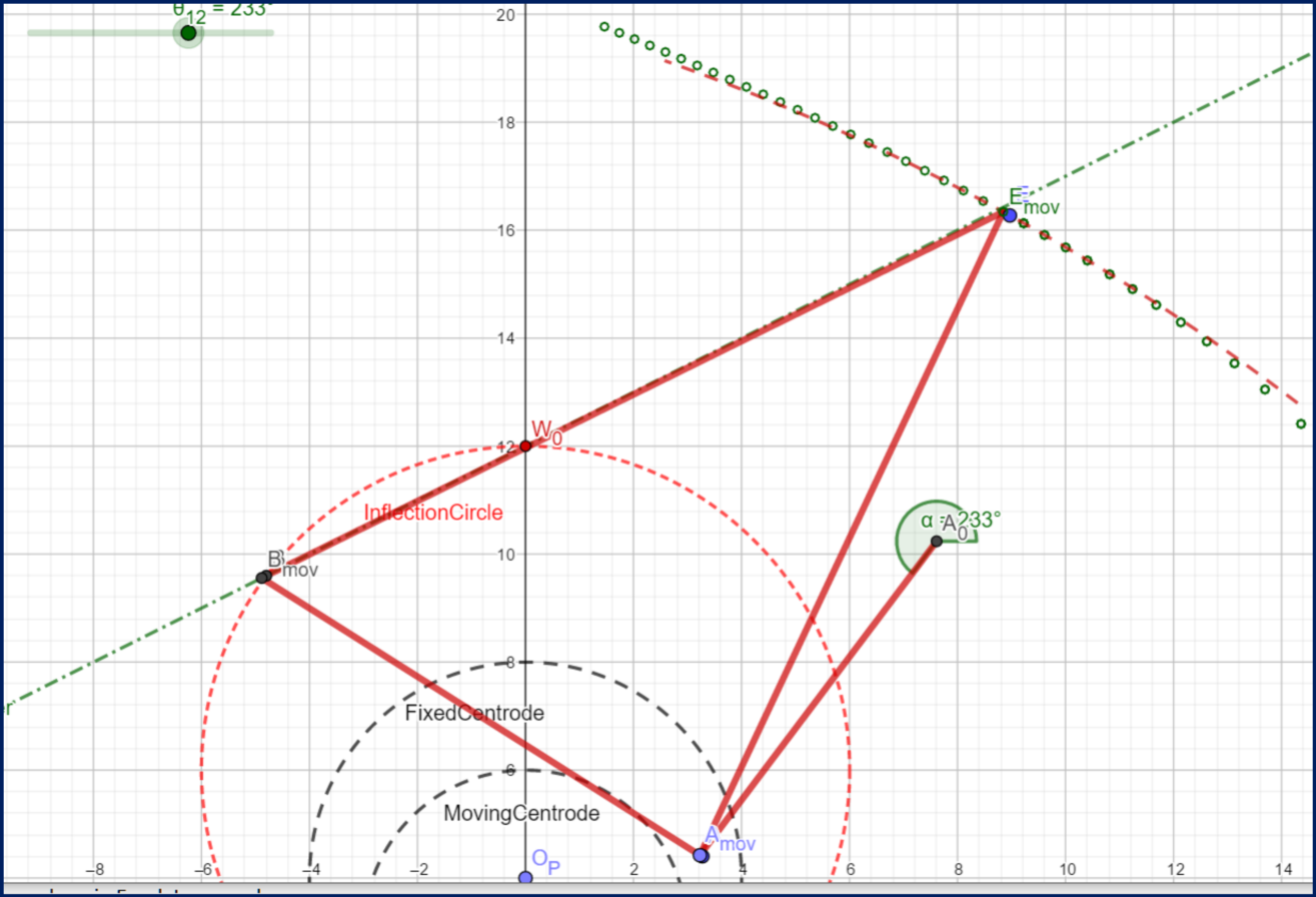
One may determine center of curvature of any point on the moving plane by Bobillier's theorem (Application Examples Case 2).

Select two arbitrary points, A and B on the moving plane and determine their centers of curvature, A_0 and B_0 which are unique.

One may also try a slider-crank. In this case B should be selected on the inflection circle (*therefore B_0 is at infinity*) ant the slider axis passes through BW_0.

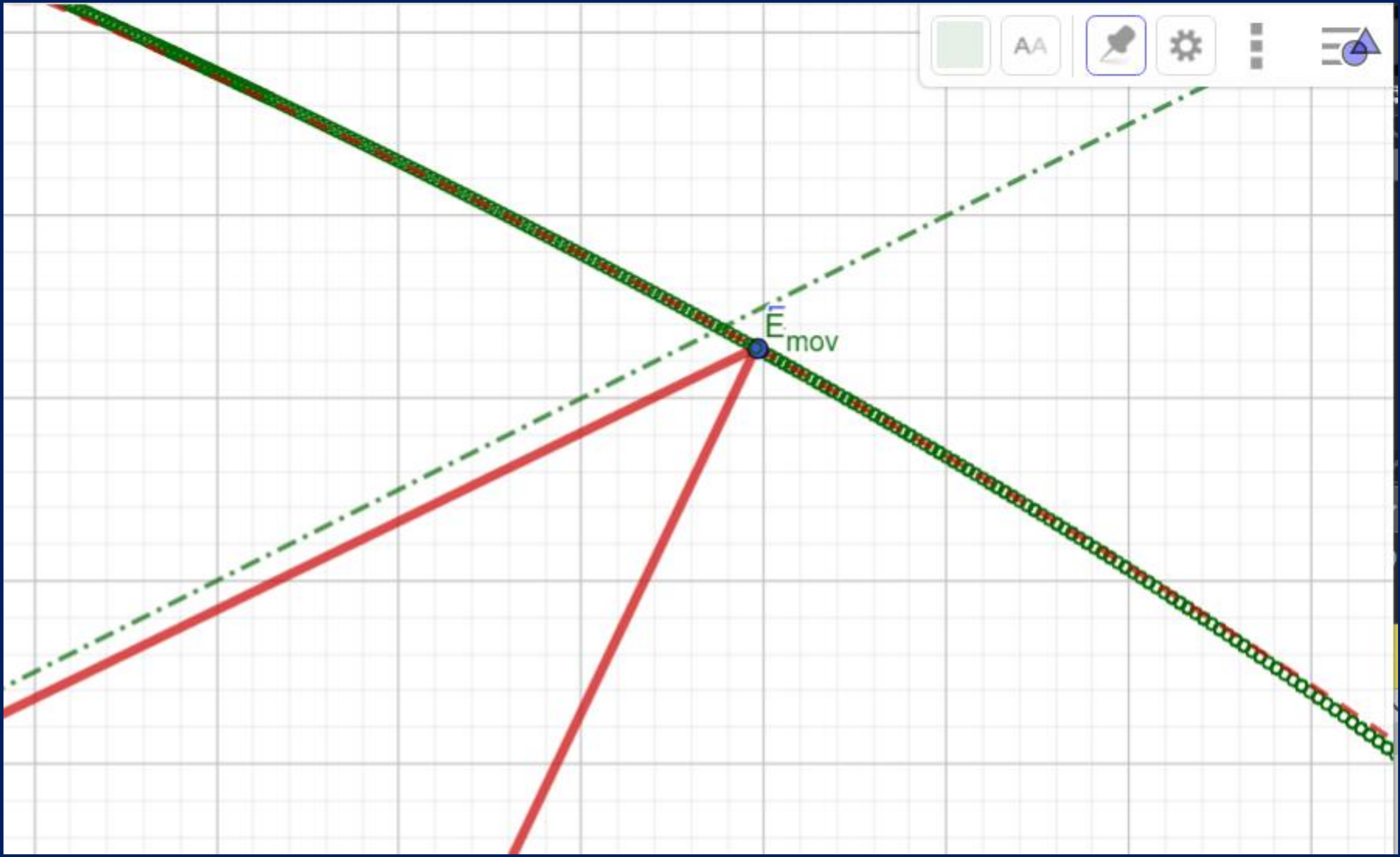
(Kinematically) Equivalent Linkages

Example:



(Kinematically) Equivalent Linkages

Example:



(Kinematically) Equivalent Linkages

Example: Consider two circles of equal diameter (say R = 1), one being fixed and other rolling around without slipping.

$$\frac{1}{\delta} = \frac{1}{r_\Pi} - \frac{1}{r_P} = \frac{1}{1} - \frac{1}{-1} = 2, \delta = \frac{1}{2}$$

To generate a straight line motion one may utilize the inflection pole, W_0.

Rather than using a planetary gear set one may approximate this motion using a four bar mechanism.

Select two arbitrary points, A and B on the moving plane and determine their centers of curvature, A_0 and B_0 using Euler-Savary equation.

(Kinematically) Equivalent Linkages

Example: Consider two circles of equal diameter (say R = 1), one being fixed and other rolling around without slipping.

Let $A\left(\sqrt{2}, 45°\right)$ and B$\left(\sqrt{2}, 135°\right)$

$$\left(\frac{1}{r} - \frac{1}{r_c}\right) sin\psi = \frac{1}{\delta}$$

For A:

$$\left(\frac{1}{\sqrt{2}} - \frac{1}{r_c}\right) sin45° = \frac{1}{1/2}, r_c = -\frac{2\sqrt{2}}{3}$$

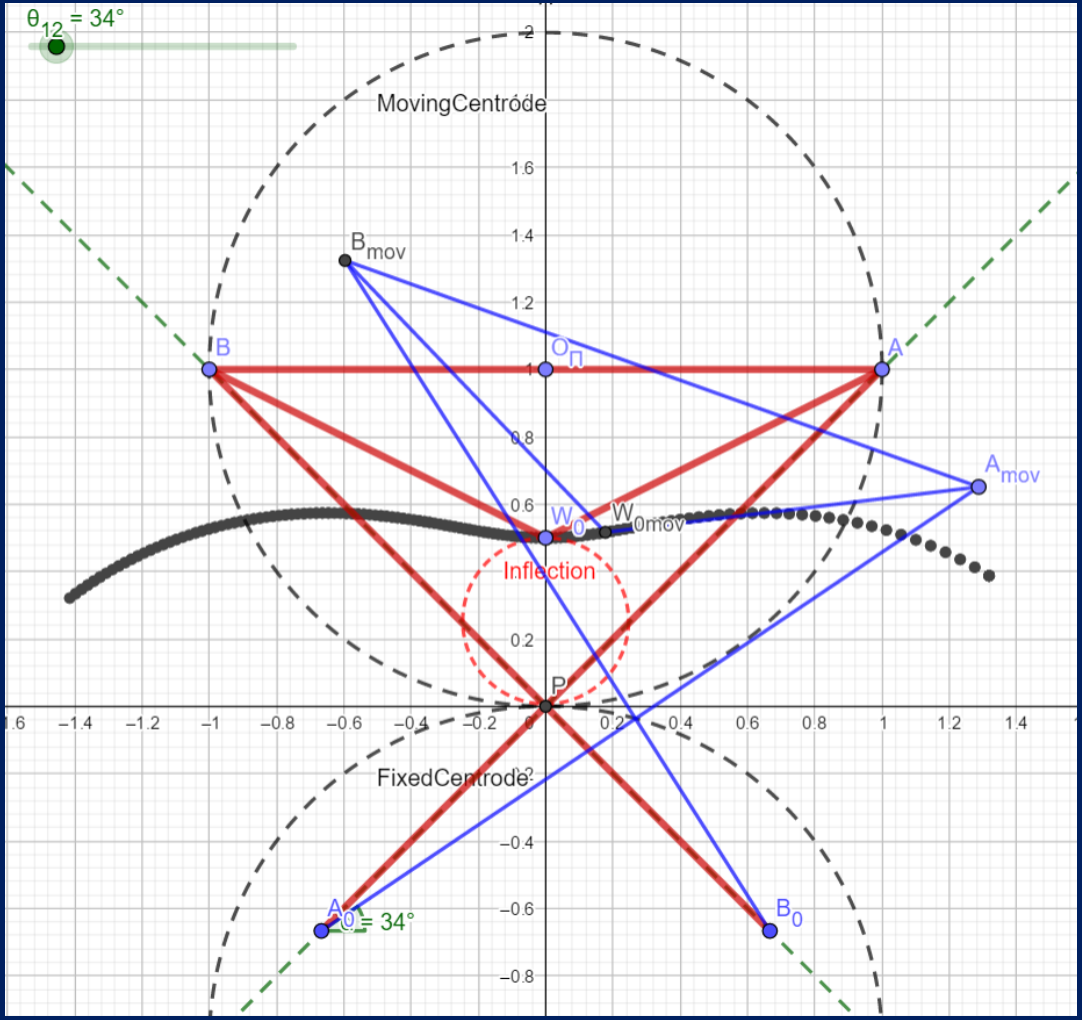
For B:

$$\left(\frac{1}{\sqrt{2}} - \frac{1}{r_c}\right) sin135° = \frac{1}{1/2}, r_c = -\frac{2\sqrt{2}}{3}$$

$$A_0\left(-\frac{2\sqrt{2}}{3}, 45°\right), B_0\left(-\frac{2\sqrt{2}}{3}, 135°\right)$$

(Kinematically) Equivalent Linkages

Example: Consider two circles of equal diameter (say R = 1), one being fixed and other rolling around without slipping.

