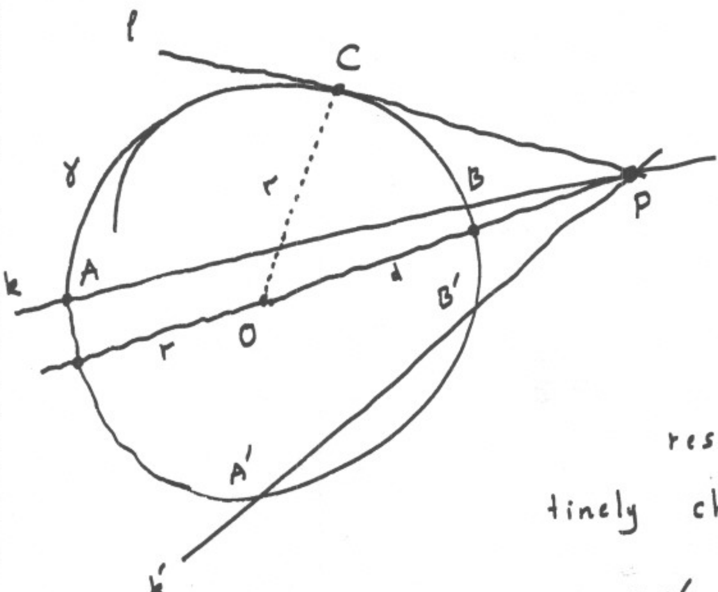
Lecture 5

A Closer Encounter with the Circle

§ 5. A Closer Encounter with the Circle

The purpose of this section is to introduce the concept of the power of a point with respect to a circle and present some important applications of it.



Let γ be a circle of center O, radius r. Consider a point P. Let lines k, k' through P intersect the circle γ in A and B, A' and B' respectively. It can be routinely checked that

$$\underbrace{PA \cdot PB}_{\text{a signed quantity}} = PA' \cdot PB'.$$

a signed quantity: To be understood to be positive if P is not between A and B!

Thus it is seen that the number $p = PA \cdot PB$ is independent of the choice of the line k through P, that p depends only on P and γ!

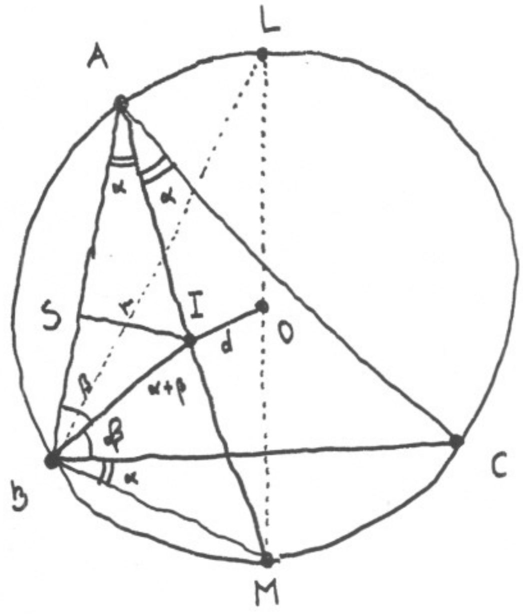
The number p is called the power of P w.r.to γ

It can be easily checked that it equals $d^2 - r^2$ where $d = |OP|$. If P is outside γ ~~then~~ and a tangent ℓ to γ through P touches γ at C then the power of P with respect to γ equals $|PC|^2$.

It should be noted that the power of P with respect to γ is $\begin{Bmatrix}\text{positive}\\ \text{negative}\\ \text{zero}\end{Bmatrix}$ iff P lies $\begin{Bmatrix}\text{outside}\\ \text{inside}\\ \text{on}\end{Bmatrix}$ γ.

Application (1): "The Euler Formula"

$$\boxed{|OI|^2 = R^2 - 2Rr.}$$



Start by noticing that MIB is isosceles. Thus

$|OI|^2 - R^2 =$ the power of I with respect to (O)

$= -|IM|\cdot|IA|$

$= -|MB|\cdot|IA|$

$= -|IS|\cdot|LM|$

$= -2Rr.$

$AIS \simeq LMB$

Remark 1: Similarly $|OI_a|^2 = R^2 + 2Rr_a$

Euler, 1707 ← Remark 2: In a "bicentric" quadrangle

Nova Acta Petropol 1795.

