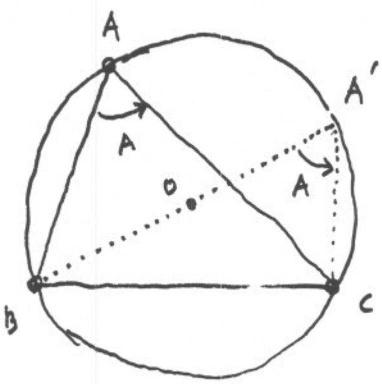
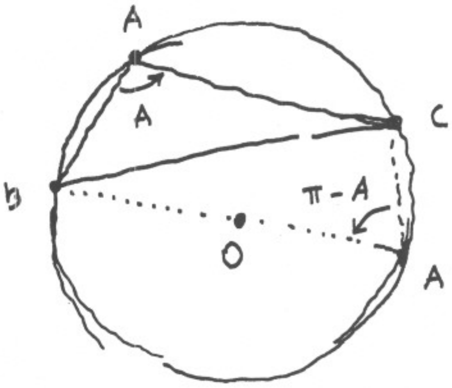
Lecture 2

Basic Computational Methods

§2 Basic Computational Methods

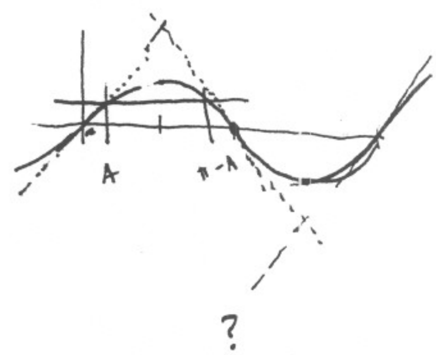
(A) The Sine Rule :





In a positively oriented triangle

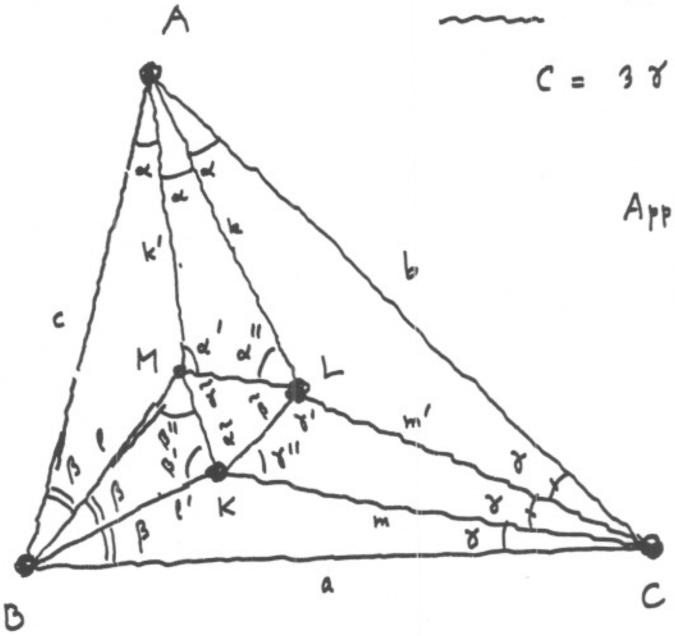
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$



Application : (The Morley "Trisector" Theorem, 1905)

Theorem: Let half lines k, k' and ℓ, ℓ' and m, m' trisect the angles A and B and C respectively in counterclockwise fashion. The points $K = \ell' \cdot m$, $L = m' \cdot k$, $M = k' \cdot \ell$ constitute the vertices of an ~~isosceles~~ equilateral triangle.

Proof: Put $A = 3\alpha$, $B = 3\beta$, $C = 3\gamma$ for convenience.



