Lecture 3

Theorems of Menelaus and Ceva

§3. Theorems of Menelaus and Ceva :

† 100, Alexandria

* Giovanni Ceva
G. C.
(1648–1734, Mantua)

I like to think of these theorems as generalisations of the Th. of Thales !

Preliminaries on the theorem of Thales (Miletus, −585)

On the notation (once again!) $\frac{AB}{CD}$.

Important special cases, $\frac{AB}{AC} = -2, -1, \frac{1}{2}, 2$

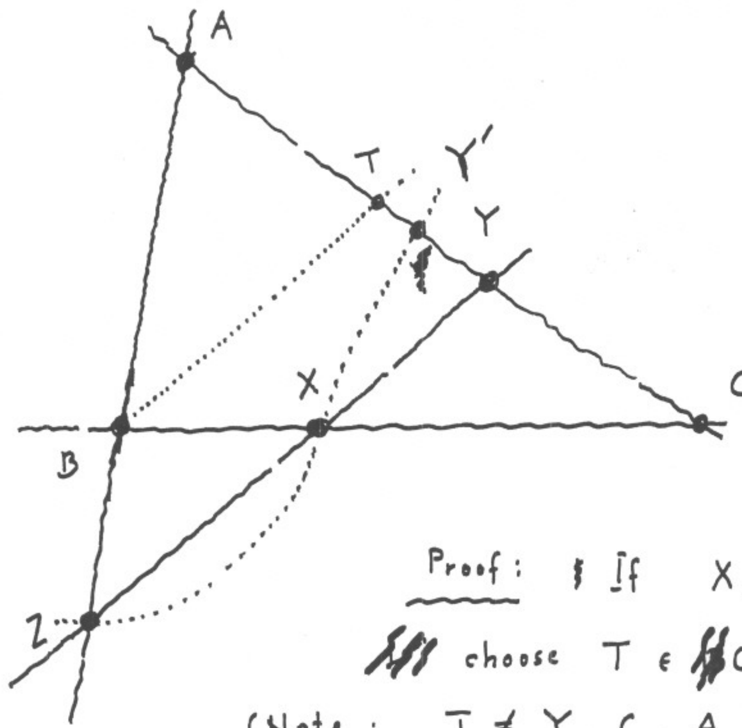
~~Ask about~~ Inquire into the case (impossible) $\frac{AB}{AC} = 1$

Talk about the "point at infinity" ..!

Theorem : ("Theorem of Menelaus")

Given any triangle ABC and points $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$, in order for X, Y, Z to be collinear, it is necessary and sufficient that

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = +1 . \qquad (*)$$



Proof: If X, Y, Z are collinear, choose $T \in CA$ with $BT \parallel XY$

(Note: $T \neq Y, C, A$! Why ?)

We obtain

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \frac{YT}{YC} \cdot \frac{YC}{YA} \cdot \frac{YA}{YT} = +1$$

Conversely, suppose that $(*)$ holds for some $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$: XZ intersects CA in a point $Y' \neq C, A$. (Why ?)

By the first part of this proof, we conclude

$$\frac{XB}{XC} \cdot \frac{Y'C}{Y'A} \cdot \frac{ZA}{ZB} = +1$$

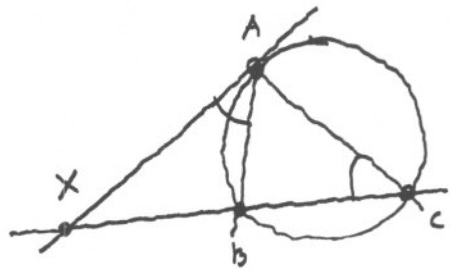
which gives, upon comparison with $(*)$ $\dfrac{YC}{YA} = \dfrac{Y'C}{Y'A}$ hence $Y = Y'$. Therefore

$X, Y' = Y, Z$ are collinear.

Application 1 : A very classical problem.

"In a triangle ABC, the tangents to (O) at A, B, C intersect BC, CA, AB in X, Y, Z respectively. Prove that X, Y, Z are collinear!"

Solution:



$AXB \sim CXA$

hence

$$\frac{|XB|}{|XA|} = \frac{|AB|}{|AC|}$$

$$\frac{|XA|}{|XC|} = \frac{|AB|}{|AC|}$$

Thus $$\frac{XB}{XC} \overset{!}{=} \frac{|XB|}{|XC|} = \frac{|AB|^2}{|AC|^2} \longrightarrow \text{Menelaus...}$$

