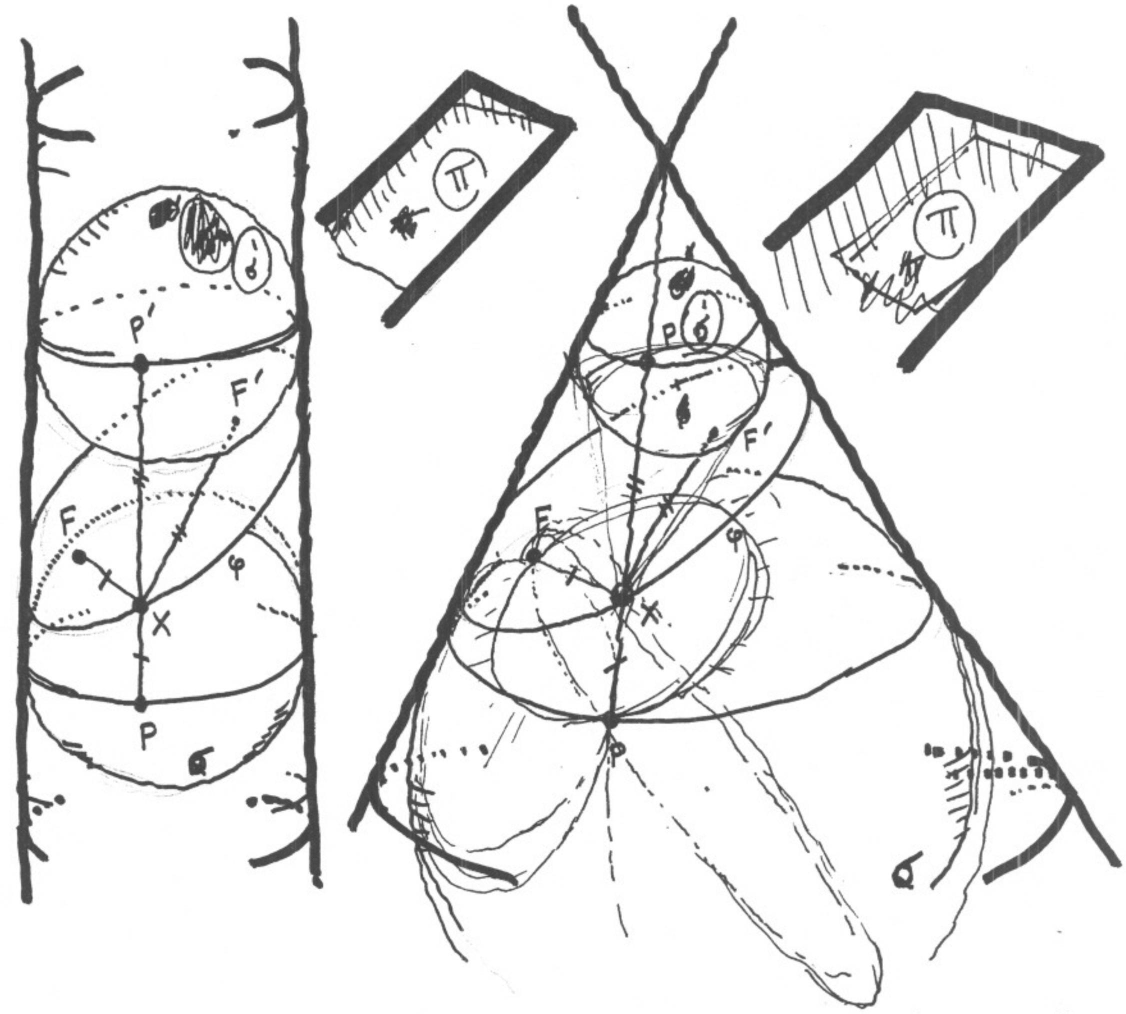
Lecture 6

Conic Sections

§6. Conic Sections

Intersection of the cone with a plane π



Characterisation I : Describe the inscribed spheres σ, σ' which touch the transversal plane in F, F'. Note that for any point X on the curve φ

$$|XF| + |XF'| = |XP| + |XP'| = |PP'|$$

independent of the position of X on φ.