Applying the sine rule in the triangle ALC

$$\frac{|AL|}{\sin\gamma} = \frac{b}{\sin(\alpha+\gamma)} \longrightarrow |AL| = \frac{b\sin\gamma}{\sin(\alpha+\gamma)}$$

$$= \frac{b\sin\gamma}{\sin(60^\circ-\beta)}.$$

$$= \frac{2R\sin(3\beta)\sin\gamma}{\sin(60^\circ-\beta)}$$

$$= \frac{2R\sin\beta\,[3\cos^2\beta-\sin^2\beta]\sin\gamma}{\frac{1}{2}\,[\sqrt{3}\cos\beta-\sin\beta]}$$

hence

$$|AL| = 8R\sin\beta\sin\gamma\sin(60^\circ+\beta)$$

Similarly

$$|AM| = 8R\sin\beta\sin\gamma\sin(60^\circ+\gamma)$$

Denoting the internal angles of AML at vertices M and L by α' and α'' respectively and applying the sine rule in AML, we obtain

$$\frac{\sin\alpha'}{\sin\alpha''} = \frac{|AL|}{|AM|} = \frac{\sin(60^\circ+\beta)}{\sin(60^\circ+\gamma)} .$$

Hence we conclude

$$\left.\begin{aligned} \alpha' &= 60^\circ+\beta \\ \alpha'' &= 60^\circ+\gamma \end{aligned}\right\} \text{ in view of } \alpha + (60^\circ+\beta) + (60^\circ+\gamma) = 180^\circ.$$

Denoting the internal angles of BKM, CLK, KLM at vertices K and M, L and K, K and L and M by β' and β'', γ' and γ'', $\tilde{\alpha}$ and $\tilde{\beta}$ and $\tilde{\gamma}$ we similarly obtain

$$\begin{aligned} \beta' &= 60^\circ+\gamma \\ \beta'' &= 60^\circ+\alpha \end{aligned}$$

and

$$\begin{aligned} \gamma' &= 60^\circ+\alpha \\ \gamma'' &= 60^\circ+\beta \end{aligned}$$

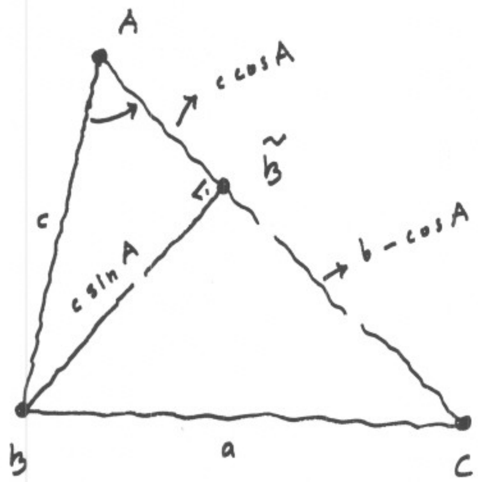
We conclude

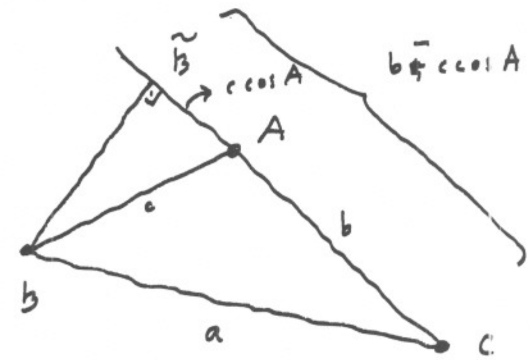
$$\begin{aligned} \tilde{\alpha} &= 360^\circ - (180^\circ-\beta-\gamma) - \beta' - \gamma'' \\ &= 180^\circ + \beta + \gamma - (60^\circ+\gamma) - (60^\circ+\beta) \\ &= 60^\circ . \end{aligned}$$

Similarly $\tilde{\beta} = \tilde{\gamma} = 60^\circ$.

B) The Cosine Rule :

— An Interlude on the Theorem of Pythagoras —





$$a^2 = (c \sin A)^2 + (b - c\cos A)^2$$

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

Application ① The Morley sidelength:

$$|LM| = |MK| = |KL| = 8R \sin\alpha \sin\beta \sin\gamma$$

Application ② The "Heron" Formula :