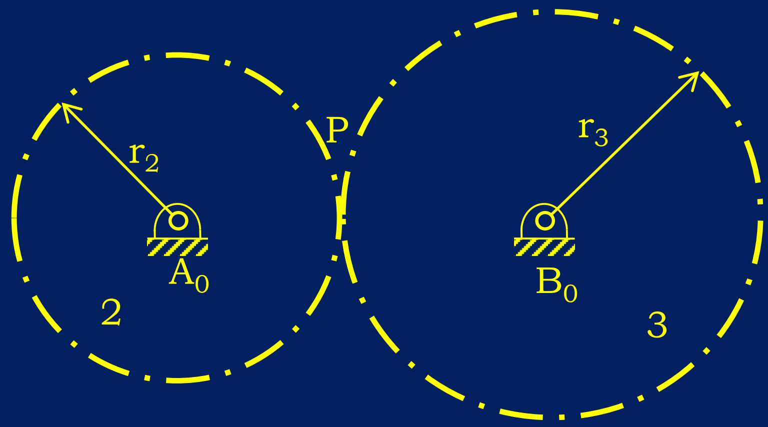


(Kinematically) Equivalent Linkages

Example: Consider two circles with a diameter ratio of n, both connected to the fixed link by revolute joints at their centers and rolling without slipping (simple gears). It is desired to replace them with a four-bar linkage for a limited range of rotation.

$$n = R_{23} = -\frac{r_2}{r_3} = -\frac{T_2}{T_3} = \frac{\omega_3}{\omega_2}$$

$$\delta = \frac{n}{(1+n)^2}$$



3. Cubic of Stationary Curvature (Four Infinitesimally Separated Positions)

Suppose there are two different moving planes and superimposed such that:

- Poles of both planes are coincident. Then, all points on both planes are coincident at this instant and this is called <u>one point contact</u>.
- Poles and pole tangents of both planes are coincident. Then coincident points on two planes share the same path tangent (or linear velocity for the same angular velocity of both planes) and this is called <u>two point contact</u>.
- Poles, pole tangents and inflection circle diameters are equal. Then coincident points on two planes have the same radius of curvature (or acceleration for the same angular velocity and acceleration of both planes) and this is called <u>three point contact</u>.

Is it possible to have the same rate of change of curvature for both planes or <u>four point contact</u>?

3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

Is it possible to have the same rate of change of curvature for both planes or <u>four point contact</u>?

Consider quadratic form of the Euler-Savary equation

$$\rho = \frac{r^2}{\delta sin\psi - r}$$

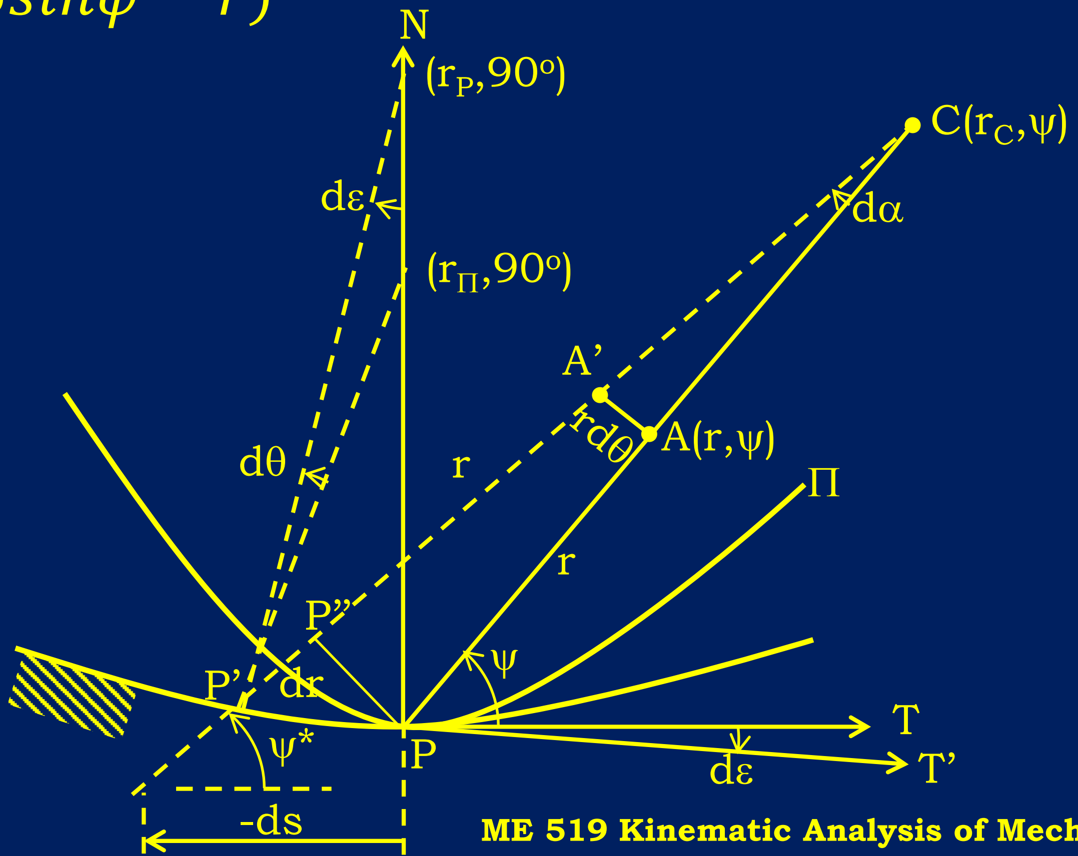
Differentiating this equation with respect to pole displacement yields:

$$\frac{d\rho}{ds} = \frac{2r\frac{dr}{ds}(\delta sin\psi - r) - r^2\left(\frac{d\delta}{ds}sin\psi + \delta cos\psi\frac{d\psi}{ds} - \frac{dr}{ds}\right)}{(\delta sin\psi - r)^2}$$

3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

Is it possible to have the same rate of change of curvature for both planes or four point contact?

$$\frac{d\rho}{ds} = \frac{2r\frac{dr}{ds}(\delta sin\psi - r) - r^2\left(\frac{d\delta}{ds}sin\psi + \delta cos\psi\frac{d\psi}{ds} - \frac{dr}{ds}\right)}{(\delta sin\psi - r)^2}$$



3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

$$\psi^* = \psi + d\psi$$

$$|P'P''| = dr$$

$$-\frac{dr}{ds} = \frac{|P'P''|}{|PP'|} = cos\psi^*$$

$$cos\psi^* = cos\psi cos d\psi - sin\psi sin d\psi$$

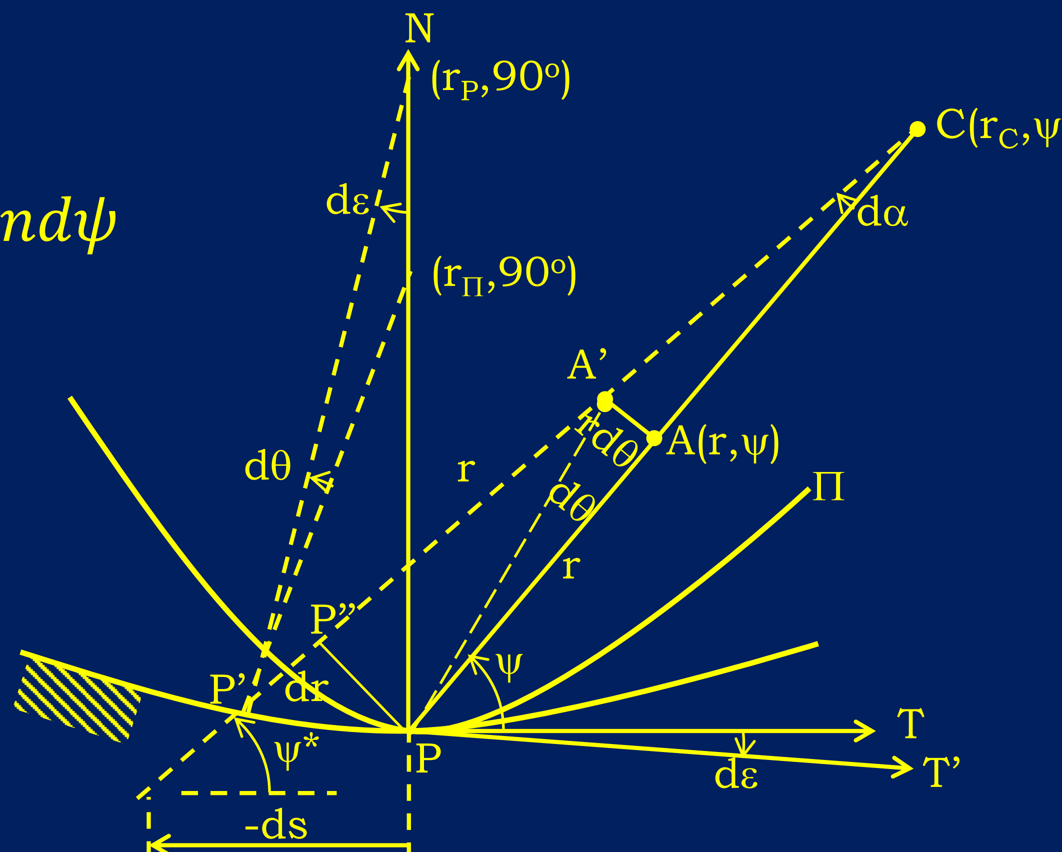
$$cos d\psi = 1, sin d\psi = d\psi$$

$$cos\psi \gg d\psi$$

$$-\frac{dr}{ds} = cos\psi$$

$$|AA'| = \rho d\alpha = -r d\theta$$

$$d\psi = d\alpha - d\varepsilon = -\frac{r}{\rho} d\theta - d\varepsilon$$



3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

$$d\psi = d\alpha - d\varepsilon = -\frac{r}{\rho}d\theta - d\varepsilon$$

$$\frac{d\psi}{ds} = -\frac{r}{\rho}\frac{d\theta}{ds} - \frac{d\varepsilon}{ds}, \frac{d\theta}{ds} = -\frac{1}{\delta}$$

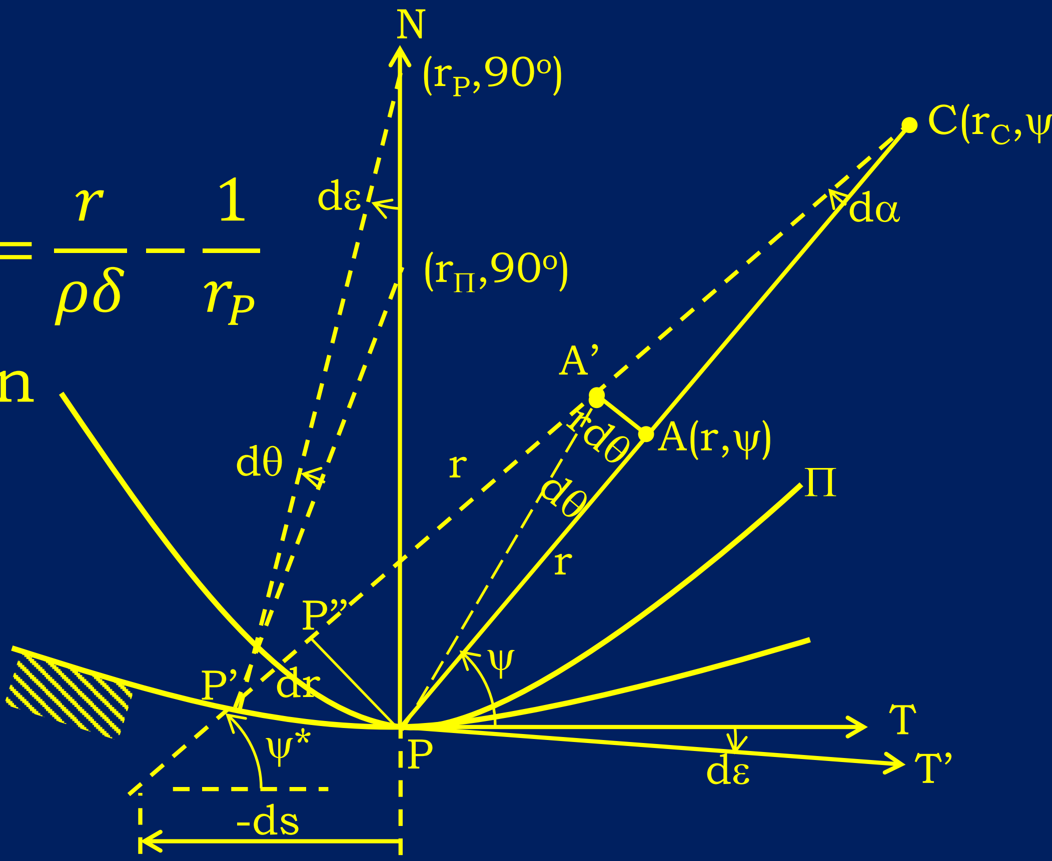
$$-r_P d\varepsilon = -ds, \frac{d\varepsilon}{ds} = \frac{1}{r_P} \; so \; \frac{d\psi}{ds} = \frac{r}{\rho\delta} - \frac{1}{r_P}$$

from Euler-Savary equation

$$\frac{d\psi}{ds} = \frac{sin\psi}{r} - \frac{1}{r_P} - \frac{1}{\delta}$$

$$\frac{1}{\delta} = \frac{1}{r_\Pi} - \frac{1}{r_P}$$

$$\frac{d\psi}{ds} = \frac{sin\psi}{r} - \frac{1}{r_\Pi}$$



3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

Substitution yields:

$$\frac{d\rho}{ds}=\frac{r}{(\delta sin\psi-r)^2}\left[-rsin\psi\frac{d\delta}{ds}-3\delta sin\psi cos\psi+rcos\psi\left(1+\frac{\delta}{r_\Pi}\right)\right]$$

$$\frac{1}{\delta}=\frac{1}{r_\Pi}-\frac{1}{r_P},1=\frac{\delta}{r_\Pi}-\frac{\delta}{r_P},\frac{\delta}{r_\Pi}=1+\frac{\delta}{r_P}$$

$$\frac{d\rho}{ds}=\frac{r}{(\delta sin\psi-r)^2}\left[-rsin\psi\frac{d\delta}{ds}-3\delta sin\psi cos\psi+rcos\psi\left(2+\frac{\delta}{r_P}\right)\right]$$

Define

$$m=-\frac{3\delta}{d\delta/ds},\frac{1}{\ell}=\frac{2}{3r_\Pi}-\frac{1}{3r_P}=\frac{1}{3}\left(\frac{2}{\delta}+\frac{1}{r_P}\right)$$

m and ℓ are only functions of the moving plane, independent of selection of point A.

3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

Substitution yields:

$$\frac{d\rho}{ds} = \frac{3r^2\delta}{(\delta sin\psi - r)^2}\left[\frac{sin\psi}{m} + \frac{cos\psi}{\ell} - \frac{sin\psi cos\psi}{r}\right]$$

$$\frac{d\rho}{ds} = \frac{3r^2\delta}{(\delta sin\psi - r)^2} K(r,\psi)$$

$K(r,\psi)$ is known as *cubic of stationary curvature* only a function of moving plane.