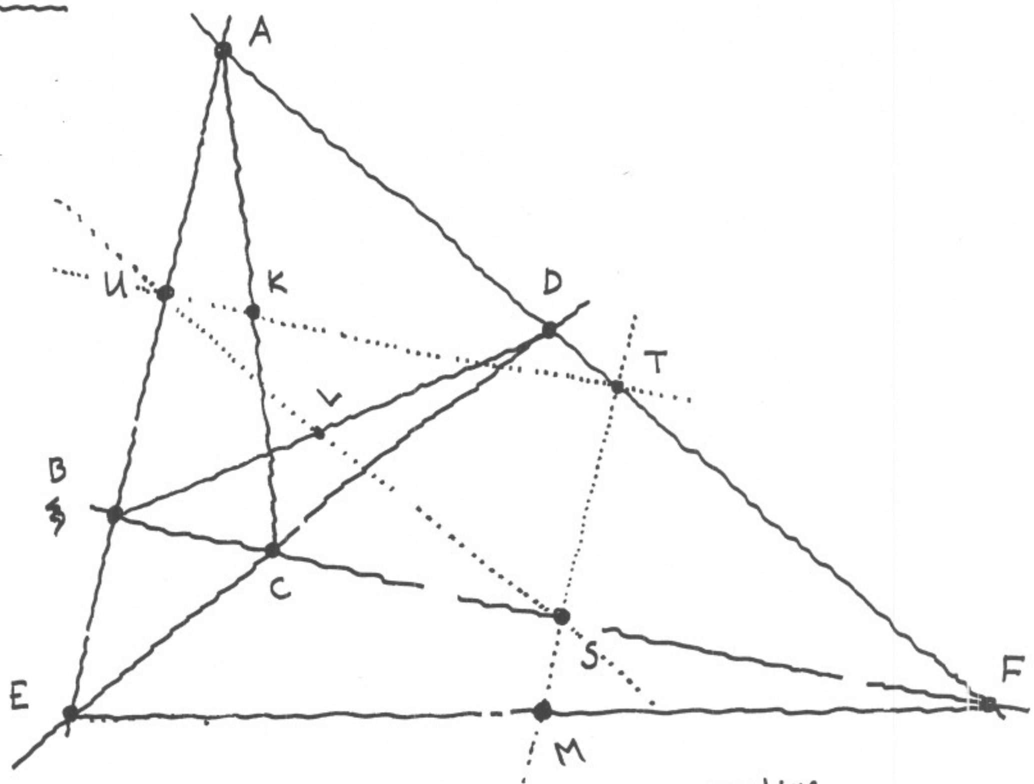
Application 2 : (The "Newton Line")

Interlude on "complete quadrilaterals" + their "sides", "vertices", "diagonals", "diagonal segments"

Theorem: In a complete quadrilateral, the midpoints of diagonal segments are collinear.

o be visited!

Proof:



Let K, L, M be the midpoints of the respective diagonal segments $[AC]$, $[BD]$, $[EF]$. Let S, T, U be midpoints of $[BF]$, $[FA]$, $[AB]$ respectively.

Observe that $K \in UT - \{U,T\}$, $L \in SU - \{S,U\}$, $M \in ST - \{S,T\}$. We shall employ "Menelaus" in STU:

$$\cancel{\frac{KU}{KL} \cdot \frac{MT}{MS}}.$$

$$\frac{KT}{KU} \cdot \frac{LU}{LS} \cdot \frac{MS}{MT} = \frac{CF}{CB} \cdot \frac{DA}{DF} \cdot \frac{EB}{EF}$$

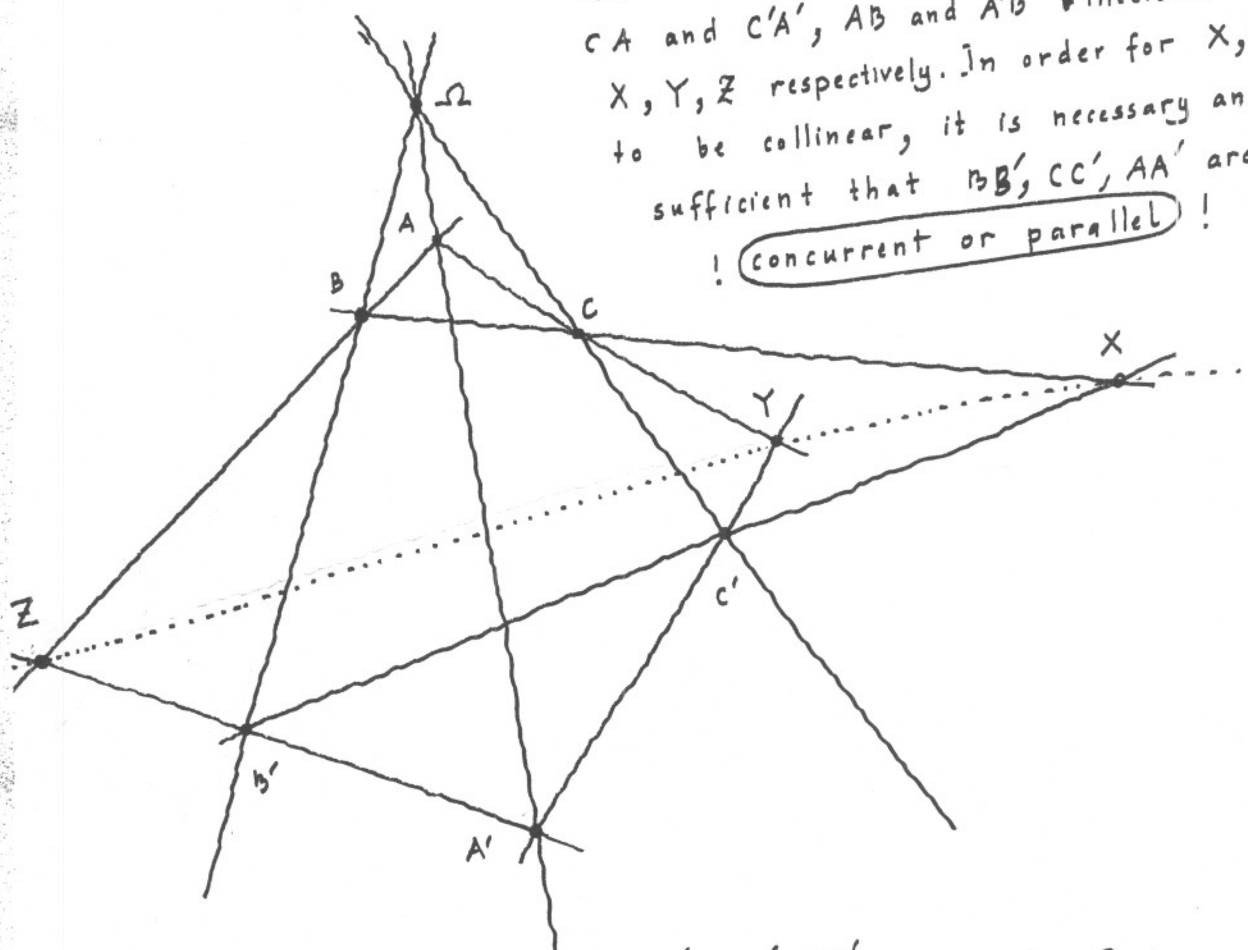
The right hand side equals $+1$ by "Menelaus".