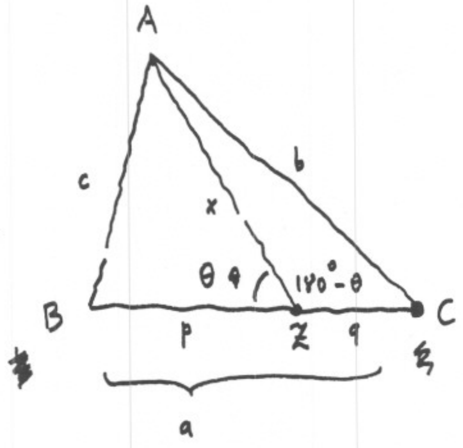
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta^2 = \frac{1}{4} b^2c^2 \sin^2 A = \frac{1}{4} b^2c^2 \left[1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right] = \frac{1}{16}\left[4b^2c^2 - (b^2+c^2-a^2)^2\right]$$

$$= \frac{1}{16}\left[2bc - b^2 - c^2 + a^2\right]\left[2bc + b^2 + c^2 - a^2\right] = \frac{1}{16}\left[a^2 - (b-c)^2\right]\left[(b+c)^2 - a^2\right]$$

$$= \frac{1}{16}(a-b+c)(a+b-c)(b+c-a)(b+c+a) \quad ✓$$

Application (3) "Stewart's Relation"



By the cosine rule in ZAB and ZCA

$$c^2 = p^2 + x^2 - 2px\cos\theta$$

$$b^2 = q^2 + x^2 + 2qx\cos\theta$$

$$qc^2 + pb^2 = pq\underbrace{(p+q)}_{a} + x^2\underbrace{(p+q)}_{a}$$

$$\boxed{ax^2 = pb^2 + qc^2 - apq}$$

Easier to memorise ?

$$apq = pb^2 + qc^2 - ax^2$$

"between"

Example : Compute the lengths of medians :

$$m_a = \sqrt{\frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}}$$

etc.

Example : Compute the length of internal bisectors :

— Interlude on the ratio in which angle bisectors divide the opposite side ... —

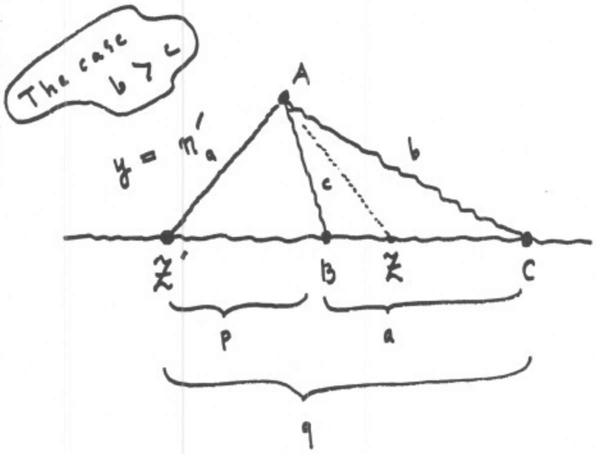
$$ax^2 = \frac{ca}{b+c}b^2 + \frac{ba}{b+c}c^2 - \frac{abca^2}{(b+c)^2}$$

$$x^2 = \frac{bc(b+c)^2 - bca^2}{(b+c)^2} = \frac{bc(b+c+a)(b+c-a)}{(b+c)^2}$$

$$n_a = \frac{2}{b+c}\sqrt{s(s-a)bc}$$

etc.

Example : External bisectors :



$$\left.\begin{array}{l} q = kb \\ p = kc \end{array}\right\} \quad a = q - p = k(b-c)$$

$$k = \frac{a}{b-c}$$

$$p = \frac{ac}{b-c}$$

$$q = \frac{ab}{b-c}$$

$$pb^2 + ay^2 - qc^2 = apq$$

$$ay^2 = -pb^2 + qc^2 + apq$$

$$= -\frac{acb^2}{b-c} + \frac{abc^2}{b-c} + \frac{a\,a^2bc}{(b-c)^2}$$

$$y^2 = \frac{(b-c)(-cb^2+bc^2) + a^2bc}{(b-c)^2} = \frac{bc}{(b-c)^2}\left[-(b-c)^2 + a^2\right]$$

$$\left[-(b-c)^2 + a^2\right] = (a+b-c)(a-b+c)$$

$$\boxed{n'_a = \frac{2}{b-c}\sqrt{(s-b)(s-c)\,bc}}$$

Interlude on the Steiner-Lehmus Theorem :

History & statement. Let the student do the computational proof.