3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

Define

$$\lambda_1 = \frac{1}{\rho}\frac{d\rho}{d\alpha}$$

$$|AA'| = \rho d\alpha = r d\theta$$

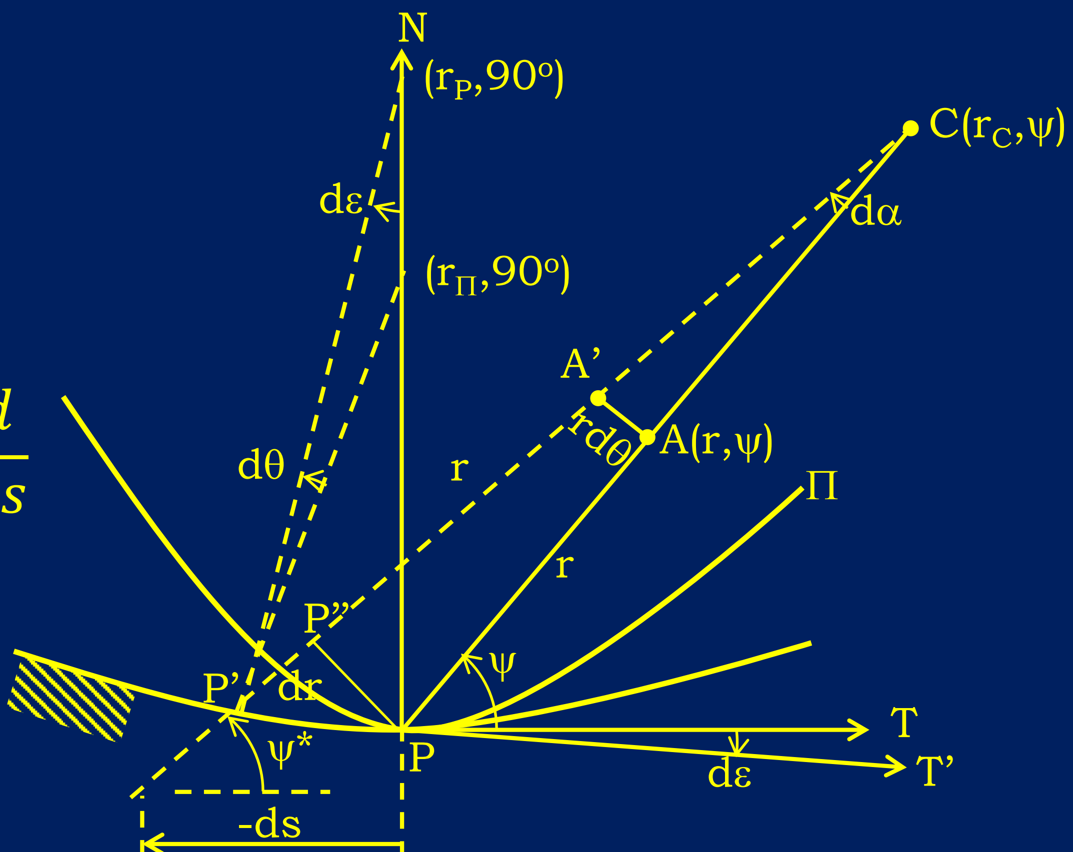
Define the operator:

$$\frac{d}{d\alpha} = -\frac{\rho}{r}\frac{d}{d\theta} = -\frac{\rho}{r}\frac{ds}{d\theta}\frac{d}{ds} = \frac{\rho\delta}{r}\frac{d}{ds}$$

$$\frac{d\rho}{ds} = \frac{3r^2\delta}{(\delta sin\psi - r)^2}K(r,\psi)$$

can be written as

$$\frac{d\rho}{d\alpha} = \frac{3\rho\delta^2 r}{(\delta sin\psi - r)^2}K(r,\psi)$$



Locus of points on the moving plane that has the same rate of change of curvature. Also known as λ_1 curve.

3. Cubic of Stationary Curvature
(Four Infinitesimally Separated Positions)

$$\frac{d\rho}{d\alpha} = \frac{3\rho\delta^2 r}{(\delta sin\psi - r)^2} K(r, \psi)$$

This equation can be expressed in Cartesian coordinates (x in T and y in N):

$$x = rcos\psi, y = rsin\psi$$

$$\lambda_1 \ell m(x^2 + y^2 - \delta y)^2 = 3\delta^2[(x^2 + y^2)(mx + \ell y) - \ell mxy]$$

This is a fourth order algebraic curve starting and ending at infinity and passing through the pole (origin of the Cartesian coordinate system) twice.

The tangents of this curve are pole tangent (x-axis) and $\lambda_1 y + 3x = 0$.

$\lambda_1 y + 3x = 0$ is known as the quartic of derivative curvature, locus of points on the moving plane having the same rate of change of curvature, λ_1.

3. Cubic of Stationary Curvature
Circular Arc Generation, 4 Point Contact

$$\frac{d\rho}{d\alpha} = \frac{3\rho\delta^2 r}{(\delta sin\psi - r)^2} K(r,\psi)$$

For point A on the moving plane to have four point contact with a circular arc, (for $\rho \neq 0$), $\frac{d\rho}{d\alpha} = 0$ therefore curvature is stationary.

Then

$$K(r,\psi) = \frac{sin\psi}{m} + \frac{cos\psi}{\ell} - \frac{sin\psi cos\psi}{r} = 0$$

in polar form or in Cartesian coordinates

$$(x^2 + y^2)(mx + \ell y) - \ell mxy = 0$$

$$m = -\frac{3\delta}{d\delta/ds}, \frac{1}{\ell} = \frac{2}{3}\left(\frac{2}{\delta} + \frac{1}{r_P}\right)$$

Locus of points on the moving plane having four point contact with a circular arc and the equation is known as *cubic of stationary curvature.*

3. Cubic of Stationary Curvature
Circular Arc Generation, 4 Point Contact

To obtain the center of curvature of all points on the cubic of stationary curvature one may utilize kinematic inversion.

Recall in inverted motion the centrodes change their roles, m remains the same, ℓ is replaced by ℓ^*:

$$\frac{1}{\ell^*} = \frac{2}{3r_P} - \frac{1}{3r_\Pi}$$

$$M(r,\psi) = \frac{sin\psi}{m} + \frac{cos\psi}{\ell^*} - \frac{sin\psi cos\psi}{r} = 0$$

in polar form or in Cartesian coordinates

$$(x^2 + y^2)(mx + \ell^* y) - \ell^* mxy = 0$$

This is known as *cubic of centers of stationary curvature*.

Recall the analogy with Burmester's K and M curves.

3. Cubic of Stationary Curvature

Straight Line Generation, 4 Point Contact: Ball's Point

In addition to stationary curvature further if straight line is required, (i.e. $\frac{1}{\rho}=0$) such point(s) must be both on cubic of stationary curvature and inflection circle, $B(r_B,\psi_B)$.

Then

$$\left.\begin{aligned} K(r_B,\psi) = \frac{sin\psi_B}{m}+\frac{cos\psi_B}{\ell}-\frac{sin\psi_B cos\psi_B}{r_B}=0 \\ r_B=\delta\, sin\psi_B \end{aligned}\right\} tan\psi_B = m\left(\frac{1}{\delta}-\frac{1}{\ell}\right)$$

$$tan\psi_B=\frac{2r_\Pi - r_P}{(r_\Pi - r_P)\frac{d\delta}{ds}}$$

This point is known as Ball's point.

Recall the analogy with Ball's point whose four finitely separated positions lay on a straight line.

For stationary inflection circle diameter inflection pole, W_0, is the Ball's point.

3. Cubic of Stationary Curvature

Stationary Inflection Circle Diameter, Symmetric Motion

$$\frac{d\rho}{d\alpha} = 0, \frac{1}{m} = 0$$

$$K(r,\psi) = cos\psi\left(\frac{1}{\ell} - \frac{sin\psi}{r}\right) = 0$$

yielding:

$$cos\psi = 0 \text{ or } sin\psi = \frac{\ell}{r}$$

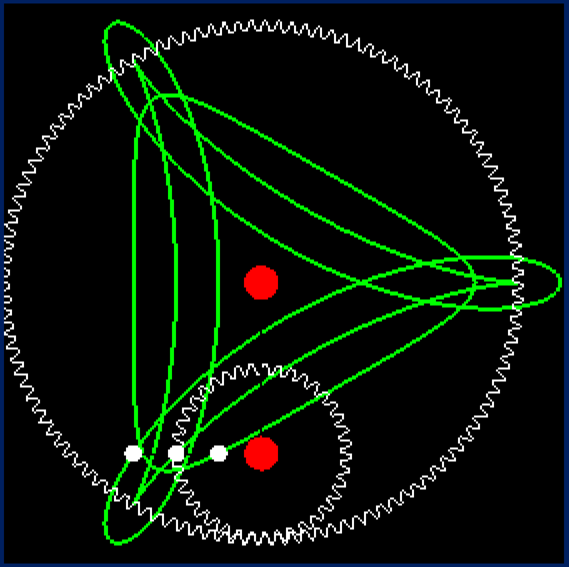
First one is the pole normal, second one is a circle of diameter ℓ ($rsin\psi = \ell$) center on pole normal and passing through the pole.

Let

$$t = \frac{r}{sin\psi}$$

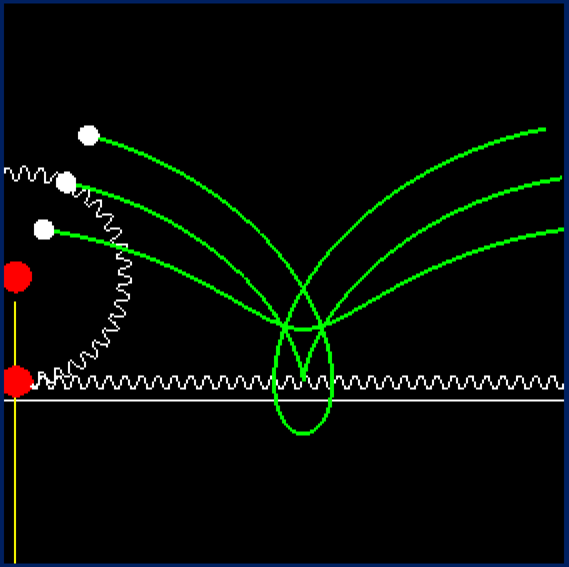
$$\lambda_1 = \frac{3\delta r^2}{(\delta sin\psi - r)^2} cos\psi\left(\frac{1}{\ell} - \frac{sin\psi}{r}\right) = \frac{3}{tan\psi}\left(\frac{\delta}{\delta - t}\right)^2\left(\frac{t}{\ell} - 1\right)$$

Example: Cycloidal Motion Mechanisms
Hypocycloid



https://www.psmotion.com/news/cycloidal-curves-gears-and-racks

Example: Cycloidal Motion Mechanisms
Cycloid



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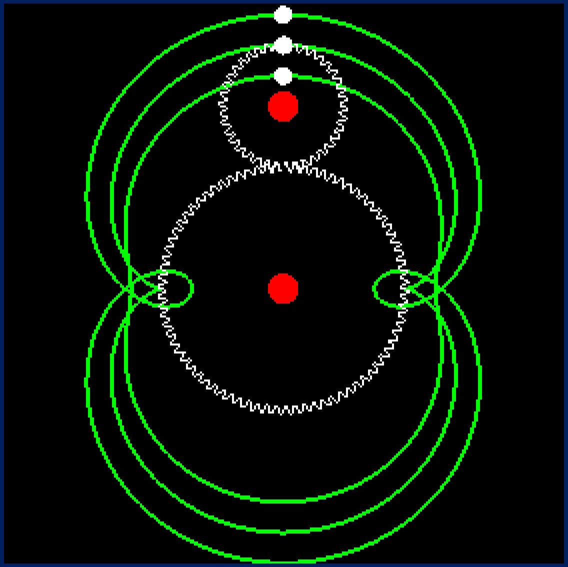
Example: Cycloidal Motion Mechanisms
Epicycloid



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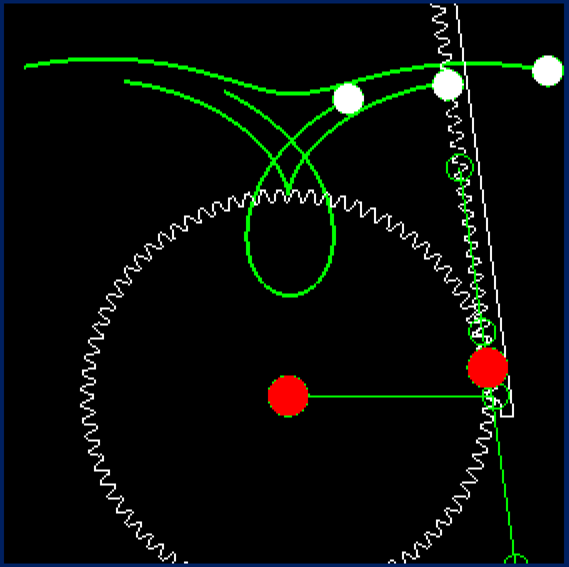
Example: Cycloidal Motion Mechanisms Evolvent



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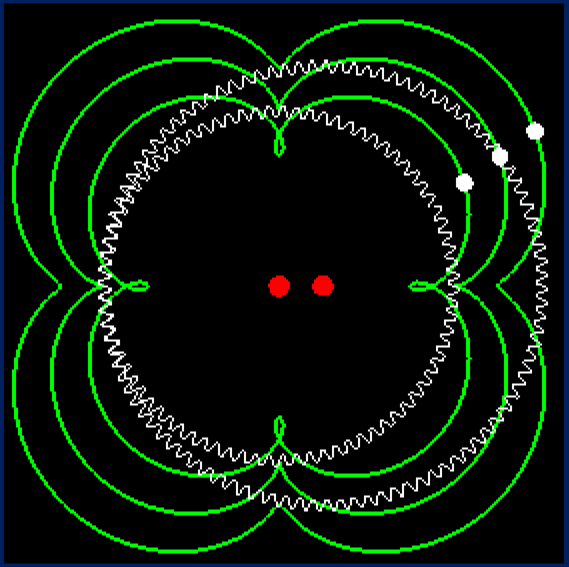
Example: Cycloidal Motion Mechanisms
Pericycloid



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Example: Cycloidal Motion Mechanism