Application ③ (The "incomplete" Theorem of Desargues)

Short interlude on the use of the word "complete" in mathematics.
$\}$ $\mathbb{Q} \subseteq \mathbb{R}$, metric spaces etc.

Gérard Desargues (1591–1661)

<u>Theorem</u>: Given triangles ABC, $A'B'C'$ with $A \neq A'$, $B \neq B'$, $C \neq C'$, let BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ intersect in X, Y, Z respectively. In order for X, Y, Z to be collinear, it is necessary and sufficient that BB', CC', AA' are ! concurrent or parallel !



<u>Proof</u>: First suppose that AA', BB', CC' concur in Ω:

"Menelaus" in ΩBC w.r.to $B'C'$:

$$\boxed{\frac{XB}{XC} \cdot \frac{C'C}{C'\Omega} \cdot \frac{B'\Omega}{B'B} = 1}$$

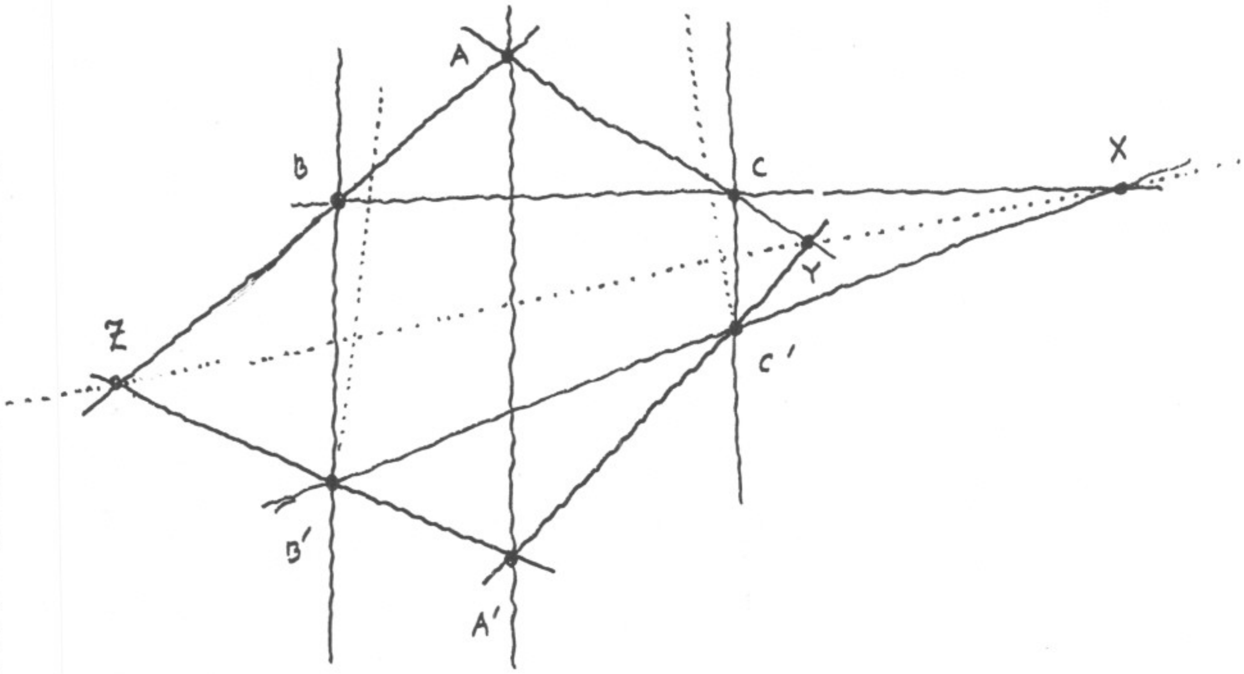
"Menelaus" in ΩCA w.r.to $C'A'$:

$$\boxed{\frac{YC}{YA} \cdot \frac{A'A}{A'\Omega} \cdot \frac{C'\Omega}{C'C} = 1}$$

"Menelaus" in ΩAB w.r.to $A'B'$:

$$\boxed{\frac{ZA}{ZB} \cdot \frac{B'B}{B'\Omega} \cdot \frac{A'\Omega}{A'A} = 1}$$

Multiplying these equalities we find $\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = +1$. By "Menelaus" in ABC we conclude that X, Y, Z are collinear.