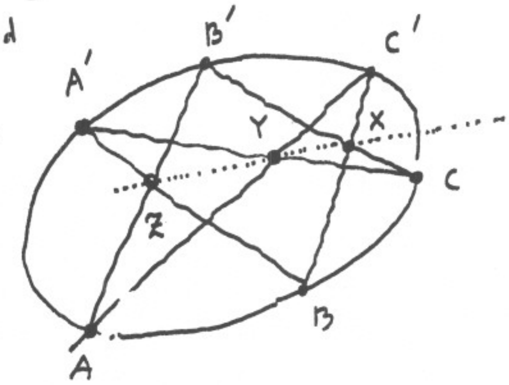
$$2r^2(R^2+d^2) = (R^2-d^2)^2 \quad \text{where } d = |OI|$$

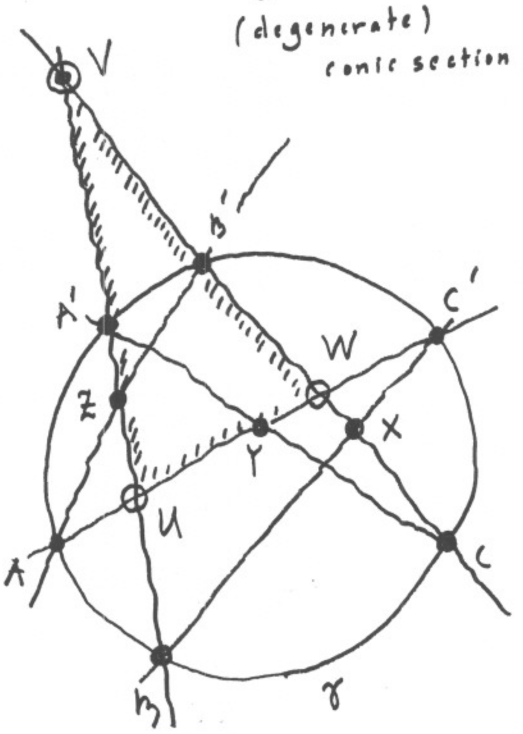
Nicholas Fuß (1755–1826) → Dörrich "Solved Famous Problems"

Application (2) "Pascal's Theorem"

Theorem: If A, B, C, A', B', C' are distinct points lying on a conic section, such that BC' and $B'C$, CA' and $C'A$, AB' and $A'B$ intersect in X, Y, Z, then the points X, Y, Z are collinear.

Remark 1 The striking similarity with the theorem of Pappus should be immediately apparent. In fact this is a generalisation thereof since a pair of straight lines may be regarded as a (degenerate) conic section





Proof: It is enough to give the proof in the special case of a circle. "Menelaus" in UVW:

$$BC': \quad \frac{XV}{XW} \cdot \frac{C'W}{C'U} \cdot \frac{BU}{BV} = +1$$

$$CA': \quad \frac{YW}{YU} \cdot \frac{A'U}{A'V} \cdot \frac{CV}{CW} = +1$$

$$AB': \quad \frac{ZU}{ZV} \cdot \frac{B'V}{B'W} \cdot \frac{AW}{AU} = +1$$

From these equations we read off $\frac{XV}{XW}\cdot\frac{YW}{YU}\cdot\frac{ZU}{ZV} = +1$

in view of the relations

$$UA'\cdot UB = UC'\cdot UA \qquad (= \text{power of } U)$$
$$VB'\cdot VC = VA'\cdot VB \qquad (= \text{power of } V)$$
$$WC'\cdot WA = WB'\cdot WC \qquad (= \text{power of } W)$$

Given two non-concentric circles, it can be routinely proven that the set of points which have equal powers with respect to the circles in question constitute a line perpendicular to the line joining the centers of the circles.

This line is called the radical axis of the circles in question.