Gear ratio:

$$R = \frac{r_\Pi}{r_P}, r_P = n r_\Pi, t = 2r_\Pi$$

Moving centrode:

$$r = t sin\psi$$

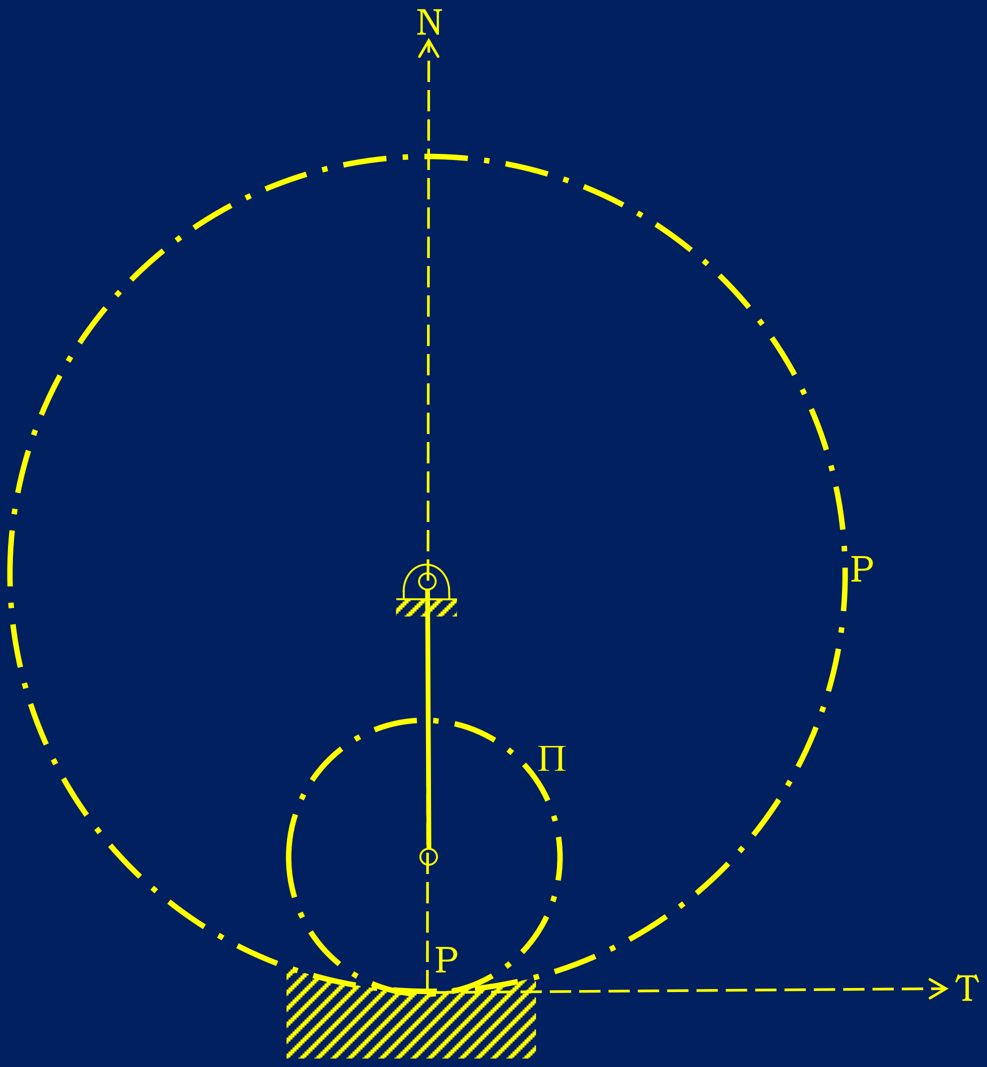
For point A on Π

$$\frac{1}{r_\Pi} - \frac{1}{r_P} = \frac{1}{\delta}, r_P = R r_\Pi$$

$$\frac{1}{\delta} = (1 - R)\frac{1}{r_\Pi}$$

$$\frac{1}{\ell} = \frac{2}{3r_\Pi} - \frac{1}{3r_P} = \frac{2 - R}{3r_\Pi}$$

$$\delta = \frac{r_\Pi}{1 - R} = \frac{r_\Pi r_P}{r_P - r_\Pi}$$



Example: Cycloidal Motion Mechanism

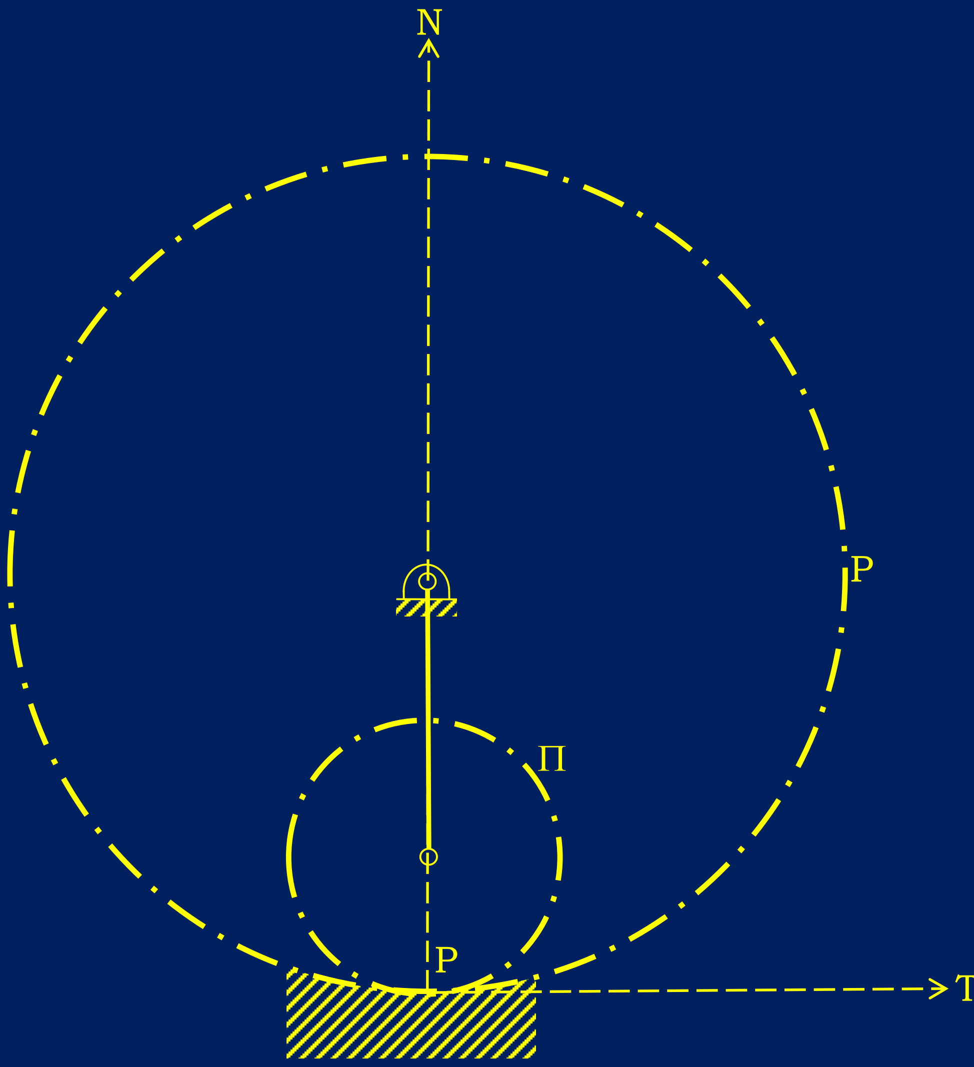
$$\lambda_1 = \frac{1}{(1-2n)tan\psi} = \frac{1}{\rho}\frac{d\rho}{d\alpha}$$

$$\rho = \frac{r^2}{\delta sin\psi - r}$$

$$\rho = \frac{4{r_\Pi}^2 sin^2\psi}{\delta sin\psi - 2r_\Pi sin\psi}$$

$$\delta = \frac{r_\Pi}{1-R}$$

$$\rho = \frac{4(1-R)r_\Pi sin\psi}{2R-1}$$



Example: Square with Rounded Corners

$$n = \frac{1}{R} = \frac{r_P}{r_\Pi} \epsilon \mathbb{N}$$

$$n = \frac{\omega_{planet} - \omega_{arm}}{\omega_{ring} - \omega_{arm}} = \frac{r_P}{r_\Pi}$$

Centrodes being circles:

$$\frac{d\delta}{ds} = 0, \frac{1}{m} = 0$$

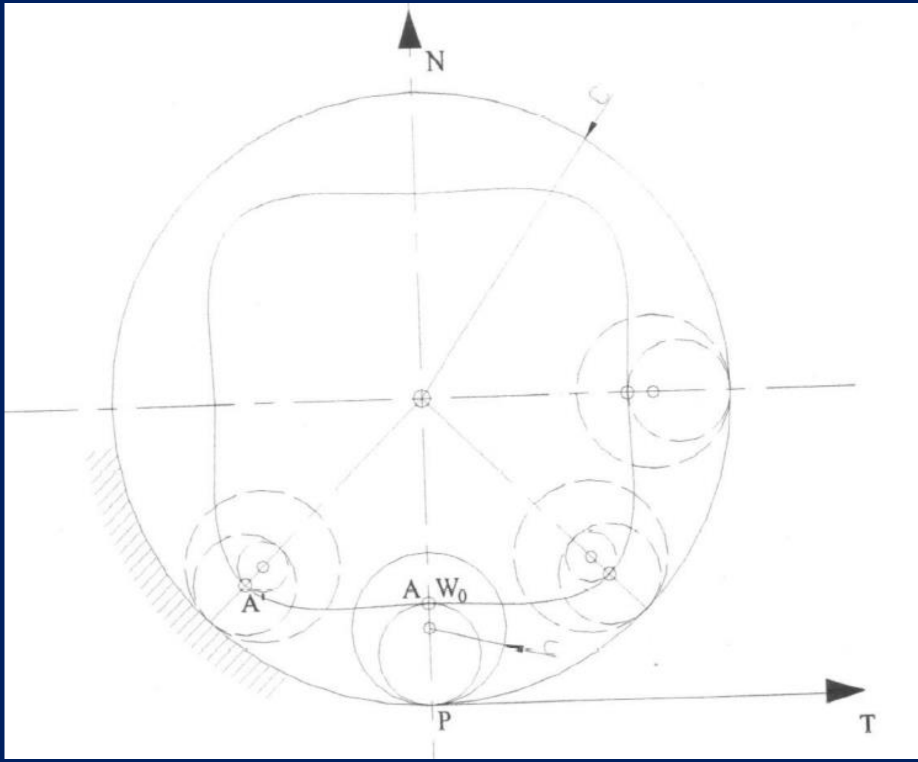
$$K(r, \psi) = cos\psi \left(\frac{1}{\ell} - \frac{sin\psi}{r} \right)$$

$$\frac{1}{\ell} = \frac{2}{3r_\Pi} - \frac{1}{3nr_\Pi} = \frac{(2n-1)(n-1)}{2nr_\Pi}$$

$$cos\psi = 0$$

$$r = \frac{3nr_\Pi}{2n-1} sin\psi$$

Circle!



Eres Söylemez (unpublished ME 519 Lecture Notes)

Example: Square with Rounded Corners

Four point contact with a straight line requires the point to be:

- Cubic of stationary curvature
- Inflection circle

Since inflection circle diameter is constant Ball's point is the inflection pole.

$$\frac{1}{\delta} = \frac{1}{r_\Pi} - \frac{1}{r_P} = \frac{1}{r_\Pi} - \frac{1}{nr_\Pi} = \frac{n-1}{nr_\Pi}$$

Consider n = 4

$$\ell = \frac{12r_\Pi}{7}, \delta = \frac{4r_\Pi}{3}$$

The inflection pole is $\frac{r_\Pi}{3}$ above the center of the planet and this point at the design position describes an approximate straight line deviating from it as it goes away.

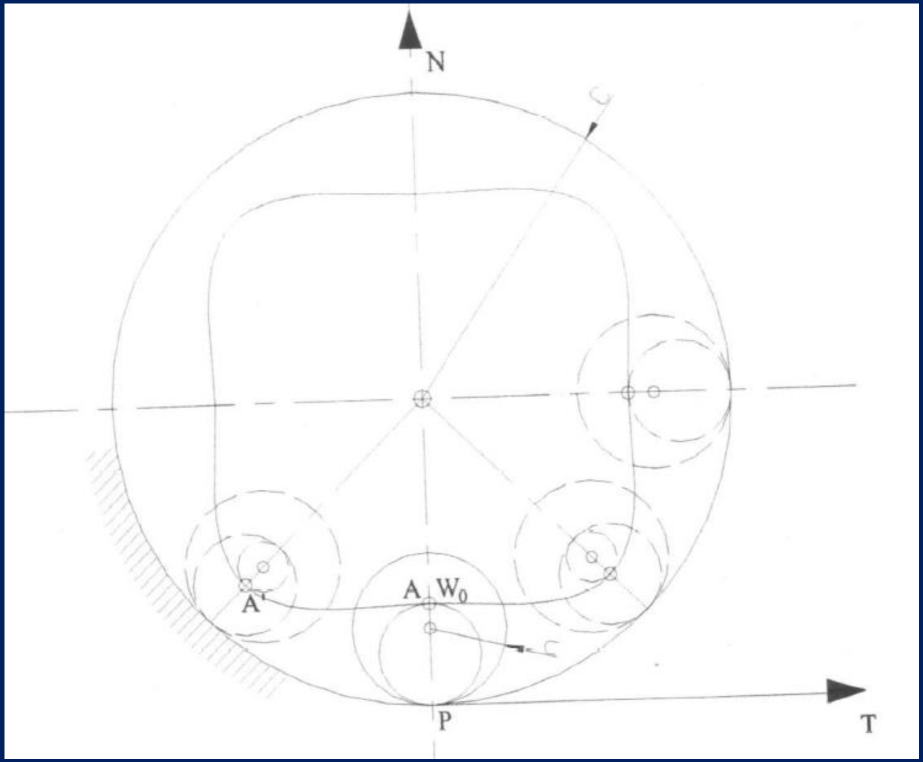
Example: Square with Rounded Corners

When the arm rotates $\frac{\pi}{4}$ the planet rotates π radians relative to the arm and has the minimum radius of curvature.