Now suppose that AA', BB', CC' are parallel. Since moving the lines BB', CC' slightly while leaving B', C' fixed we can make these lines intersect on AA', we conclude that

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \frac{ZA}{ZB} > 0 \quad \text{"by continuity" !}$$

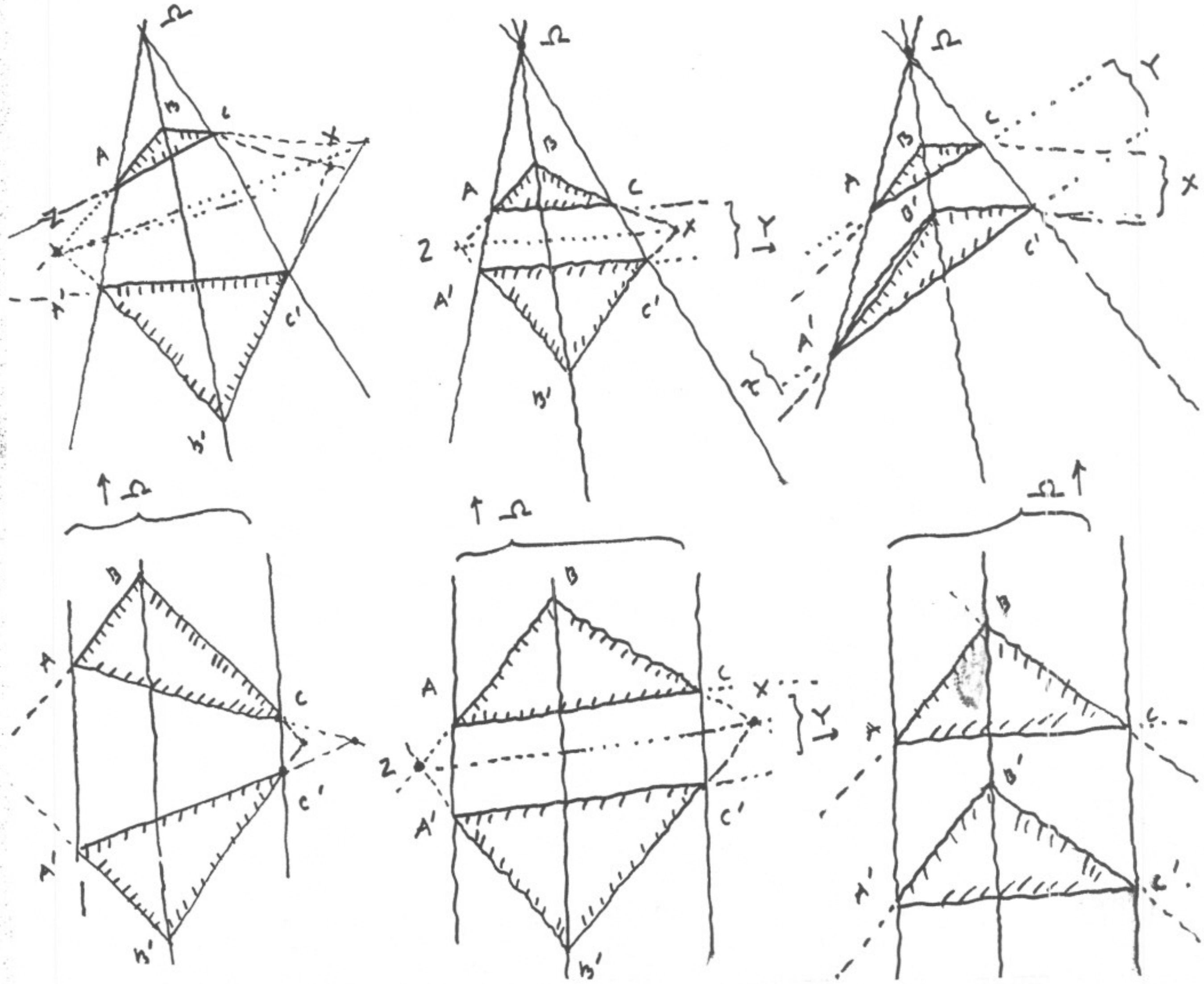
Consequently

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \frac{|XB|}{|XC|} \cdot \frac{|YC|}{|YA|} \cdot \frac{|ZA|}{|ZB|} = \frac{|BB'|}{|CC'|} \cdot \frac{|CC'|}{|AA'|} \cdot \frac{|AA'|}{|BB'|} = 1.$$

Again we conclude that X, Y, Z are collinear.