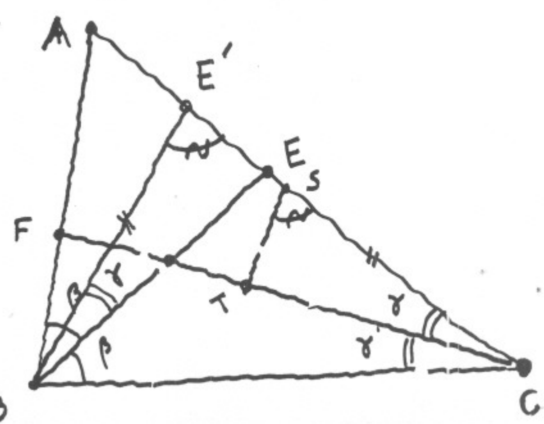
Alternative proof : Coxeter + Greitzer.

Absolute proof : Suppose $C < B$, hence $\gamma < \beta$ and $\gamma + \gamma < \gamma + \beta$

Conclude : $|BE'| < |CE'|$. Choose S between A and C with $|SC| = |BE'|$.

Conclude T between C and F and

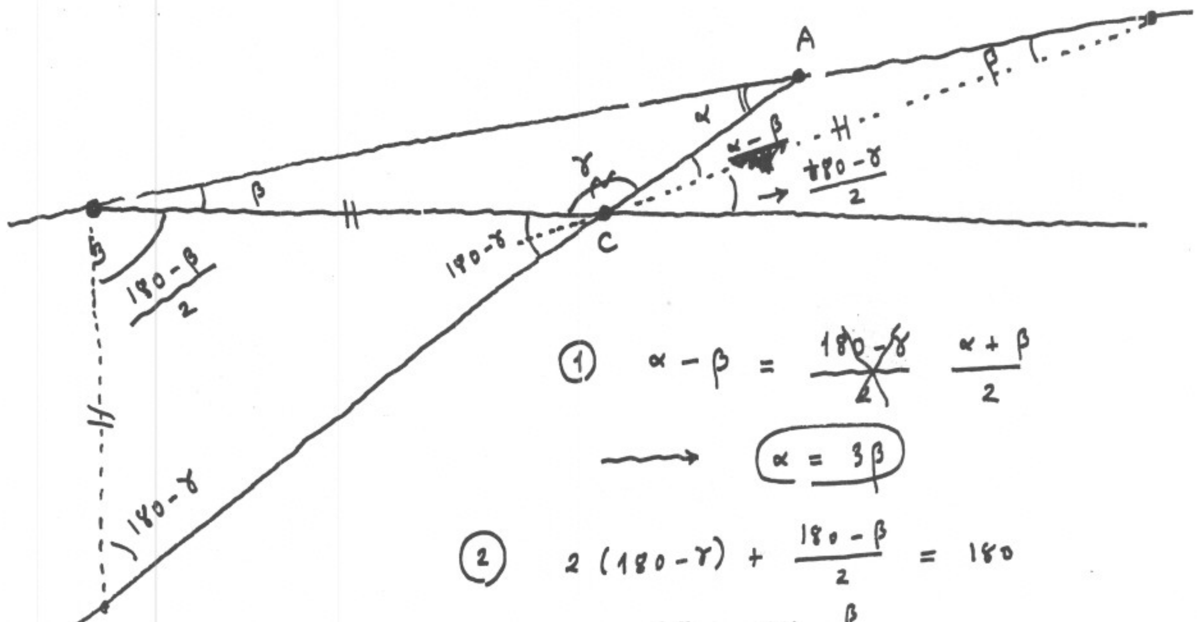
$E'EB \equiv STC$ and $|BE| = |CT| < |CF|$.



Interlude on the Bottema Triangle :

Theorem : There exists "a" (unique !) (up to similarity ...) triangle ABC with

$$h_b' = |BC| = h_c' \quad !$$



① $\alpha - \beta = \frac{\cancel{180-\gamma}}{\cancel{2}} \quad \frac{\alpha+\beta}{2}$

$\rightsquigarrow \boxed{\alpha = 3\beta}$

② $2(180-\gamma) + \frac{180-\beta}{2} = 180$

$270 - 2\gamma - \frac{\beta}{2} = 0$

$4\gamma + \beta = 540$

$\boxed{\gamma = 135 - \frac{\beta}{4}}$

Thus :

$\alpha + \beta + \gamma = 180$

$3\beta + \beta + 135 - \frac{\beta}{4} = 180$

$\frac{15\beta}{4} = 45 \quad \longrightarrow \beta = 12^\circ, \ \gamma = 132^\circ, \ \alpha = 36^\circ$

This is uniqueness.