$$r = r_{\Pi} - \frac{r_{\Pi}}{3} = \frac{2r_{\Pi}}{3}$$

$$\frac{1}{\delta} = \frac{1}{r} - \frac{1}{r_c}$$

$$\rho_{min} = \frac{2r_{\Pi}}{3}$$



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Example: Production of Circular Arcs on Lathes

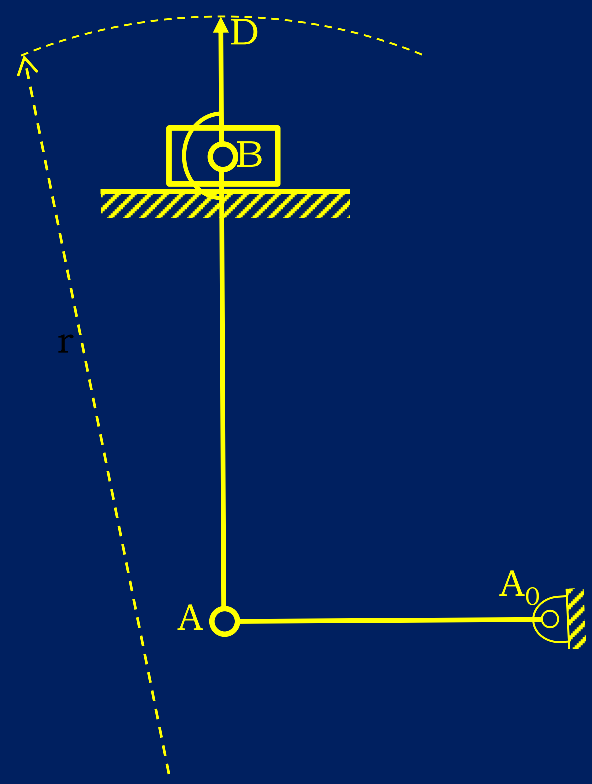
At the design position the mechanism is symmetric therefore

$$\frac{1}{m} = 0$$

$$K(r,\psi) = cos\psi\left(\frac{1}{\ell} - \frac{sin\psi}{r}\right) = 0$$

degenerates into pole normal and circle.

$$\rho_D = \frac{r^2}{\delta - r} = \frac{|AD|^2}{|BD|}, \delta = |AB|$$



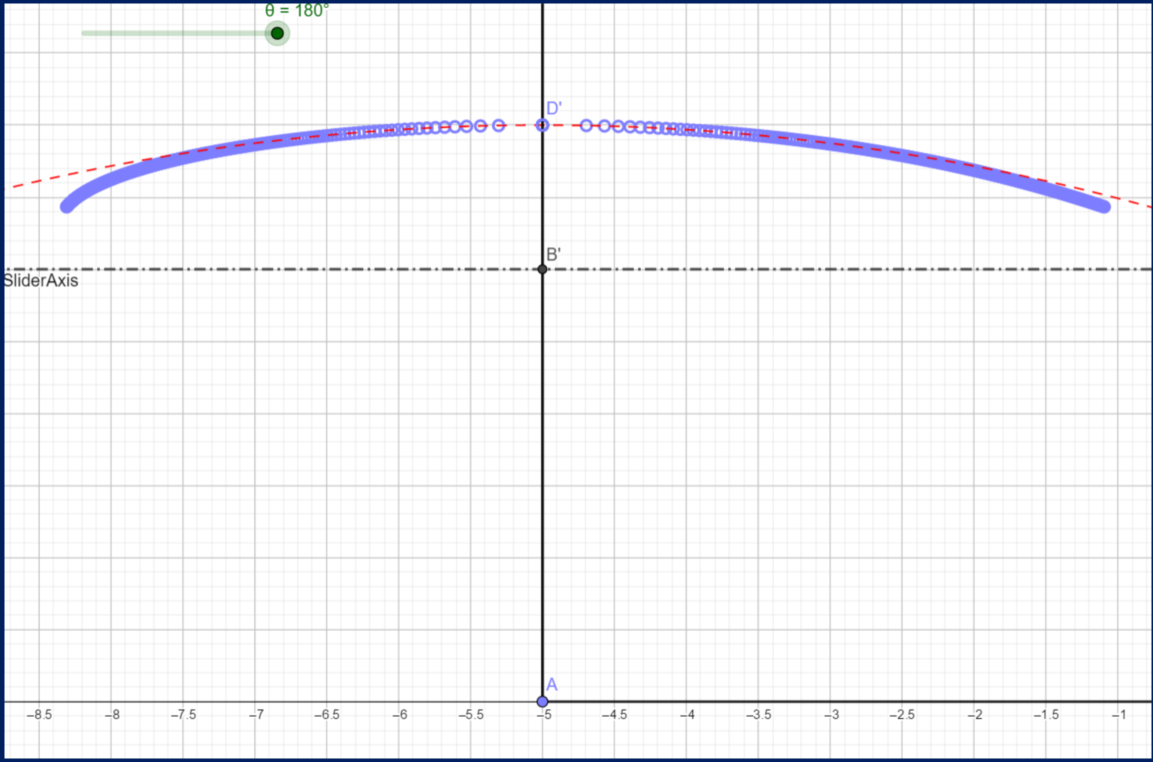
Example: Production of Circular Arcs on Lathes

$$\rho_D = \frac{r^2}{\delta - r} = \frac{|AD|^2}{|BD|}$$

$$|AD| = 4$$

$$|BD| = 1$$

$$\rho_D = \frac{4^2}{1} = 16\ length\ units$$



Cubic of Stationary Curvature of a Four Bar

In polar form

$$K(r,\psi) = \frac{sin\psi}{m} + \frac{cos\psi}{\ell} - \frac{sin\psi cos\psi}{r} = 0$$

$$\frac{r/cos\psi}{m} + \frac{r/sin\psi}{\ell} = 1$$

Define

$$\mu = \frac{r}{cos\psi}, \lambda = \frac{r}{sin\psi}$$

$$\frac{\mu}{m} + \frac{\lambda}{\ell} = 1$$

In Cartesian coordinates

$$\mu = \frac{x^2+y^2}{x}, \lambda = \frac{x^2+y^2}{y}$$

$$K_\mu = x^2 + y^2 - \mu x = 0, K_\lambda = x^2 + y^2 - \lambda y = 0$$

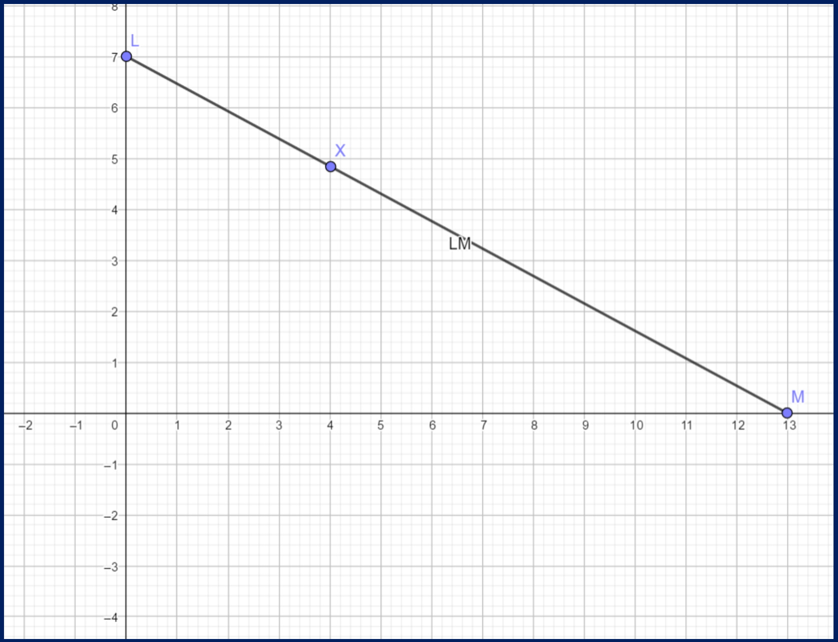
Two circles with centers on pole tangent and normal respectively.

Cubic of Stationary Curvature of a Four Bar

$K_\mu = x^2 + y^2 - \mu x = 0, K_\lambda = x^2 + y^2 - \lambda y = 0$

This system of equations form the pencils of circles. The intersection point of pencils of circles yield a point on $K(r, \psi)$.

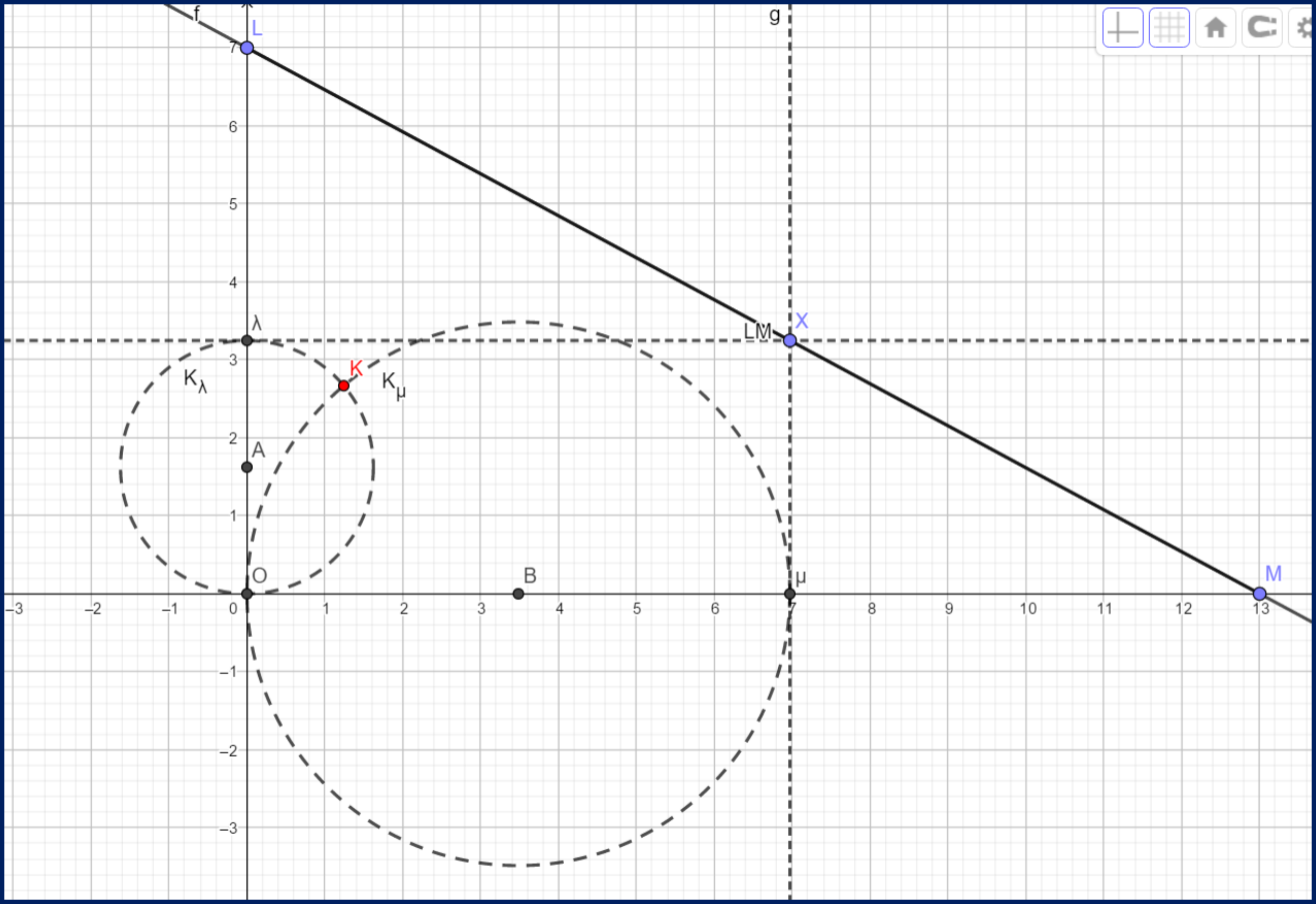
The idea is for known ℓ (say 7) and m (say 13) draw line LM. Select any point on LM, say X.



Cubic of Stationary Curvature of a Four Bar

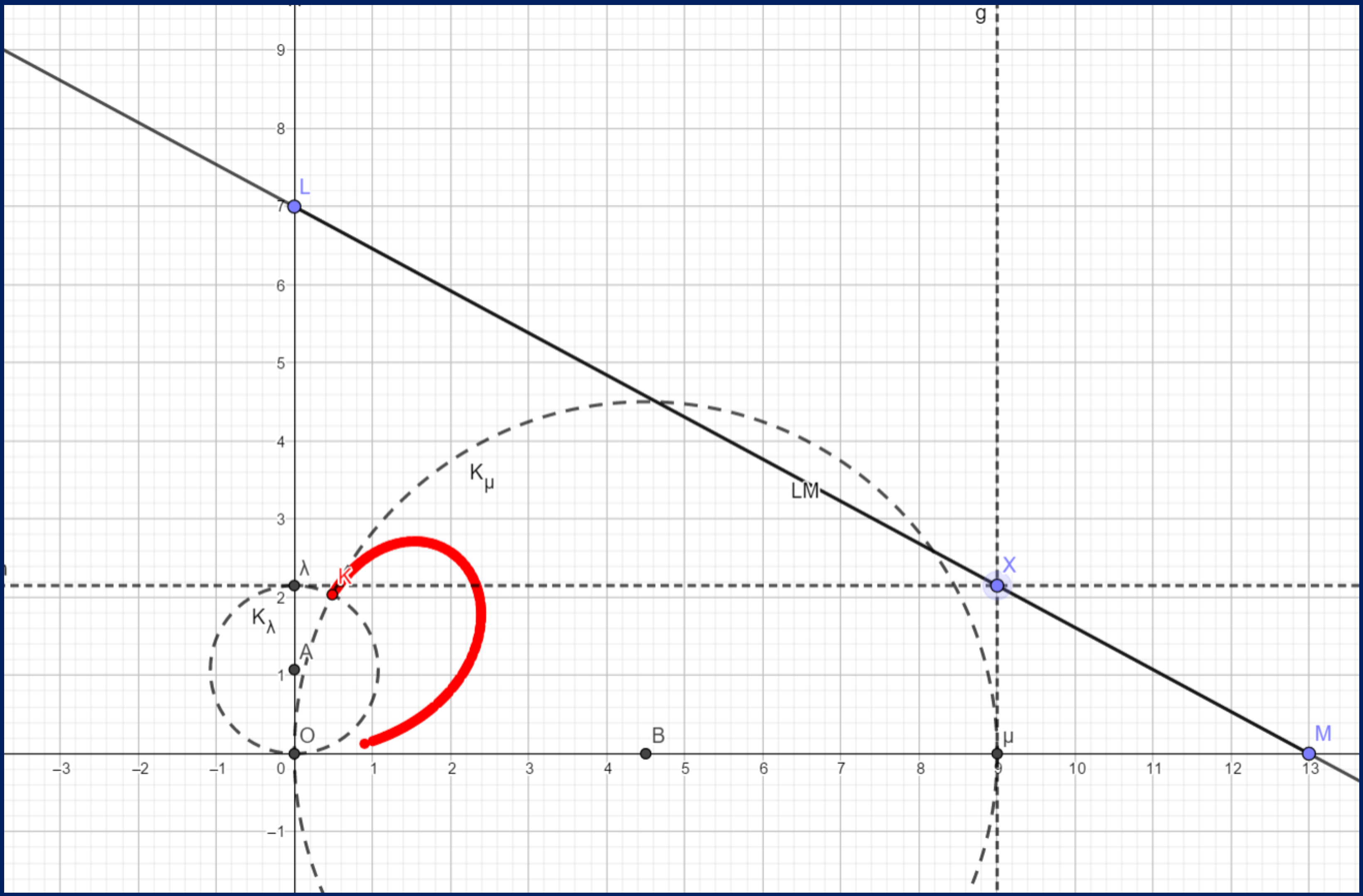
Draw horizontal and vertical lines through X intersecting PN at λ and PT at μ.