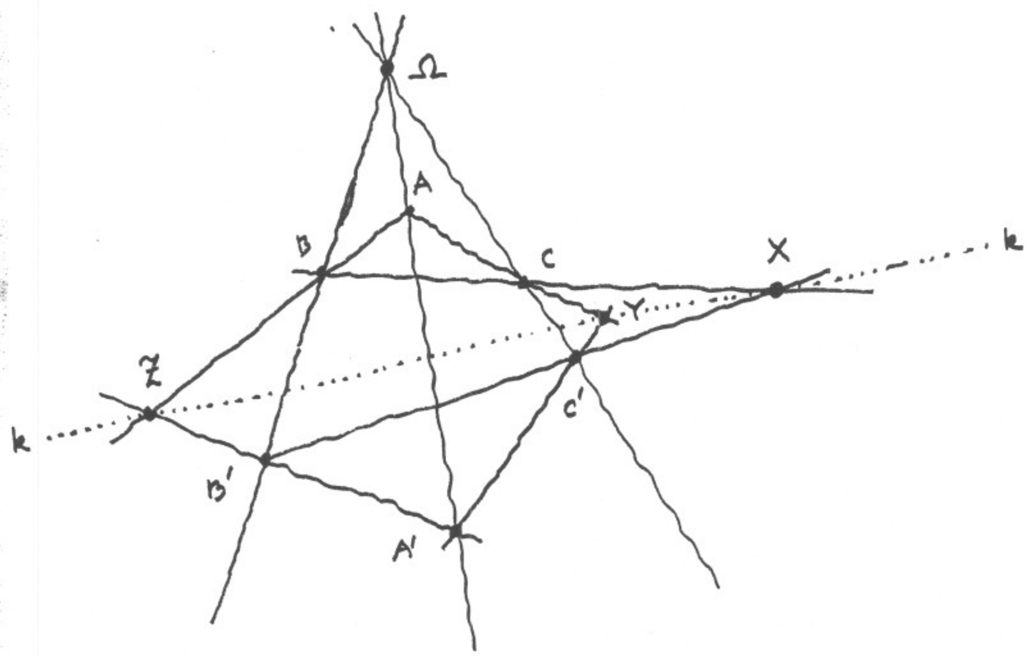
Conversely, suppose that X, Y, Z are collinear.
If AA', BB', CC' are parallel, there is nothing to prove.
If not we may assume w.l.o.g. that BB', CC' intersect in a point, say Ω. Consider the triangles ZBB' and YCC': The sides BB' and CC', ZB and YC, ZB' and YC' intersect in Ω, A, A' respectively. But BC, $B'C'$ and ~~Y~~ ZY concur in X! Hence we conclude by the first part of this proof that Ω, A, A' are collinear; equivalently BB', CC' and AA' concur (in Ω!)

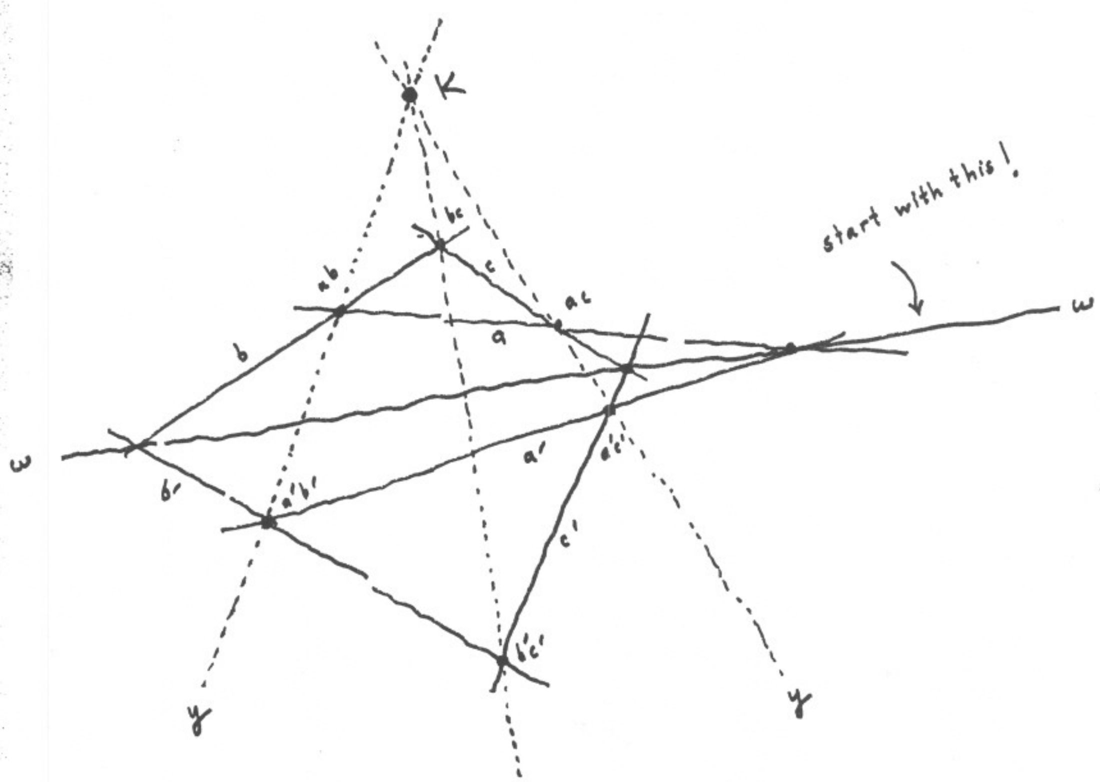
- End of the proof } Now ask: What is it that we miss in the above "incomplete" theorem?



Interlude on duality :

The theorem of Desargues is self-dual !