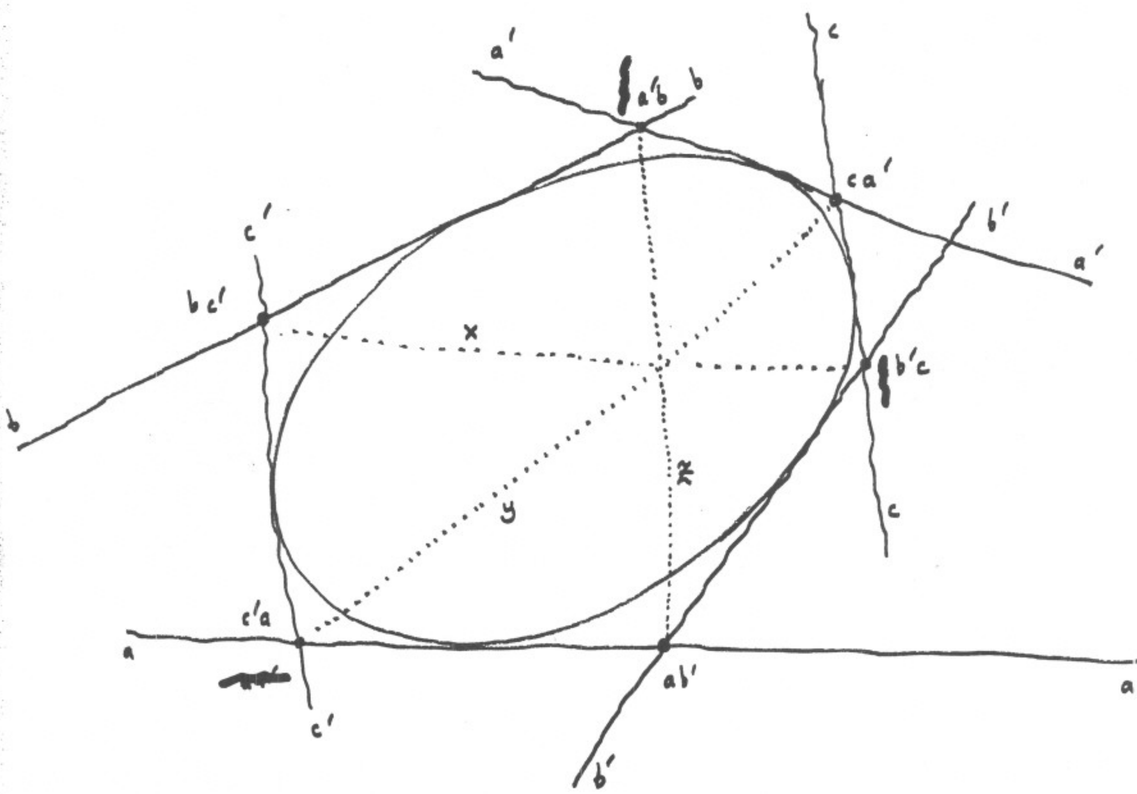
Remarks: On the construction of the radical axis.

Let α, β, γ be three circles with non-collinear centres. (Thus no two may be concentric!). Let k, ℓ, m be the radical axes of β and γ, γ and α, α and β. ℓ and m intersect (why?) in some P. Observe that P must lie on k, too. Thus P has the same power w.r.t. α, β, γ.

By the above construction, For any triple of circles with non-collinear ~~points~~ centers there exists a unique point which has the same power with respect to all three. This point is called the radical center of the circles in question.

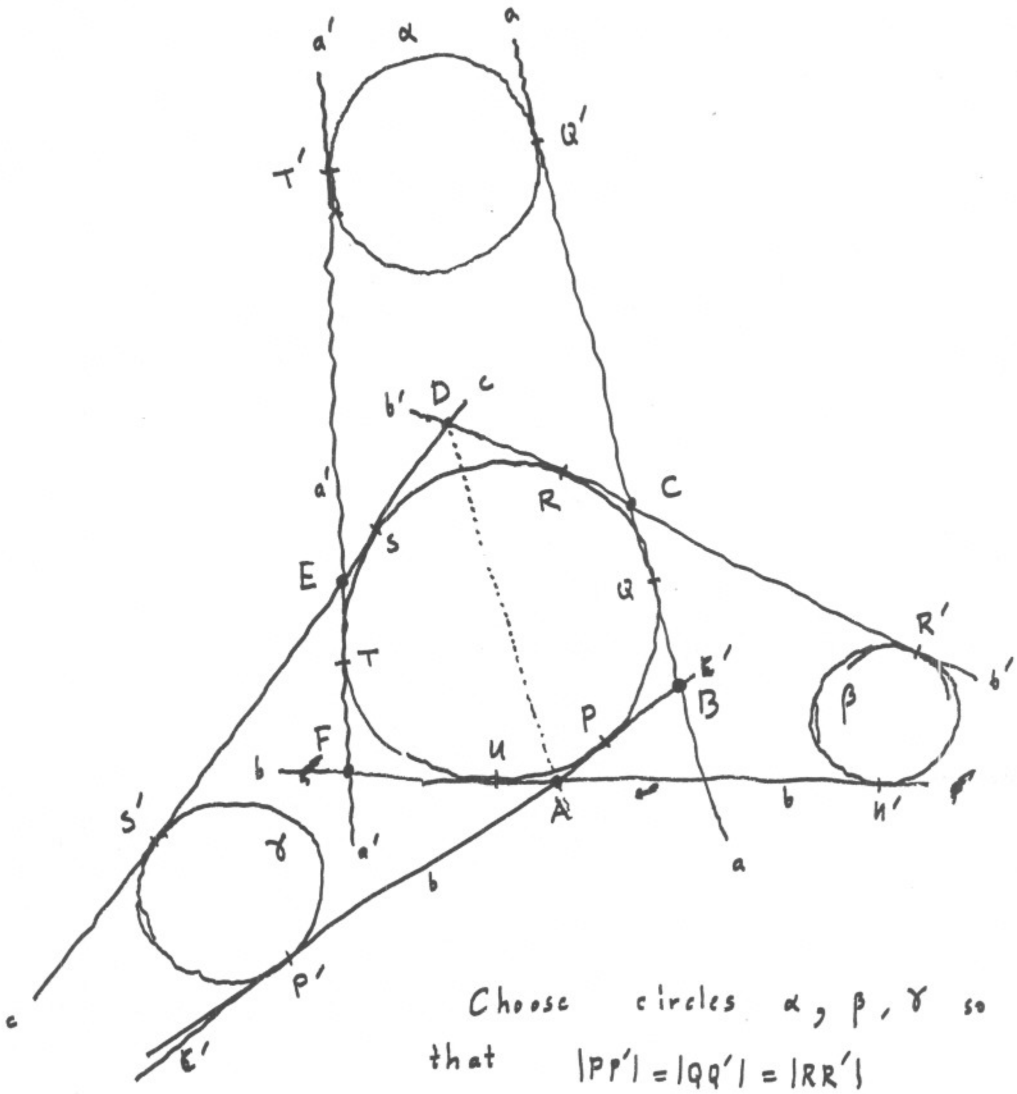
Application (3) "Brianchon's Theorem"