Draw K_λ and K_μ, intersection yields a point on $K(r, \psi)$.



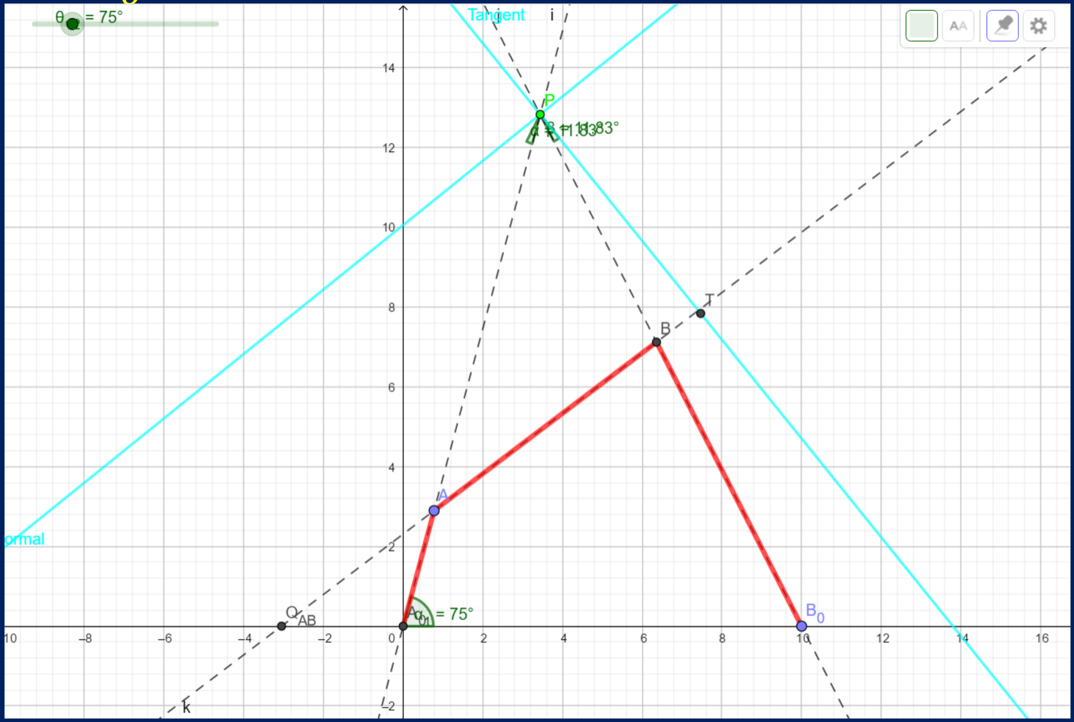
Cubic of Stationary Curvature of a Four Bar

Change X and obtain another point on $K(r, \psi)$.



Procedure for the Four Bar

1. Determine the pole, pole tangent, pole normal and inflection pole utilizing Bobillier's construction.
   - A_0A and B_0B form two pole rays therefore P is at intersection,
   - Q_{AB} is at the intersection of A_0B_0 and AB,
   - $\sphericalangle Q_{AB}PA = \sphericalangle BPT = \alpha$
   - $\left(\frac{1}{r} - \frac{1}{r_c}\right) sin\psi = \frac{1}{\delta}$ should yield the same δ for the pair A A_0 or B B_0.



Procedure for the Four Bar

1. Determine the pole, pole tangent, pole normal and inflection pole utilizing Bobillier's construction.
   - A_0A and B_0B form two pole rays therefore P is at intersection,
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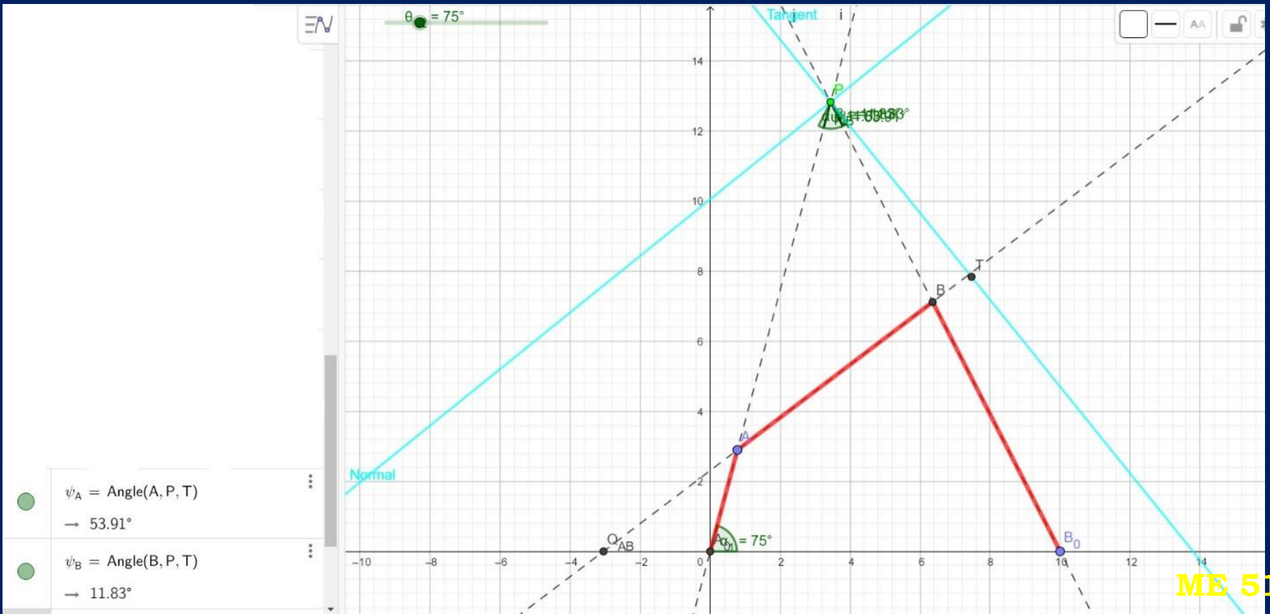
   - $\sphericalangle Q_{AB}PA = \sphericalangle BPT = \alpha$
   - $\left(\frac{1}{r} - \frac{1}{r_c}\right) sin\psi = \frac{1}{\delta}$ should yield the same δ for the pair A-A_0 or B-B_0.

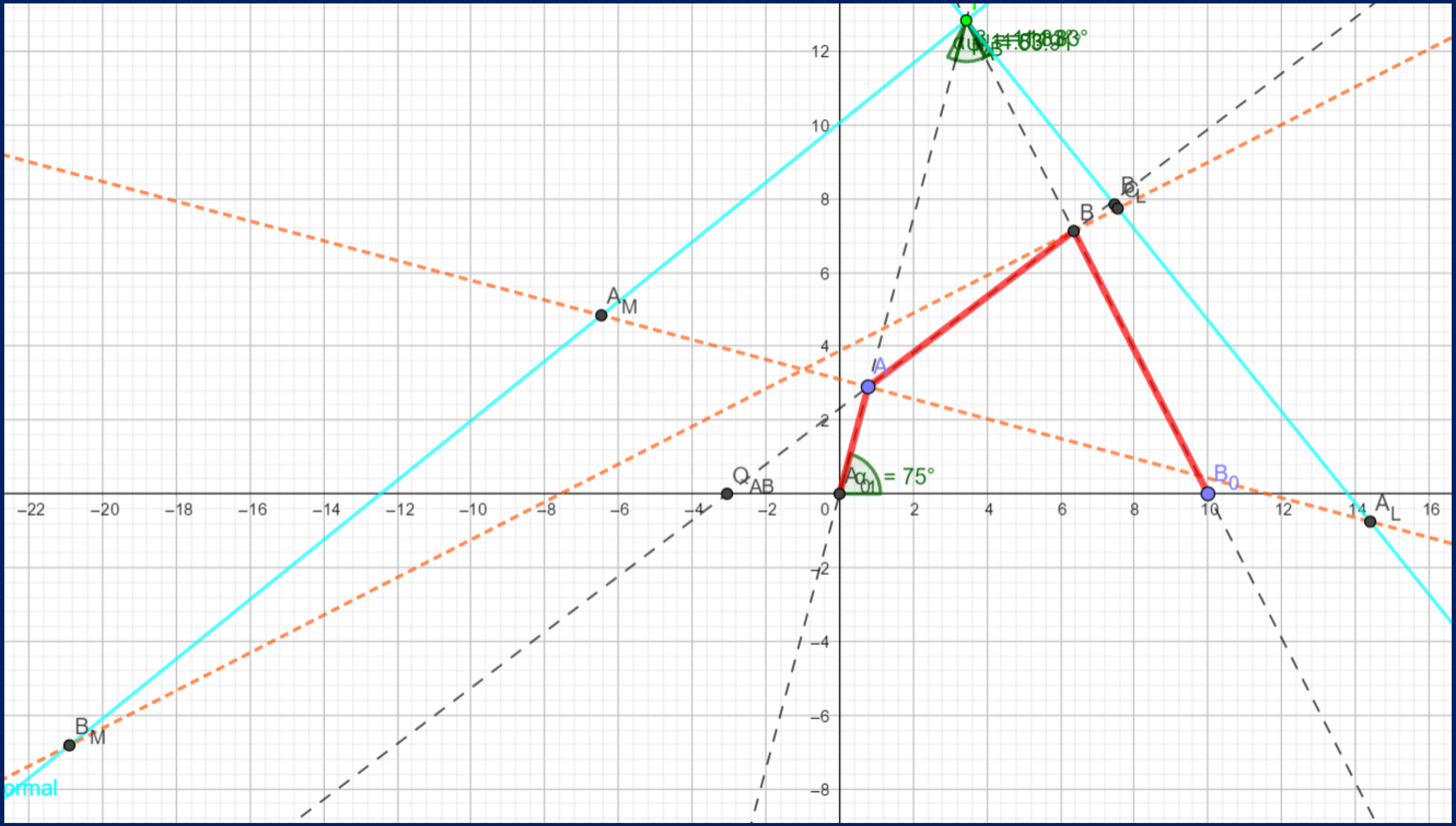
   $\psi_A = 53.91°, r_A = 10.28, r_{A_0} = 13.28 \rightarrow \delta = 56.31$