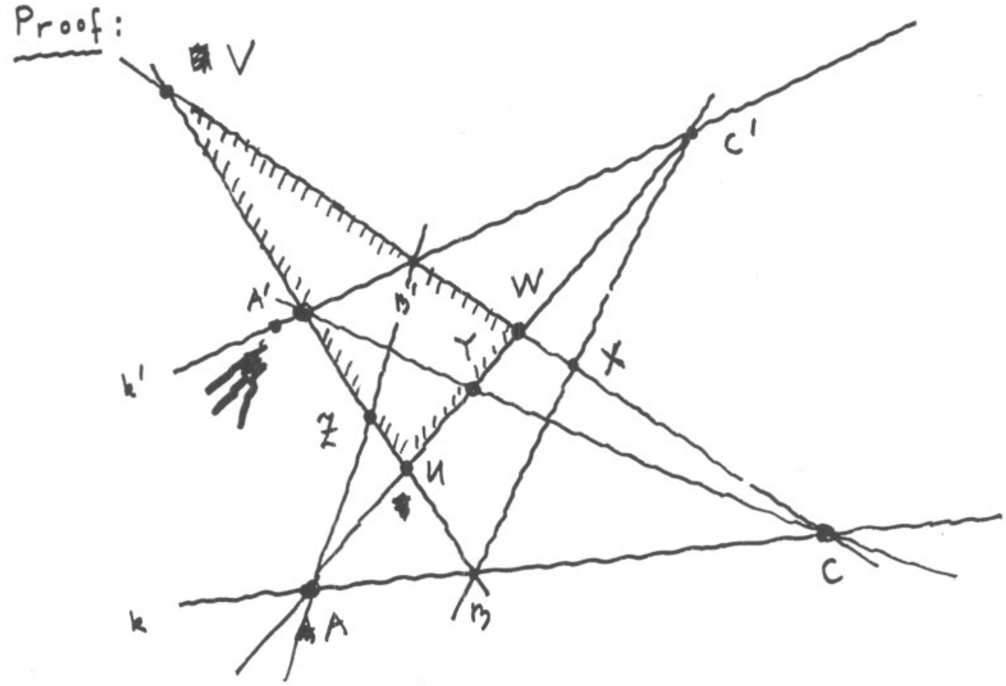
Application [redacted] of Desargues: Pr. 8

(3.5) + 320, Alexandria...

Application (4) The Theorem of Pappus

Theorem: Given distinct lines k, k' and distinct points $A, B, C \in k$, $A', B', C' \in k'$, [redacted] if BC' and $B'C$, CA' and $C'A$, AB' and $A'B$ intersect in X, Y, Z, then X, Y, Z are collinear.

Proof:



"Menelaus" in UVW: $B'C, C'A, A'B$

w.r.to $BC' = XB$: $\frac{XV}{XW} \cdot \frac{C'W}{C'U} \cdot \frac{BU}{BV} = 1$

$CA' = YC$: $\frac{YW}{YU} \cdot \frac{A'U}{A'V} \cdot \frac{CV}{CW} = 1$

AB' ZA : $\frac{ZU}{ZV} \cdot \frac{B'V}{B'W} \cdot \frac{AW}{AU} = 1$

and

[redacted]

k : $\frac{CV}{CW} \cdot \frac{AW}{AU} \cdot \frac{BU}{BV} = 1$

k' : $\frac{B'V}{B'W} \cdot \frac{C'W}{C'U} \cdot \frac{A'U}{A'V} = 1$

From these we can easily derive

$$\frac{XV}{XW}\cdot\frac{YW}{YU}\cdot\frac{ZU}{ZV}=1$$

from which we conclude by "Menelaus" in UVW again, that X, Y, Z are collinear.

— Another interlude on duality —

Here we formulate the dual of Pappus' theorem and prove the dual by employing the theorem itself!

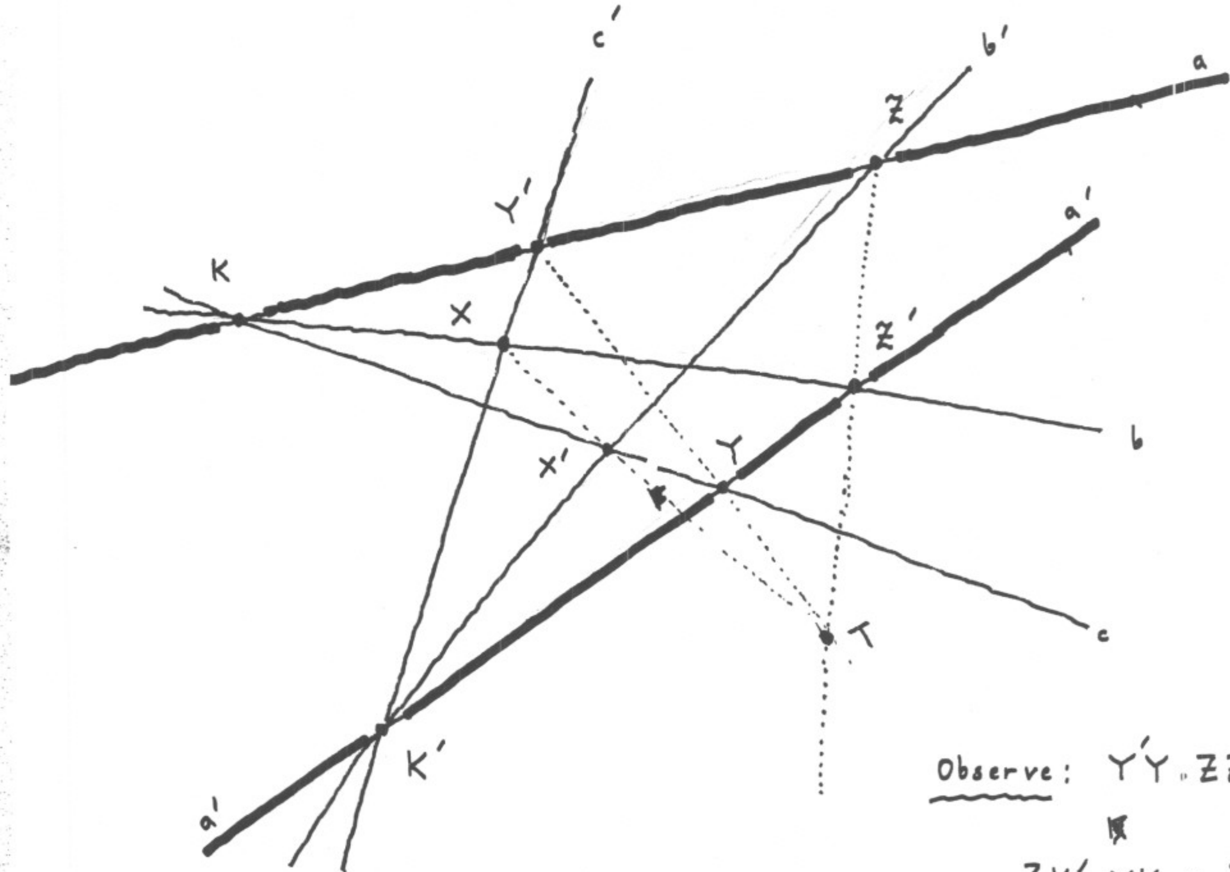
Let us do it <u>first verbally</u>: "Theorem" in p. 23 reads in the dual:

"Given distinct points K, K' and distinct lines a, b, c, concurring in K, a', b', c' concurring in K', if the points bc' and $b'c$, ca' and $c'a$, ab' and $a'b$ determine lines x, y, z then x, y, z are concurrent."

<u>Theorem</u>: Given distinct points K, K' and distinct lines a, b, c through K, a', b', c' through K' such that b and c', b' and c, c and a', c' and a, a and b', a' and b intersect in X, X', Y, Y', Z, Z', the lines XX, YY', ZZ' are concurrent or parallel.

<u>Proof</u>: Apply Pappus:

$$\begin{array}{llll} a \text{ —} & K & Y' & Z \\ a' \text{ —} & K' & Z' & Y \end{array}$$



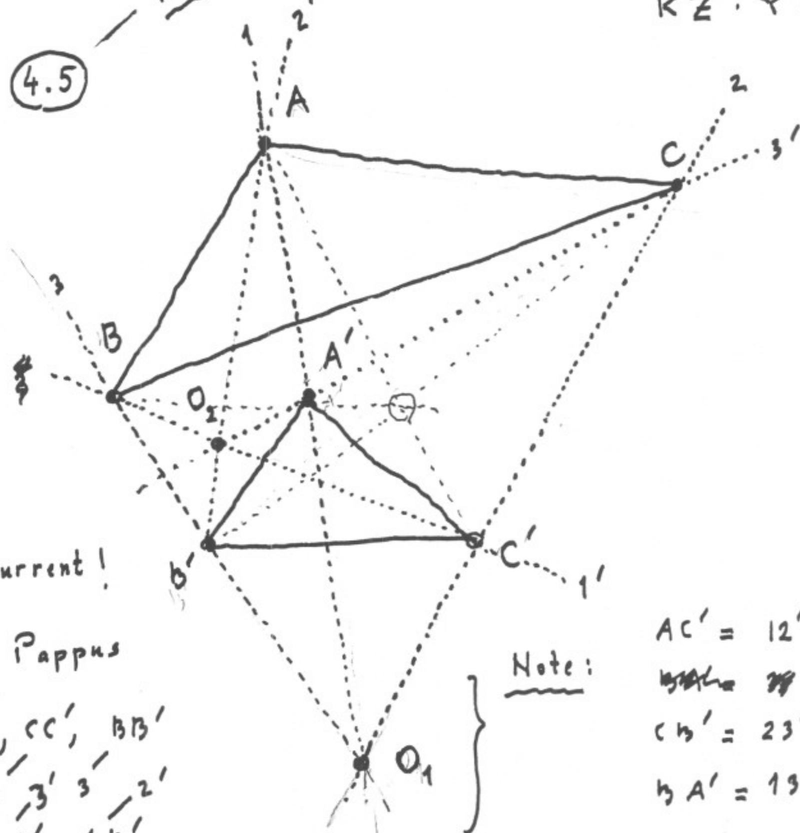
Observe: $Y'Y \cdot ZZ' = T$

$ZK' \cdot YK = X'$

$KZ' \cdot Y'K' = X$

Pr 11.

Application (4.5)



AA', BB', CC' in O_1

AB', BC', CA' in O_2

$\Downarrow$

AC', BA', CB' concurrent!

Solution: Dual of Pappus

O_1 : AA', CC', BB'

1 1' 2 3' 3 2'

O_2 : BC', CA', AB'

Note:

$AC' = 12' \cdot 1'2$

$CB' = 23' \cdot 2'3$

$BA' = 13' \cdot 1'3$

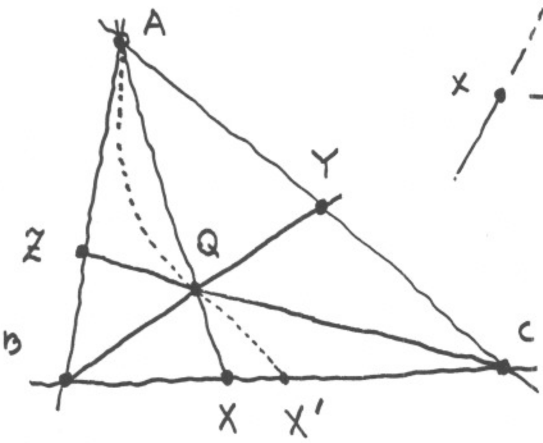
Theorem : ("Theorem of Ceva")