This is the dual of the Pascal theorem:



Coxeter-Greitzer

Proof: Again sufficient to establish for a circle...



Choose circles α, β, γ so that $|PP'| = |QQ'| = |RR'|$
$= |SS'| = |TT'| = |UU'|$

Conclude : $\left.\begin{array}{l} |AP'| = |AU'| \\ |DS'| = |DR'| \end{array}\right\}$ hence AD is the r.a of β and γ

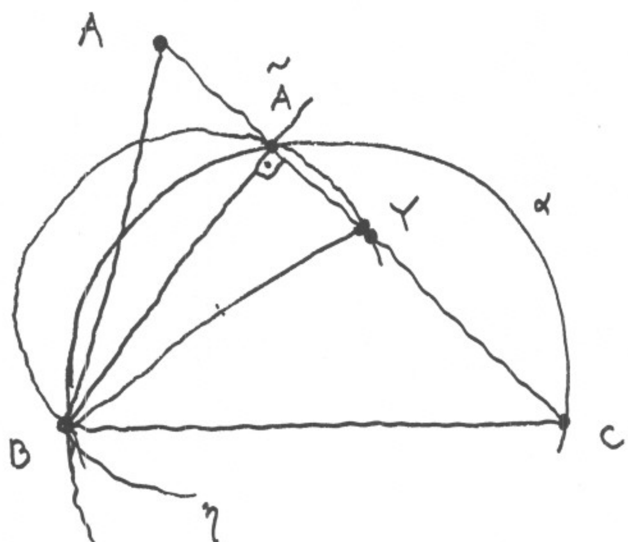
Similarly BE is the r.a. of γ and α
CF " " " " α " β.

Consequently AD, BE, CF concur in the r.c. of α, β, γ !

A set $\mathfrak{A}$ of circles is said to be <u>coaxal</u> if there exists a line k such that ~~for any distinct $\alpha, \beta \in \mathfrak{A}$~~ k is the radical axis of any distinct $\alpha, \beta \in \mathfrak{A}$.

The following lemmata and the theorem are related to this kind of phenomenon.

<u>Lemma 1</u>: In a triangle ABC, let α be the circle of diameter $[B, C]$. For any $Y \in CA - \{C\}$ and any circle η of diameter $[B, Y]$ the radical axis of α and η is the altitude through B.



<u>Proof</u>: Obvious from the diagram...

<u>Lemma 2</u>: In a triangle ABC, let $Y \in CA$, $Z \in AB$ such that $(Y, Z) \neq (C, B)$. Let η, ζ be the circles of diameter $[B, Y]$, $[C, Z]$ respectively. The radical axis of η and ζ passes through H.

by Lemma 1 / circle of diameter [B,C]

Proof: H is on the r.a. of α and η and on the r.a. of α and ζ. Therefore H has the same power w.r.to η and ζ.

Application (4) "The Gauss-Bodenmiller Theorem"

Theorem: The orthocentres of the four triangles constituted by the sides of a complete quadrilateral lie on a line perpendicular to the Newton Line of the quadrilateral.

Proof: Let α, β, γ be the circles of diameters [A,C], [B,D], [E,F]. Lemma 2 applies to α and β within the triangles ABF, AED, EBC, CFD. thereof the orthocentres have to lie on the radical axis of α and β. Similarly for β and γ, γ and α.

This shows that α, β, γ are coaxal. Their radical axis contains the four orthocentres in question. Clearly the line containing the centres of α, β, γ is the Newton Line.