   $\psi_B = 11.83°, r_B = 6.412, r_{A_0} = 14.41 \rightarrow \delta = 56.35$



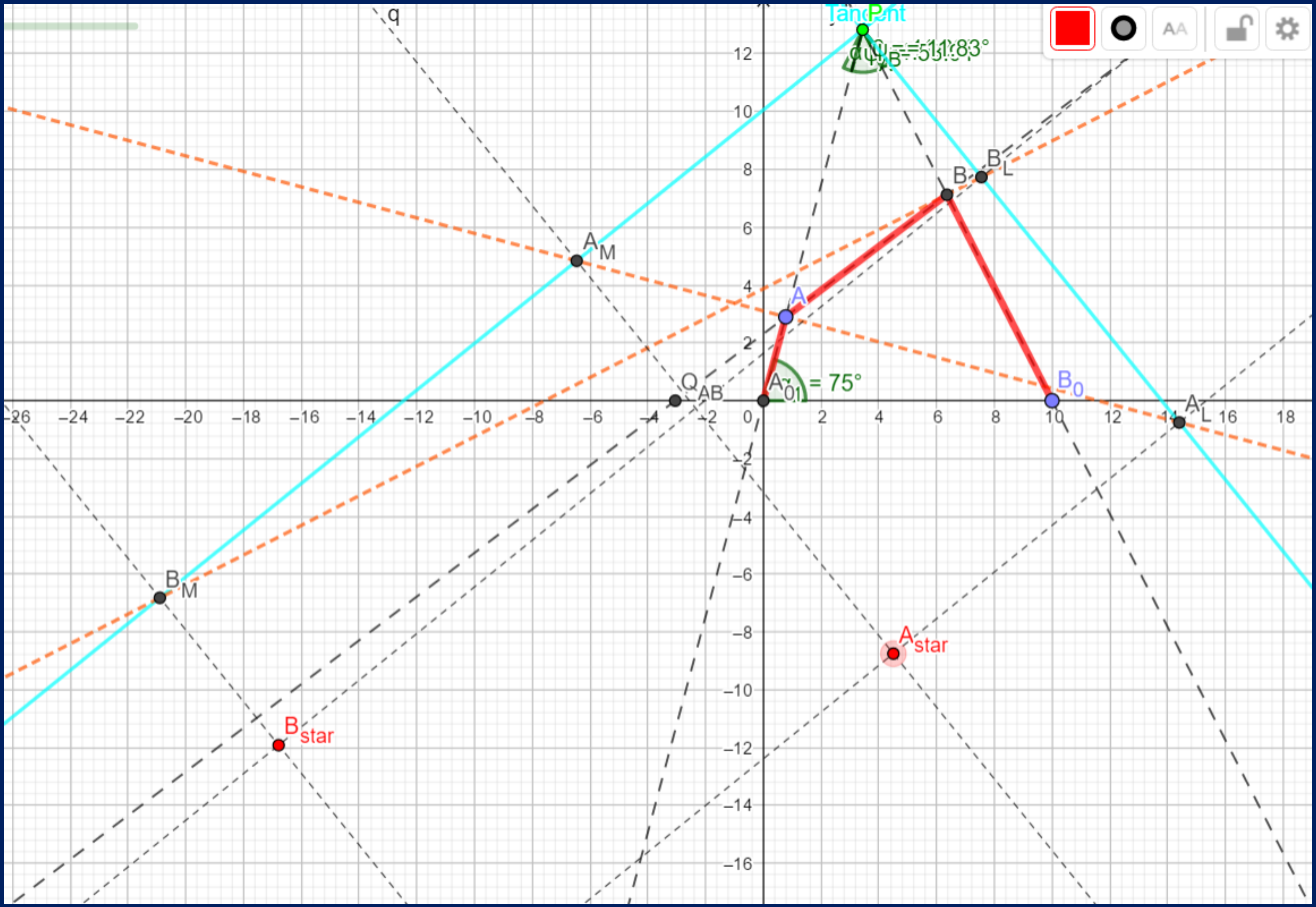
Procedure for the Four Bar

2. Draw perpendiculars to AP and BP though A and B. These perpendiculars intersect PT and PN at A_M, A_L, B_M and B_L.



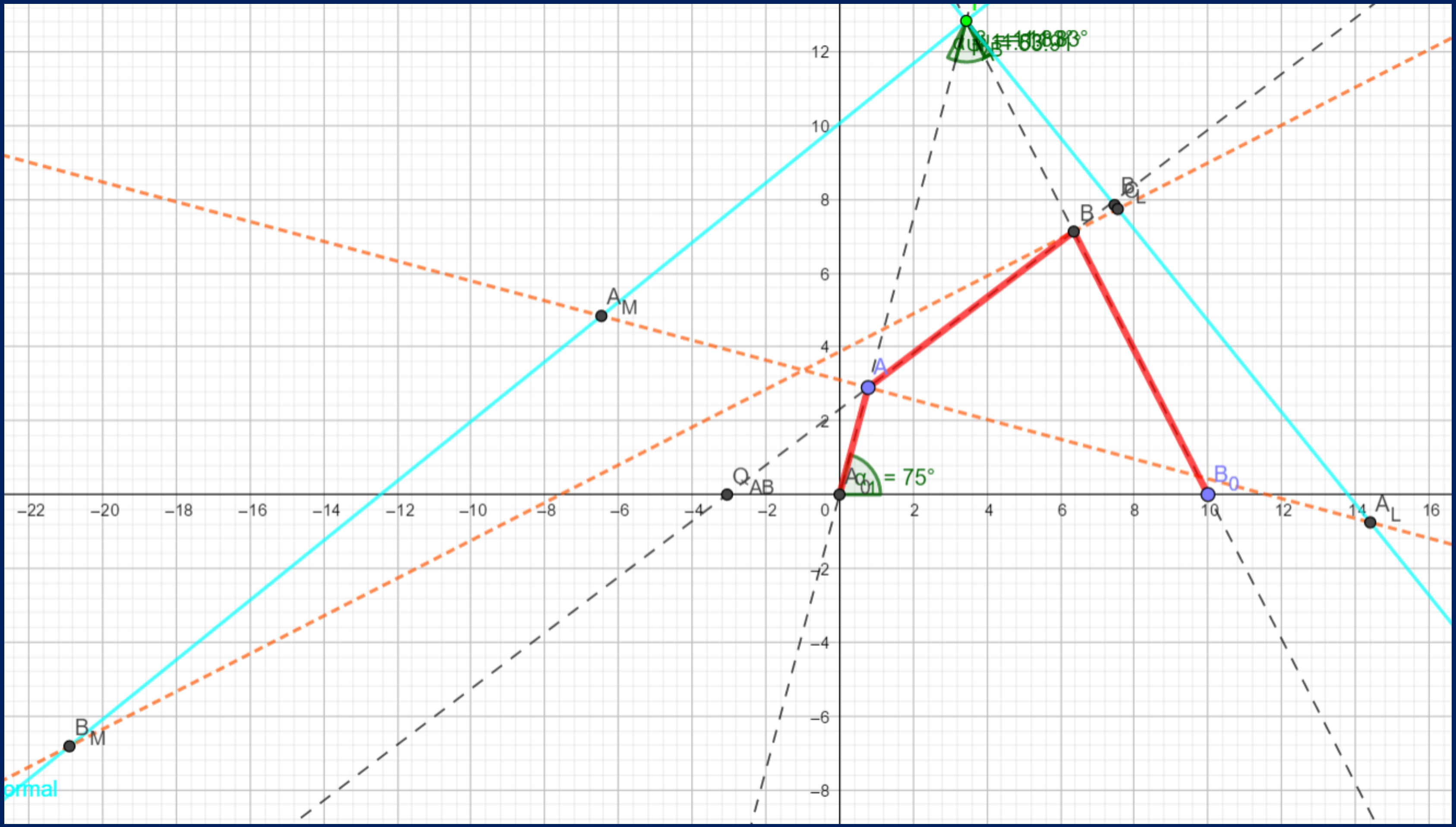
Procedure for the Four Bar

3. Complete rectangles $PA_MA_LA^*$ and $PB_MB_LB^*$.



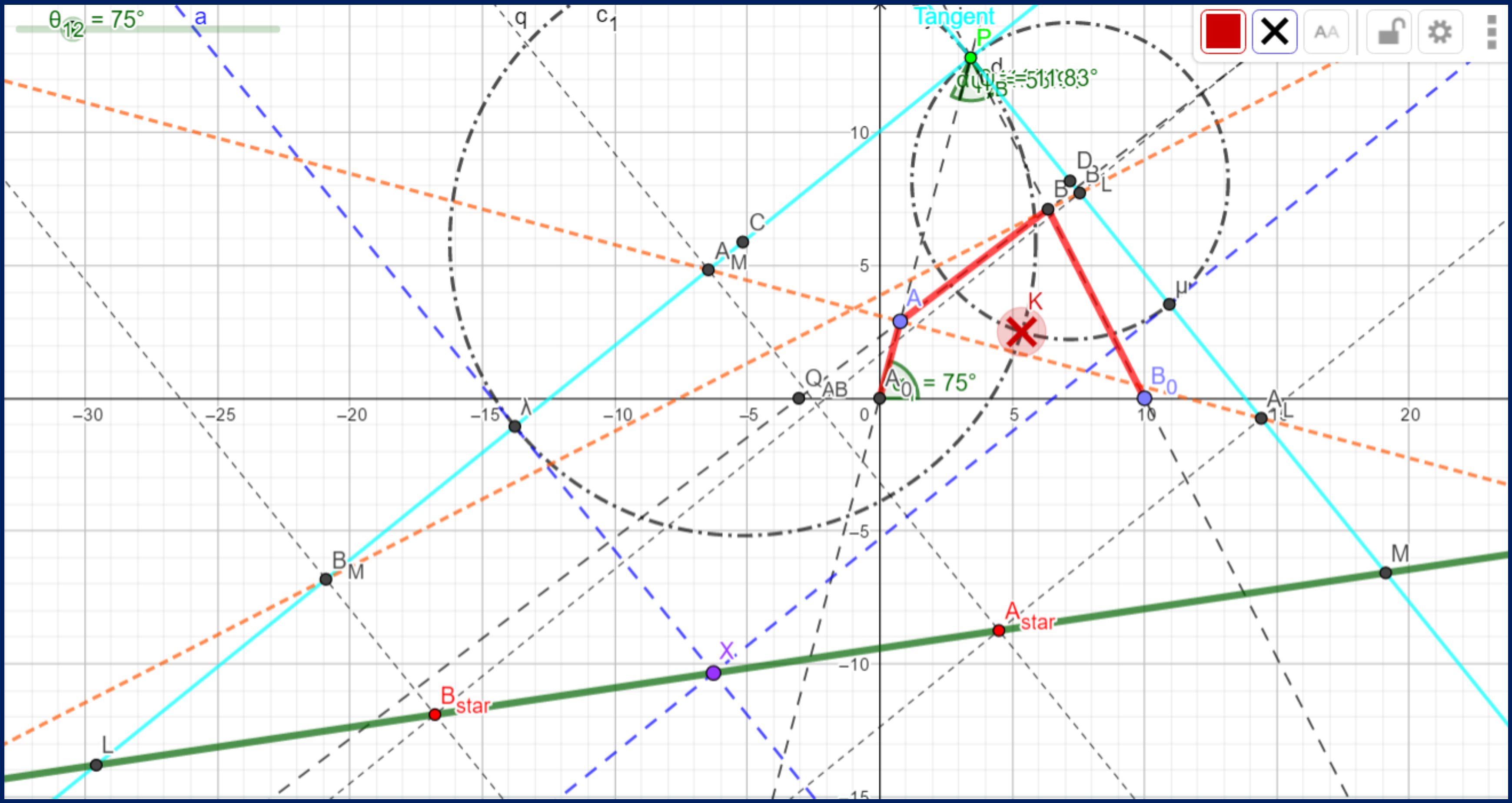
Procedure for the Four Bar

4. Line A*B* intersects N at L and T at M.



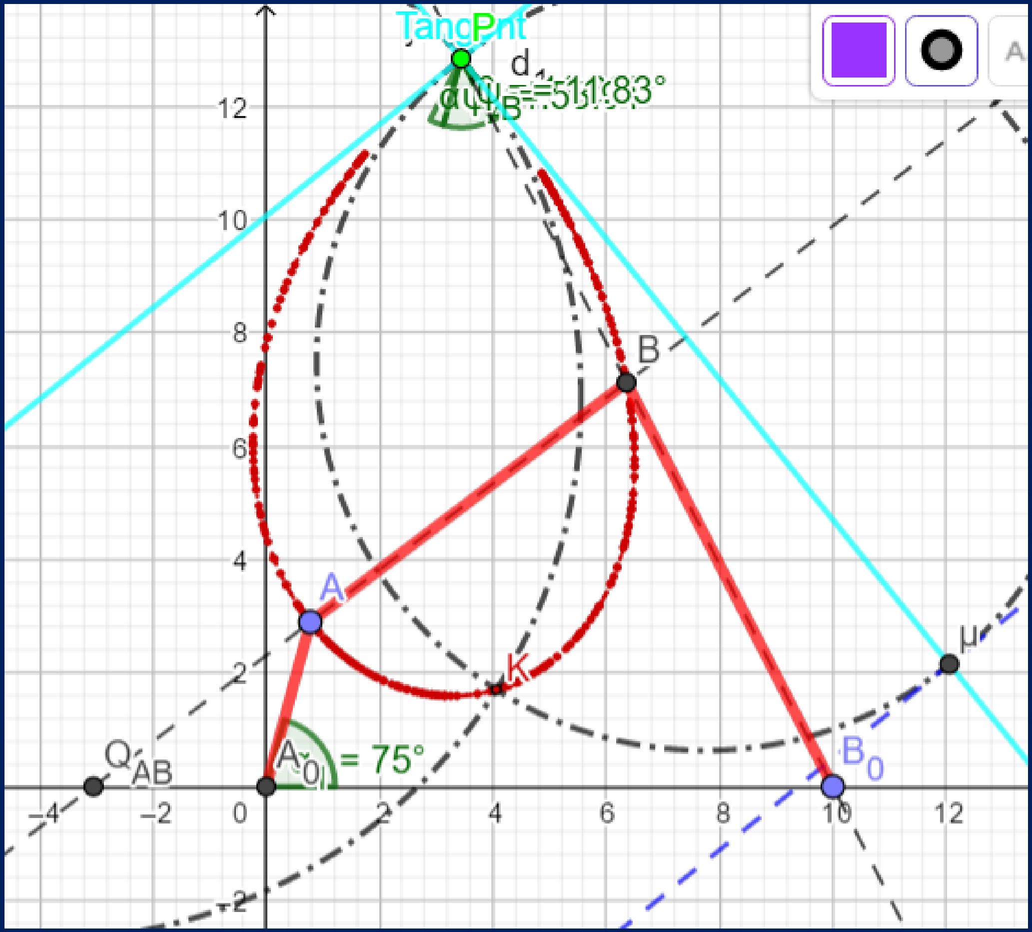
Procedure for the Four Bar

5. Select an X on LM and determine λ and μ to draw the pencils of circles. Intersection of circles yield a point on $K(r,\psi)$.



