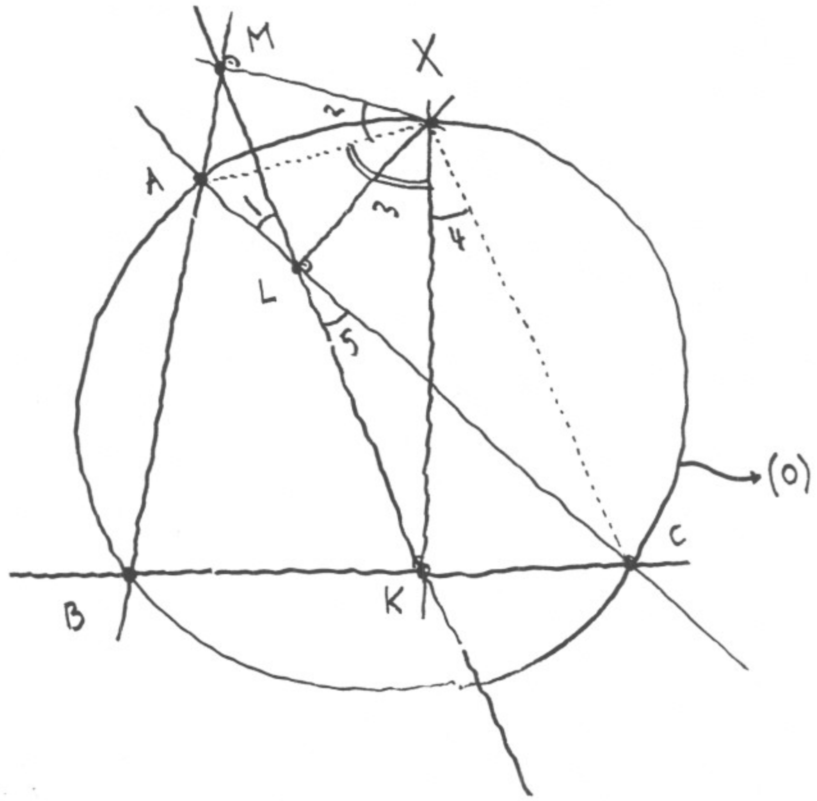
Lecture 4

Simson Lines



Theorem: The feet of perpendiculars from X on BC, CA, AB are collinear iff $X \in (O)$.

Proof: Let $X \in (O)$. Since A, L, M, X and L, K, C, X are concyclic we have

$$1 = 2 \quad \text{and} \quad 5 = 4 .$$

Since M, X, K, B and, A, X, C, B are concyclic we have

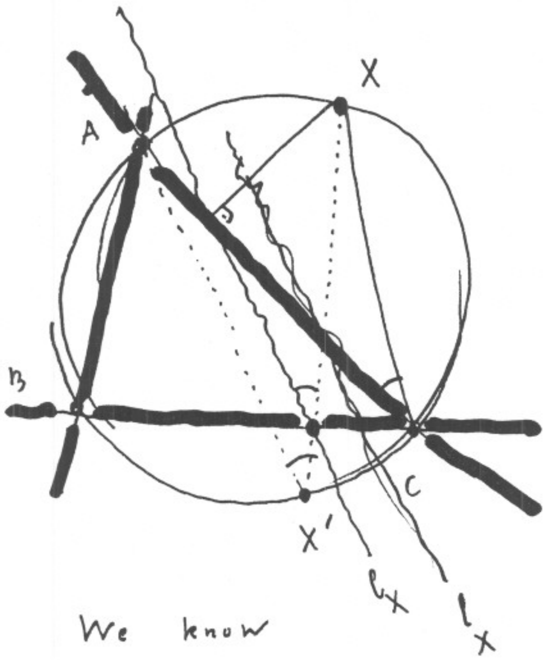
$$2 + 3 = \pi - B = 3 + 4$$

consequently $2 = 4$ hence $1 = 5$. ~~Conv~~

$\therefore$ K, L, M collinear. Converse ?

~~The Flashback~~:

$\widehat{XY}$ (always in connection with a definite circle!) is the angle subtended by the arc at any point on the circumference (mod π).