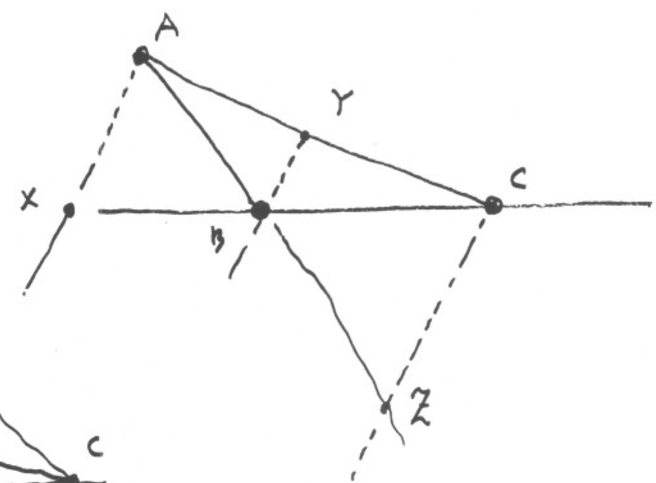
Given a triangle ABC and points $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$ in order for AX, BY, CZ to be concurrent or parallel, it is necessary and sufficient that

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \underline{-1} \qquad (*)$$

!!!!!

Proof :





If AX, BY, CZ concur in Q then

"Menelaus" ABX, CZ : $\frac{CB}{CX} \cdot \frac{QX}{QA} \cdot \frac{ZA}{ZB} = 1$

"Menelaus" AXC, BY : $\frac{BX}{BC} \cdot \frac{YC}{YA} \cdot \frac{QA}{QX} = 1$

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = -1 \longrightarrow (*) \text{ holds !}$$

If AX, BY, CZ are parallel by "Thales" :

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \frac{AY}{AC} \cdot \frac{YC}{YA} \cdot \frac{CA}{CY} = -1.$$

Conversely assume that (*) holds...

If AX, BY, CZ are parallel, there is nothing to prove. If not, assume w.l.o.g that BY, CZ intersect in Q. In this case AQ intersects BC in some $X' \in BC - \{B, C\}$. (<u>If not</u>, $AQ \parallel BC$ which gives (coupling "Thales" and (*)!) $\frac{XB}{XC} = 1$ which is impossible!) Thus by the first part

$$\frac{X'B}{X'C} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = -1$$

which gives upon comparison with (*)

$$\frac{X'B}{X'C} = \frac{XB}{XC} \qquad \text{hence} \qquad X = X'.$$

Therefore $AX = AX', BY, CZ$ concur in Q!

<u>Application</u>: (1) Revisit medians
(2) " angle bisectors (internal + external...)

(3) Nagel, Gergonne

& <u>Application</u> (4)

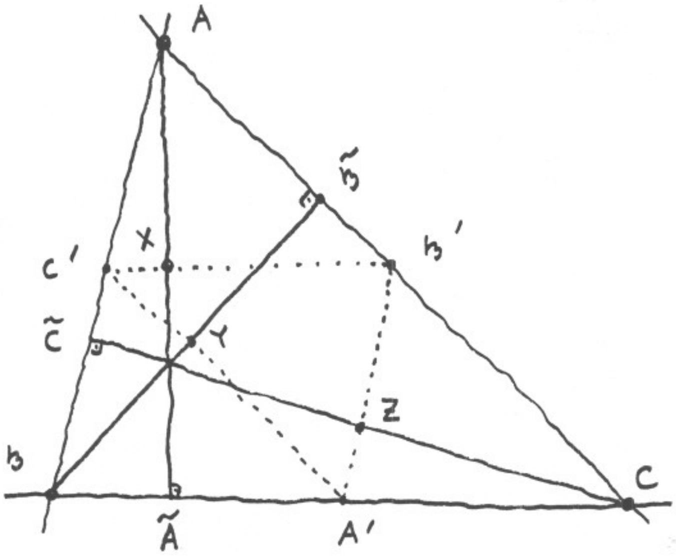
<u>Problem</u>: (H. Demir, "Proposal 1197" Mathematics Magazin 57 (1984) 238

The original text

1197. Characterize the triangles of which the midpoints of the altitudes are collinear. [*Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*]

Solution : (Cem Tezer)

In a triangle the midpoints of altitudes are collinear iff the triangle is a right triangle ...



We know

$$\frac{XB'}{XC'}\cdot\frac{YC'}{YA'}\cdot\frac{ZA'}{ZB'} = \frac{\tilde{A}B}{\tilde{A}C}\cdot\frac{\tilde{B}C}{\tilde{B}A}\cdot\frac{\tilde{C}A}{\tilde{C}B} = -1$$

If X, Y, Z are also collinear, and then $\begin{cases} X \neq B', C' \\ Y \neq C', A' \\ Z \neq A', B' \end{cases}$

$$\frac{XB'}{XC'}\cdot\frac{YC'}{YA'}\cdot\frac{ZA'}{ZB'} = +1$$

Hence $X \in \{B', C'\}$ or $Y \in \{C', A'\}$ or $Z \in \{A', B'\}$.

| w.l.o.g. $X = C'$

$AX = A\tilde{A} = AB$

$\Rightarrow AB \perp AC$