Procedure for the Four Bar

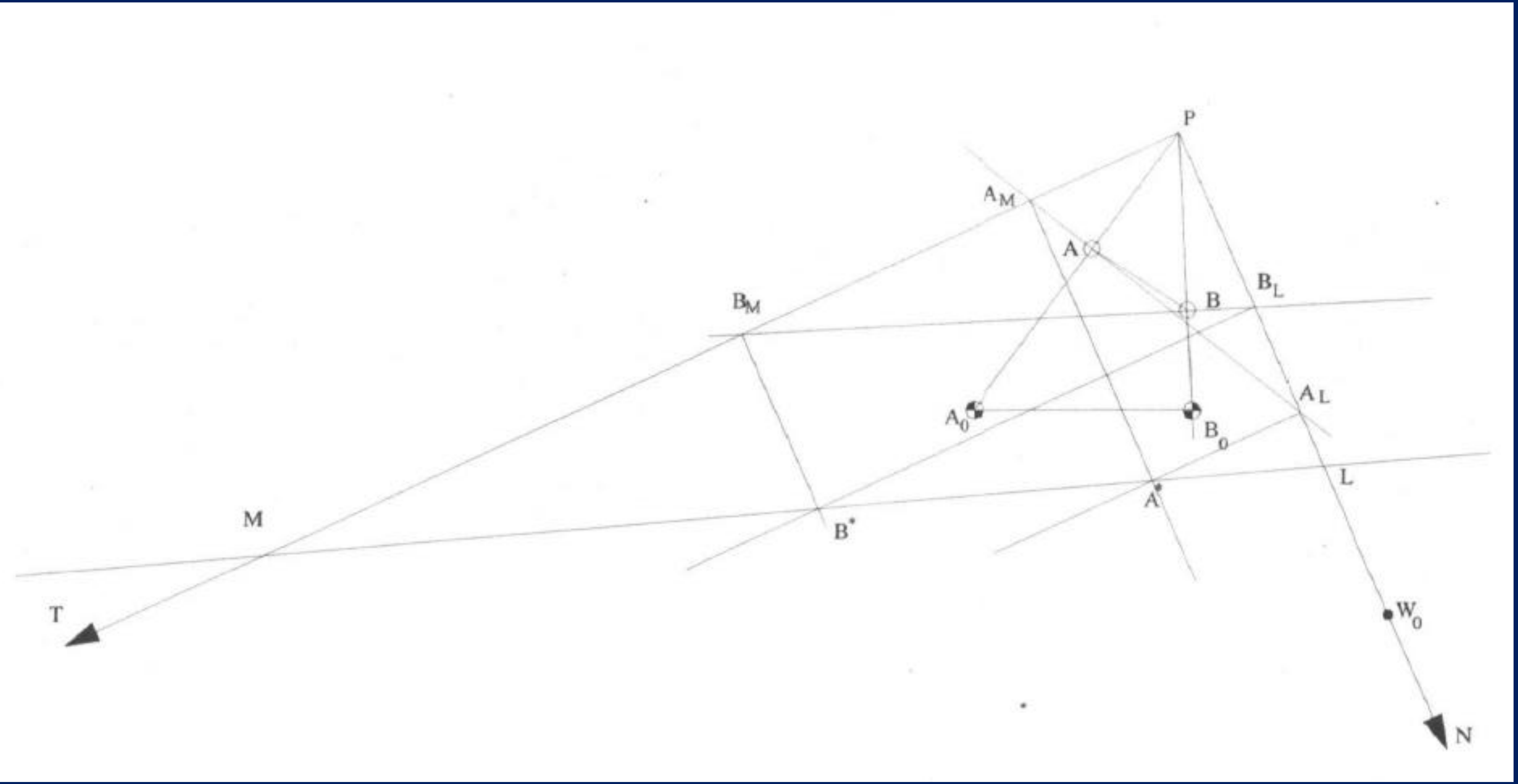
5. (cont'd) Change location of X on LM and determine new λ and μ to draw the new pencils of circles. Intersection of circles yield another point on $K(r, \psi)$.



Procedure for the Four Bar

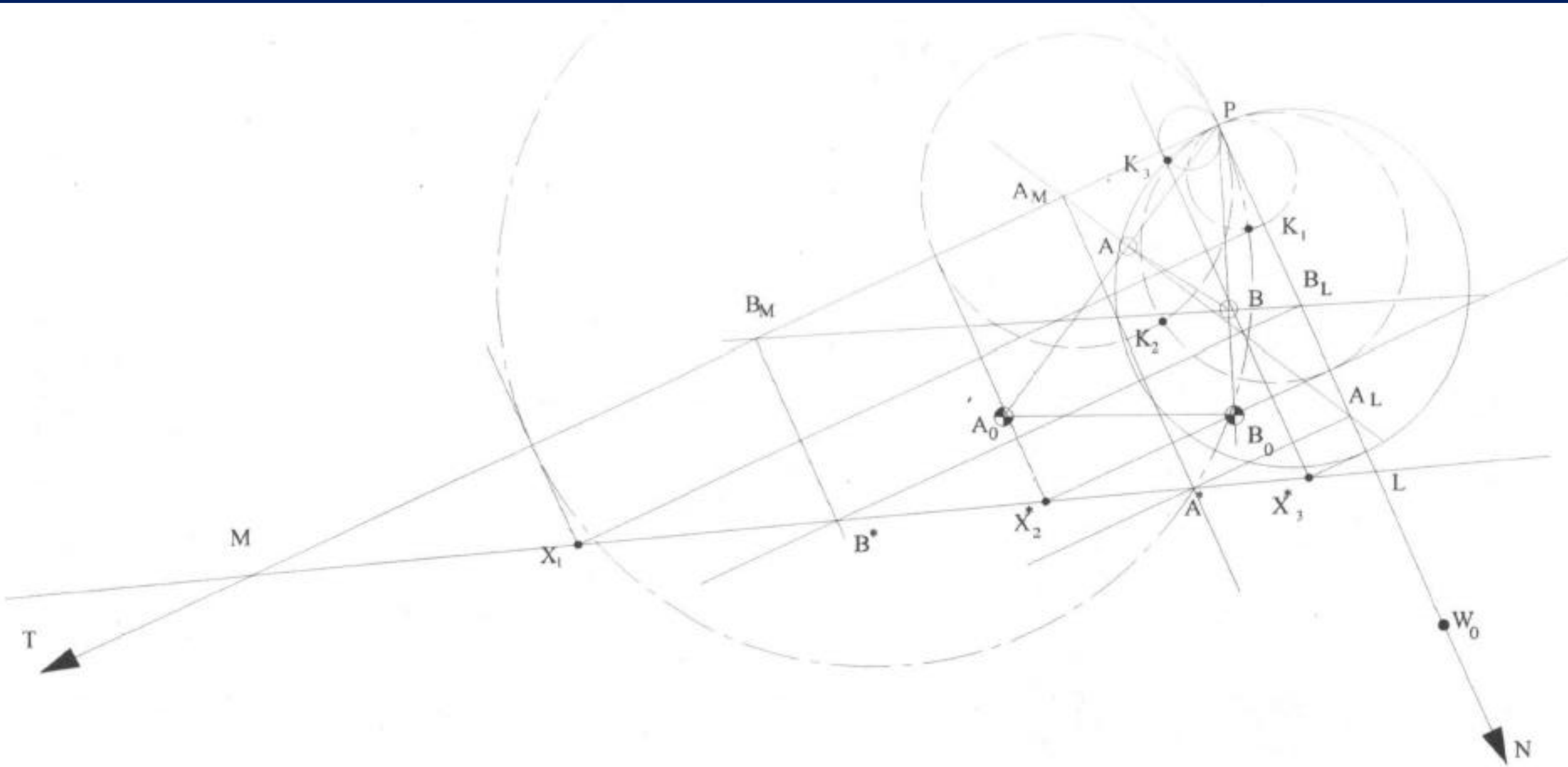
6. For M(r, ψ), center of stationary curvature, one may determine the $K(r, \psi)$ of the inverted motion.

Determination of a Point on K(r,Ψ)



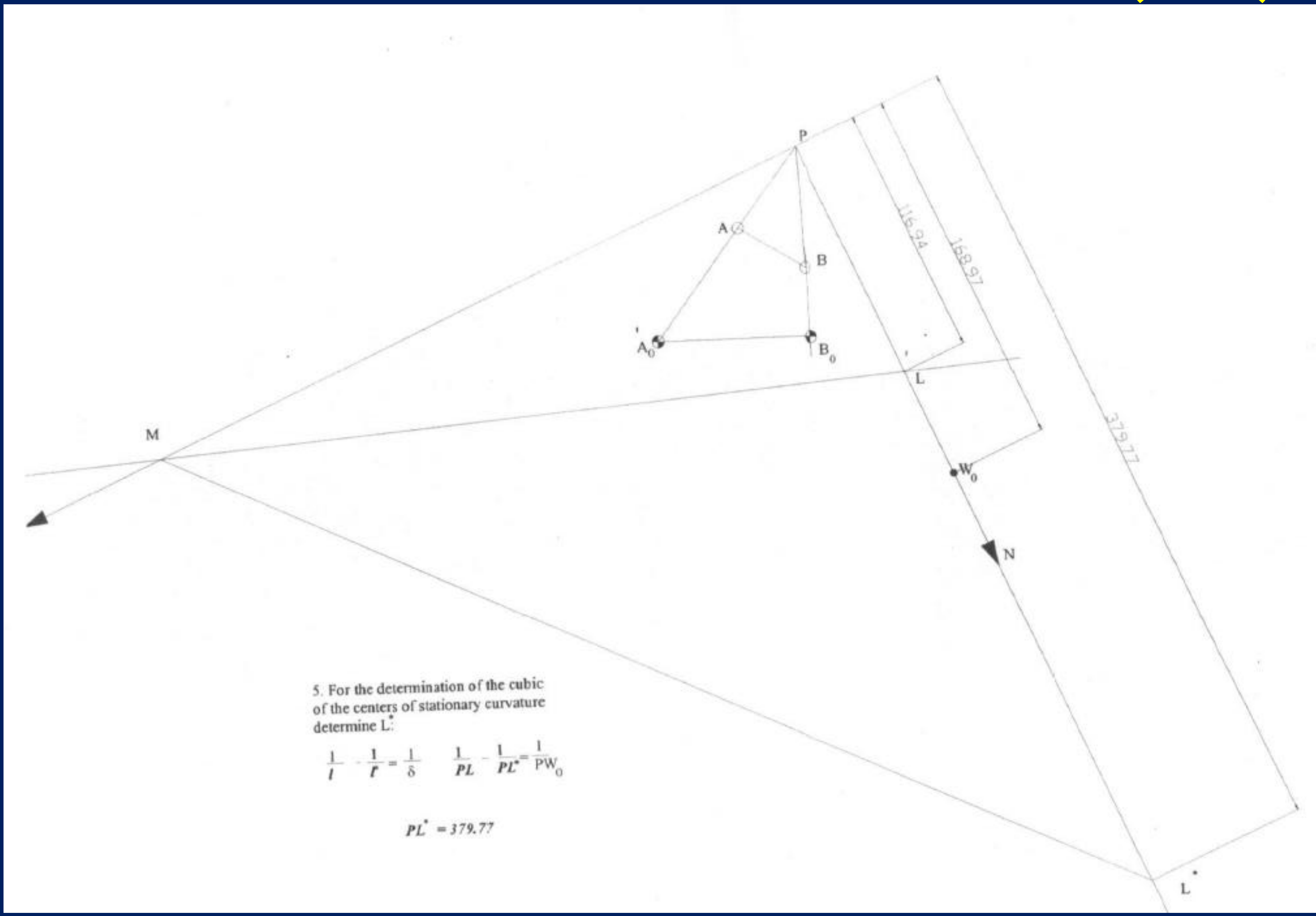
Eres Söylemez (unpublished)

Determination of a Point on K(r,Ψ)



Eres Söylemez (unpublished)

Determination of a Point on K(r,Ψ)



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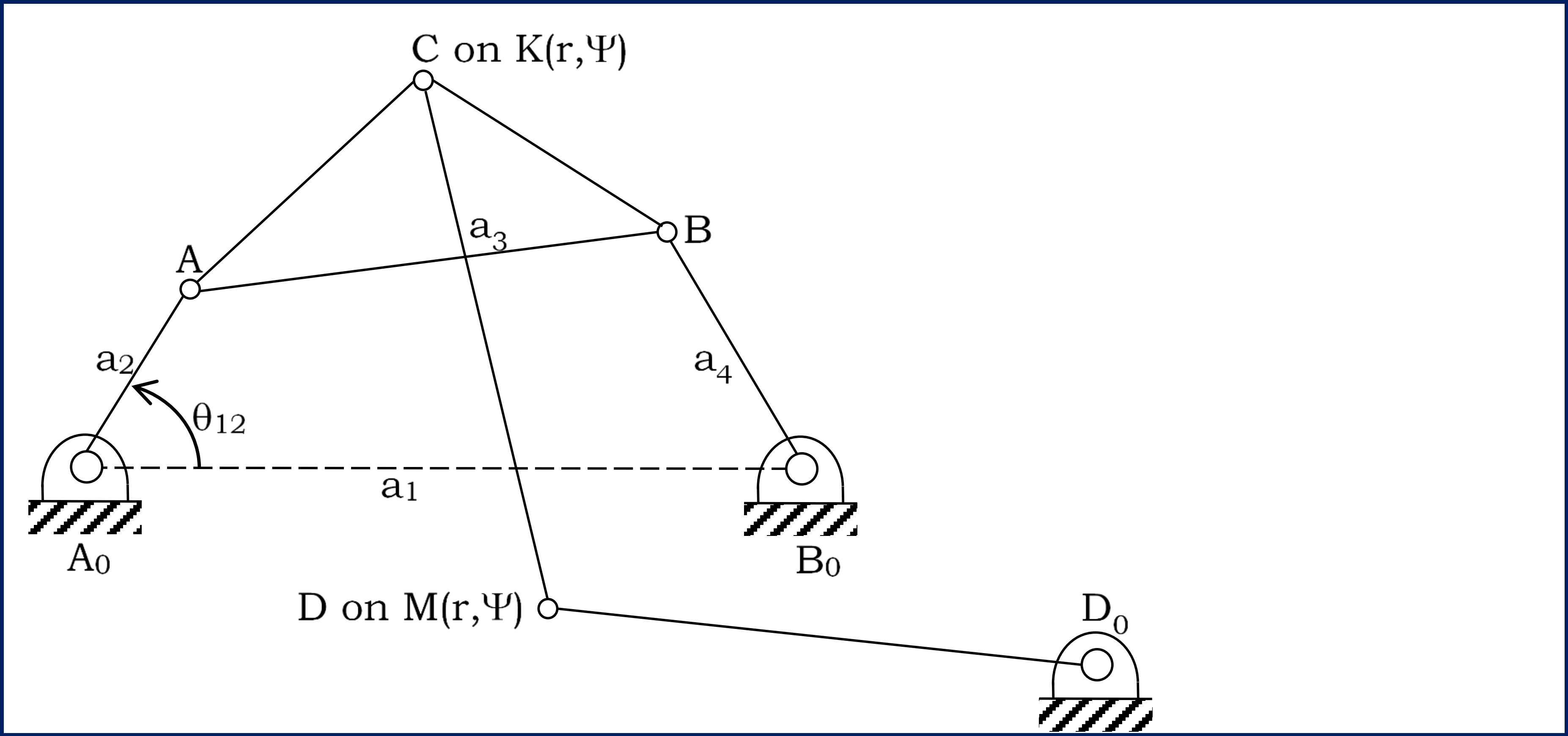
Use of K(r,Ψ) and M(r,Ψ)



At this instant path of point C approximates a circle to the fourth order (i.e. contacts circle, has the same tangent and radius of curvature and further, the rate of change of radius of curvature is zero) therefore is expected to trace an approximate circular path in the vicinity of this position. D being the center of the circle during that instant is stationary therefore D_0D is in dwell.