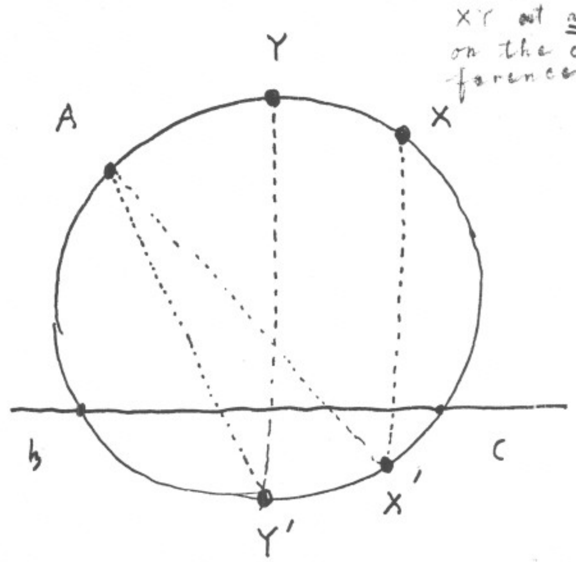
We know

$\ell_X \parallel AX'$

Application 1

Given $X, Y \in (O)$

$\measuredangle(\ell_X, \ell_Y) = -\widehat{XY}$ (angle subtended by the arc XY at any point on the circumference)



$\ell_X \parallel AX'$, $\ell_Y \parallel AY'$

Consequently

$\measuredangle(\ell_X, \ell_Y) = \measuredangle(AX', AY') = \widehat{X'Y'}$
$= -\widehat{XY}$

~~Application 2 (Steiner)~~

~~This is generalisation of an earlier theorem to the effect that the reflection of H in any side lies ..~~

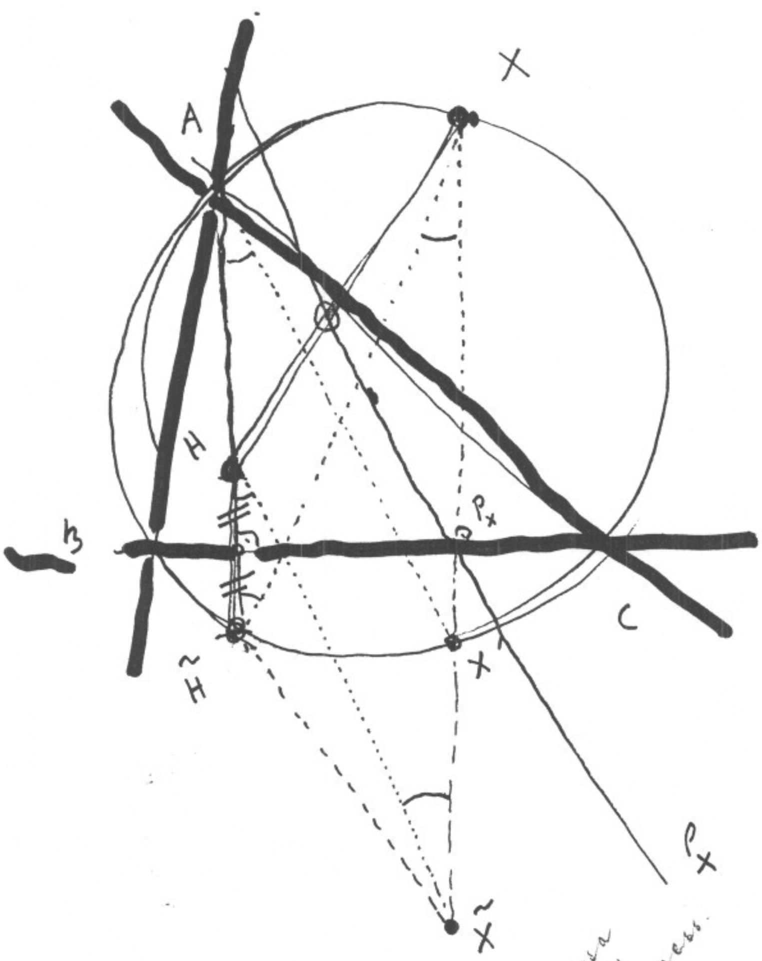
Interesting implication:

$\measuredangle$ If $[A, B]$ is a diameter of (O), then $\ell_A \perp \ell_B$!

E. H. Lockwood: "A Book of Curves"
QA 483 L62.

Application 2 (**Steiner**) (This is a beautiful generalisation of the simple theorem to the effect that the reflections of H in BC, CA, AB lie on (O) !).

Theorem: (Steiner) For each $X \in (O)$, ℓ_X bisects $[H,X]$!



Draw parallel to ℓ_X through H. It meets XX' in $\tilde{X}$.

As BC is the p.b. of $[H\tilde{H}]$ and

$$\measuredangle(AH, AX') = \widehat{\tilde{H}X'}$$

$$= -XA = \measuredangle(\tilde{H}X, \tilde{H}A)$$

we conclude that ~~$\tilde{X}$~~ BC is the p.b. of $[X\tilde{X}]$. ✓.