This is the "elliptic" case!

This argument can be easily adapted to the case of ~~the~~ a transversal plane which intersects the cone on both sides of the cone; <u>the "hyperbolic case"</u>: There is one significant change: $|XF| + |XF'|$ has to be replaced with $|XF| - |XF'|$.

This set-up gives rise to what I like to call the <u>first characterisation</u> of conic sections:

convention!

$$\text{A}\begin{cases}\text{n } \underline{\text{ellipse}} \\ \underline{\text{hyperbola}}\end{cases} \text{ of foci } F, F' \text{ and major diameter } 2a$$

plural of "focus"

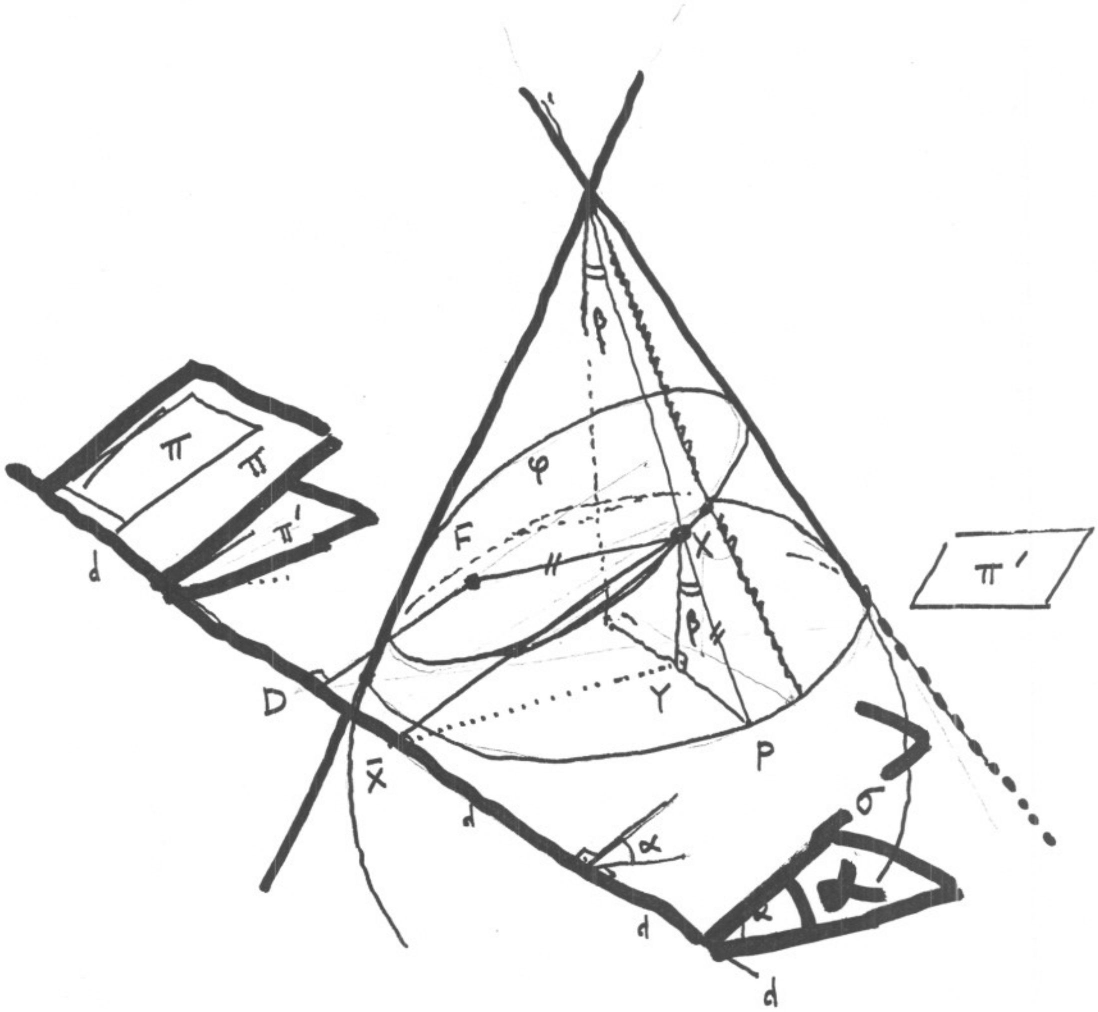
is the set

$$\left\{ X \in \mathbb{R}^2 \;\middle|\; \begin{cases} |XF| + |XF'| = 2a \\ \big| |XF| - |XF'| \big| = 2a \end{cases} \right\}.$$

<u>Remark 1</u> The hyperbola will be seen to consist of two "branches". On one $|XF| - |XF'| = 2a$ will hold whereas on the other $|XF'| - |XF| = 2a$.

<u>Remark 2</u> The special case of a circle arises when the foci coincide.