Check that the triangle determined (up to similarity !) by these angles does indeed do the trick !

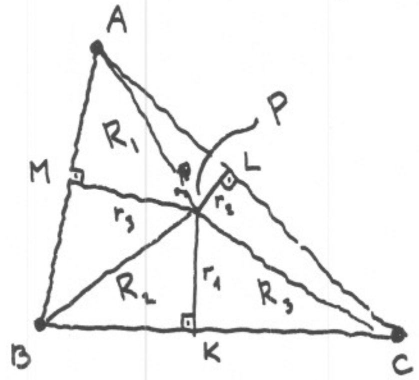
Application ④ The Erdös - Mordell Inequality

Preliminaries on Arithmetic and Geometric means :

a) $x, y \geqslant 0 \longrightarrow \dfrac{x+y}{2} \geqslant \sqrt{xy}$: Reduces to equality iff $x = y$

Proof : Look at $(\sqrt{x} - \sqrt{y})^2 \geqslant 0$

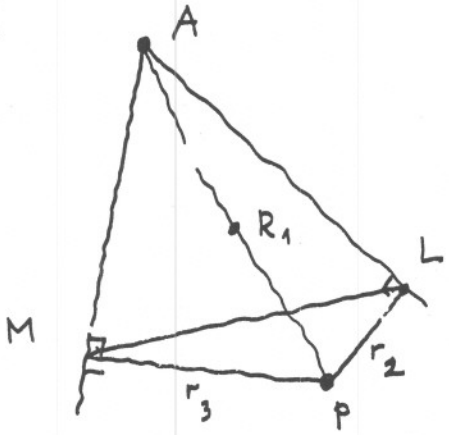
b) $x, y > 0 \rightsquigarrow \dfrac{x}{y} + \dfrac{y}{x} \geqslant 2$: Equality iff $x = y$!



Theorem: For any point P inside ABC,

$$R_1 + R_2 + R_3 \geqslant 2(r_1 + r_2 + r_3)$$

equality holding iff ABC equilateral and $P = O = H = G = I$!



Proof: The sine rule in AML which admits $[AP]$ as a diameter of its circum cirle, followed by the cosine rule in PLM give

$$R_1 = \frac{|LM|}{\sin A} = \frac{(r_2^2 + r_3^2 + 2 r_2 r_3 \cos A)^{1/2}}{\sin A}$$

$$= \frac{(r_2^2 + r_3^2 - 2 r_2 r_3 \cos(B + C))^{1/2}}{\sin A}$$

$$= \frac{\left[(r_2 \sin C + r_3 \sin B)^2 + (r_2 \cos C - r_3 \cos B)^2\right]^{1/2}}{\sin A}$$

$$\geqslant \frac{r_2 \sin C + r_3 \sin B}{\sin A}$$

Thus

$$R_1 + R_2 + R_3 \geqslant r_1 \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \right) + r_2 \left(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C} \right) + r_3 \left(\frac{\sin A}{\sin B} + \frac{\sin B}{\sin A} \right)$$

$$\geqslant 2\,(r_1 + r_2 + r_3) \quad \text{equality holding only if } A = B = C$$

and

$$r_2 \cos C - r_3 \cos B = 0 \longrightarrow r_2 = r_3 \longrightarrow r_1 = r_2 = r_3 \;!$$

For more: Scientific American, May 1998

Application (5) A Japanese Temple Problem :

Preliminaries: a) Note that two "disjoint" circles have four common tangents.