Let $\tilde{X}$ be the reflection of X in BC. BC is clearly the common bisector of $[H\tilde{H}]$ and $[X\tilde{X}]$. Since

$$\measuredangle \tilde{H}H\tilde{X} = \measuredangle X\tilde{H}A = \measuredangle \tilde{H}AX'$$

we conclude that $H\tilde{X} \parallel AX' \parallel \ell_X$

$\therefore$ ℓ_X bisects $[X,H]$.

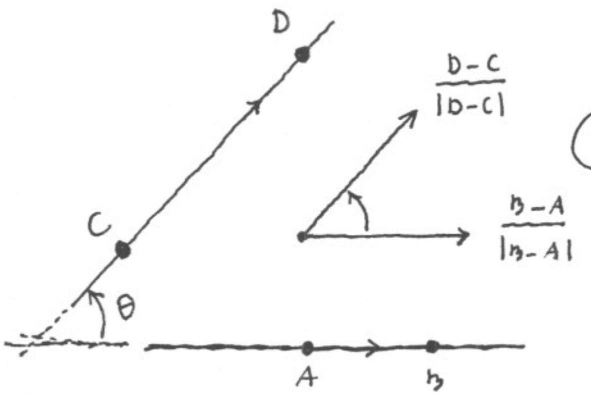
Important Consequence:

Two simson lines which are $\perp$ to one another intersect on the 9-point circle..

Remarks on angles between lines

~~§4. Angles, Applications to Circles :~~

We employ complex numbers to investigate the following issues which are rather misunderstood.



Angle between directed lines : $\overrightarrow{AB}$ and $\overrightarrow{CD}$

$$\measuredangle(\overrightarrow{AB}, \overrightarrow{CD}) = \theta \pmod{2\pi}$$

if

$$\frac{D-C}{|D-C|} = e^{i\theta} \frac{B-A}{|B-A|}$$

Angle between (undirected) lines AB and CD (order!)

! !

$$\measuredangle(AB, CD) = \measuredangle(\overrightarrow{AB}, \overrightarrow{CD}) \pmod{\pi}$$

! !

Properties : For any lines a, b, c

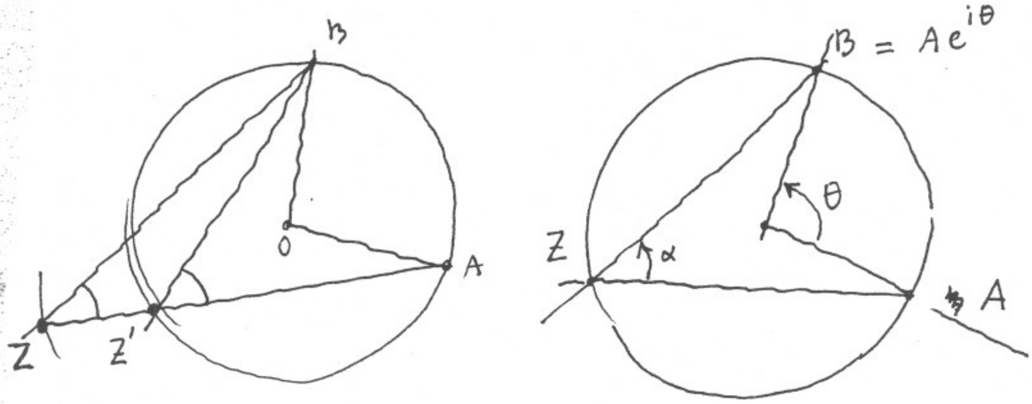
1) $\measuredangle(a,b) = 0$ iff $a \parallel b$ (including $a = b$!)

2) $\measuredangle(a,b) = -\measuredangle(b,a)$

3) $\measuredangle(a,b) + \measuredangle(b,c) = \measuredangle(a,c)$.

Theorem : Let ω be a circle of centre O, let $A, B \in \omega$:

"The angle subtended by the arc $\overset{\frown}{AB}$ at any point $Z \in \omega$ is half the angle subtended by the same arc at O"



Assume without loss of generality, that ω is the circle $z\bar{z} = 1$

$$e^{2i\alpha} = \frac{\left(\frac{A-Z}{|A-Z|}\right)^2}{\left(\frac{B-Z}{|B-Z|}\right)^2} = \frac{\frac{A-Z}{\bar{A}-\bar{Z}}}{\frac{B-Z}{\bar{B}-\bar{Z}}} = \frac{\frac{A-Z}{1/A - 1/Z}}{\frac{B-Z}{1/B - 1/Z}} = \frac{B}{A} = e^{i\theta}$$

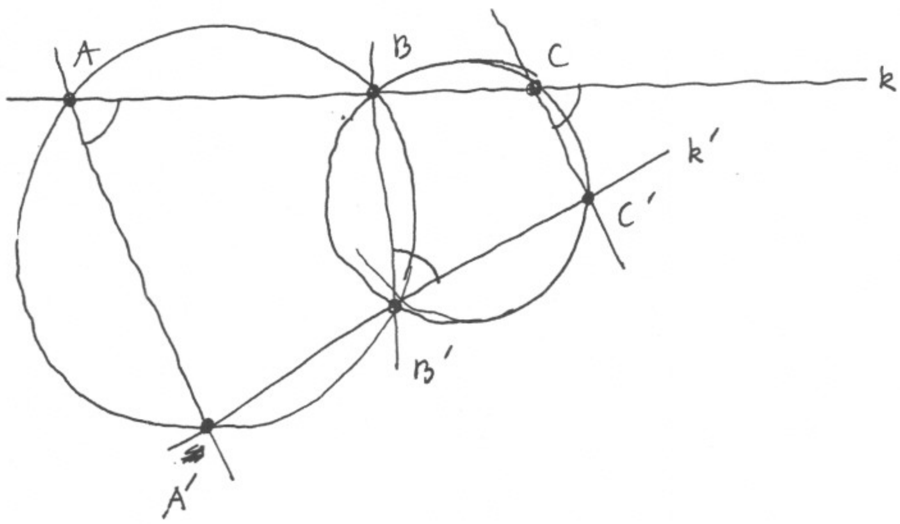
Conversely : Let $ZA \cap \omega = \{A, Z'\}$ (what if $Z' = A$)

$$\measuredangle(ZA, ZB) = \measuredangle(Z'A, Z'B) \longrightarrow Z = Z'.$$

Corollary: Given X, Y, A, B, with $\{X, Y\} \cap \{A, B\} = \emptyset$, X, Y, A, B are (collinear or concyclic) iff

$$\boxed{\measuredangle(XA, XB) = \measuredangle(YA, YB)}$$

Application (1) A classical problem :



Let A, B, A', B' be concyclic :

Prove that

B, C, B', C' are concyclic iff $AA' \parallel CC'$

Solution :

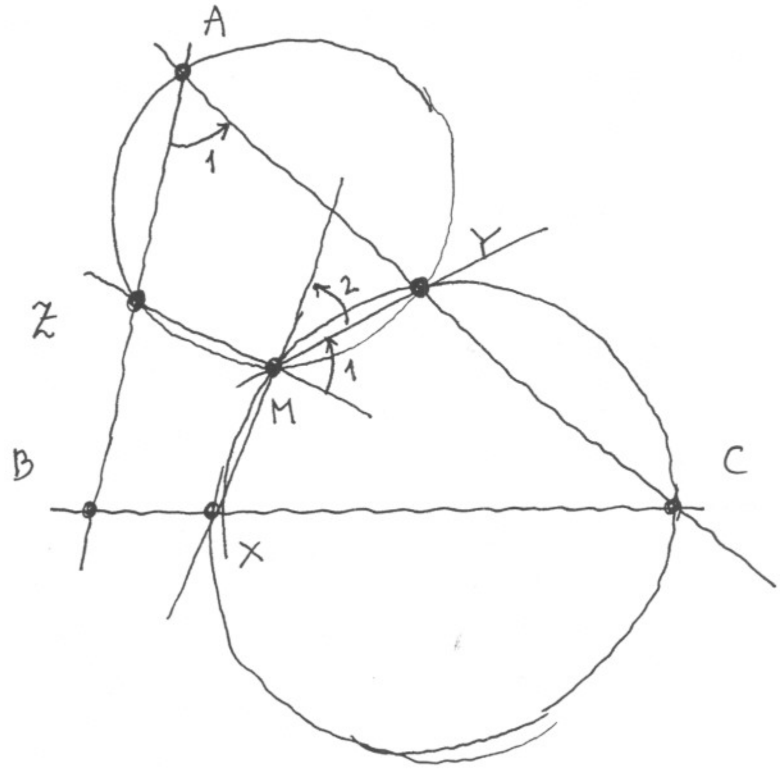
$$\measuredangle(AA', k) = \measuredangle(k', BB') = \measuredangle(CC', k)$$

Consequently,

$$\measuredangle(AA', CC') = \measuredangle(AA', k) + \measuredangle(k, CC') = 0 .$$ ✓

Application (2) The "Miguel point" of a triangle inscribed in a triangle:

Theorem: Given $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$ the circumcircles of AYZ, XBZ, XYC have a common point.