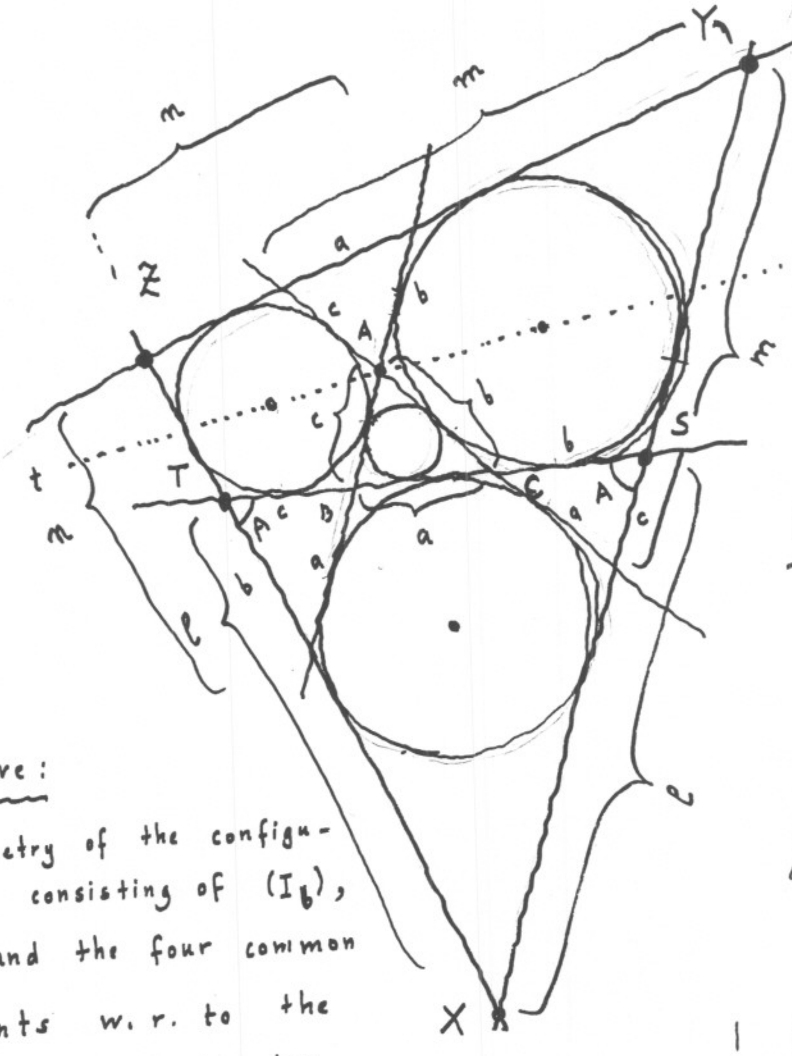
b) In a triangle ABC, the sides a, b, c are common tangents of any ~~of the~~ pair of circles from among (I_a), (I_b), (I_c). We refer to the remaining common tangents as the "fourth".

Theorem: Let ABC be an acute angled triangle.

$$r + r_a + r_b + r_c = 2\rho$$

where ρ is the inradius of the triangle constituted by the "fourth" common tangents of (I_b) and (I_c), (I_c) and (I_a), (I_a) and (I_b).

Proof:



Preliminaries:

$$r = (s-a)\tan\frac{A}{2}$$

$$!\quad 2r = (b+c-a)\tan\frac{A}{2}\quad !$$

Observe:

1) Symmetry of the configuration consisting of (I_b), (I_c) and the four common tangents w.r. to the external angle bisector through A; indicated by t in the illustration.

Similar configurations attached to the vertices B, C.

~~2 ∡ X = 180 − 2A~~

2) XST isosceles with $\measuredangle TSX = \measuredangle STX = A$

3) $X = 180 - 2A$, $\quad 2P = \dfrac{a+b+c}{\cos A} = |ST|$

$$= \frac{1}{\sin A}\left[a+b+c + (a-b-c)\cos A \right]$$

$$= \cot A \left[\underbrace{\frac{a+b+c}{\cos A}}_{2\ell} + a-b-c \right]$$

$$\vdots$$

$$= \tan\left(\frac{X}{2}\right)\left[\underbrace{(\ell + \overset{!}{m} - c)}_{|XY|} + \underbrace{(\ell + \overset{!}{n} - b)}_{|XZ|} - \underbrace{(\overset{!}{m} + \overset{!}{n} - a)}_{|YZ|}\right]$$

$$= 2\rho .$$