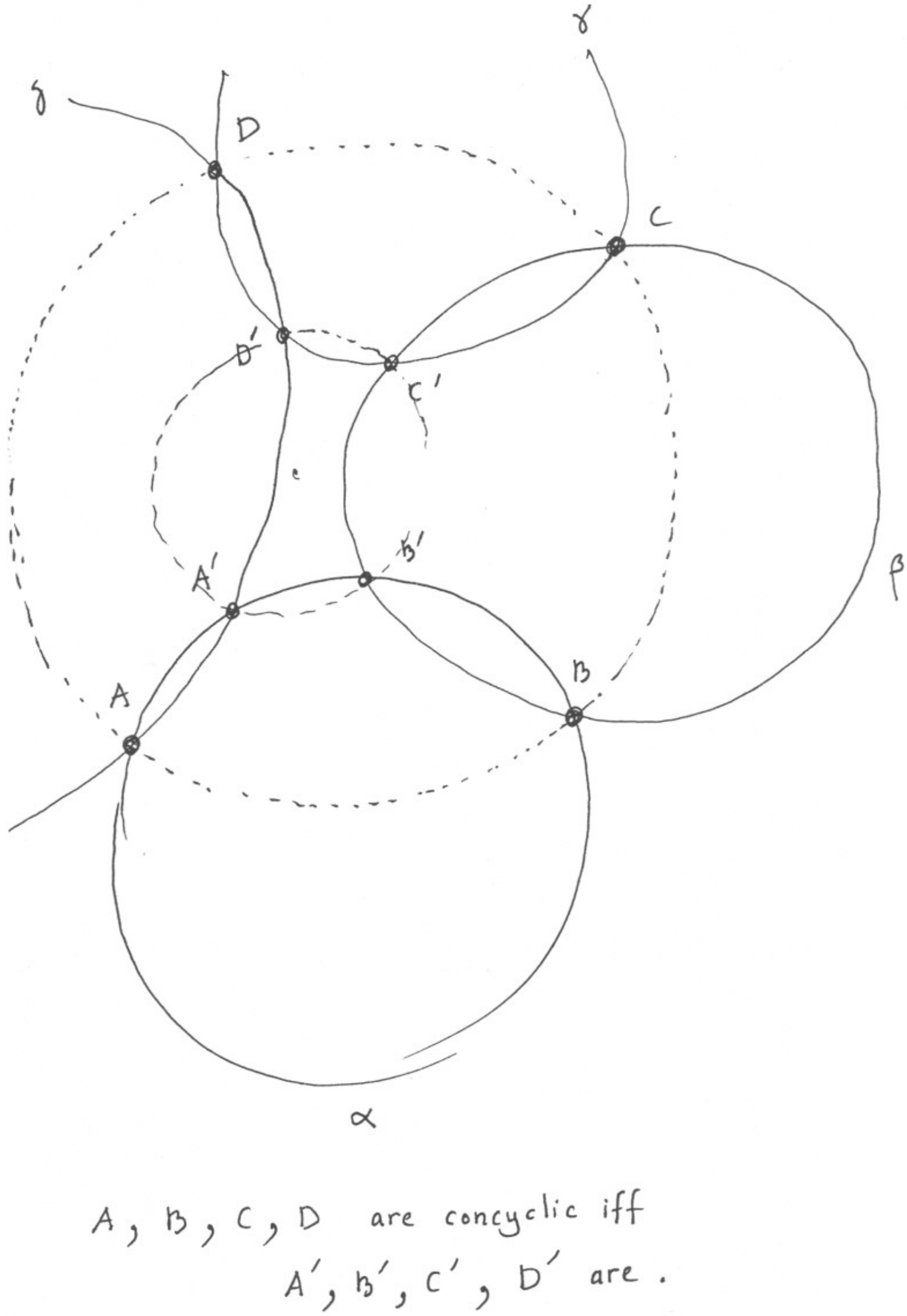


Let the circumcircles of AYZ and XBZ intersect in Y and M (what if $Y = M$?)

$$\begin{aligned}
\measuredangle(BZ, BX) &= -\measuredangle(BC, AB) = -\left(\measuredangle(BC, AC) + \measuredangle(AC, AB)\right) \\
&= \measuredangle(AC, BC) + \measuredangle(AB, AC) \\
&= \underbrace{\measuredangle(MY, MX)}_{1} + \underbrace{\measuredangle(MZ, MY)}_{2} = \measuredangle(MZ, MX)
\end{aligned}$$

Hence: B, M, Z, X are concyclic.

Application (3) A classical beauty:



A, B, C, D are concyclic iff
A', B', C', D' are.