<u>Remark 3</u> The "parabola" is missed in this characterisation.



<u>Characterisation</u> <u>II</u> : Leave out the sphere σ'.

Let π' be the plane containing the circle along which the sphere σ touches the cone. Let α be the angle between π' and π, β be the angle between the axis and any of the generators of the cone. <u>Suppose</u> π and π' intersect in the line d. Note:

$$\frac{|XF|}{|X\bar{X}|} = \frac{|XP|}{|X\bar{X}|} = \frac{|XY|/\cos\beta}{|XY|/\sin\alpha} = \frac{\sin\alpha}{\cos\beta}$$ = constant independent of the position of X.

which is independent of the position of X.

Thus, the <u>second characterisation</u> :

A <u>conic section</u> with focus F and <u>directrix</u> d $F \notin d$

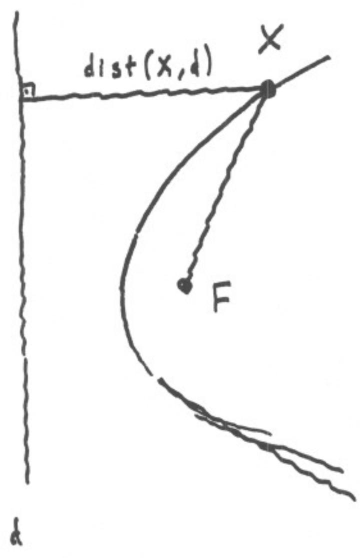
and <u>eccentricity</u> ε is the set

$$\left\{ X \in \mathbb{R}^2 \;\middle|\; \frac{|XF|}{\operatorname{dist}(X,d)} = \varepsilon \right\}$$

Ellipse $\varepsilon < 1$

parabola $\varepsilon = 1$

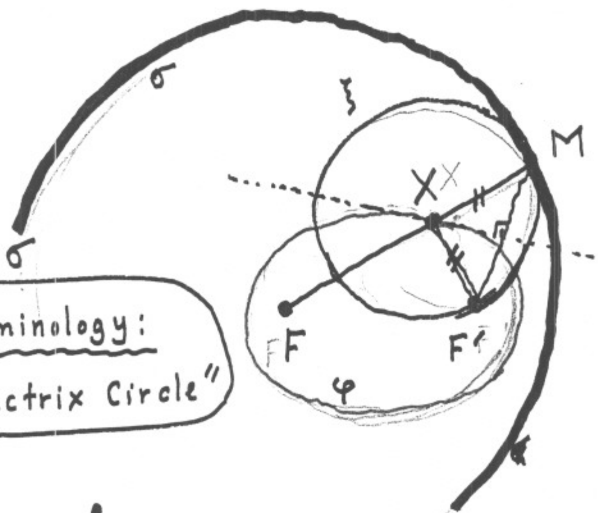
hyperbola $\varepsilon > 1$.



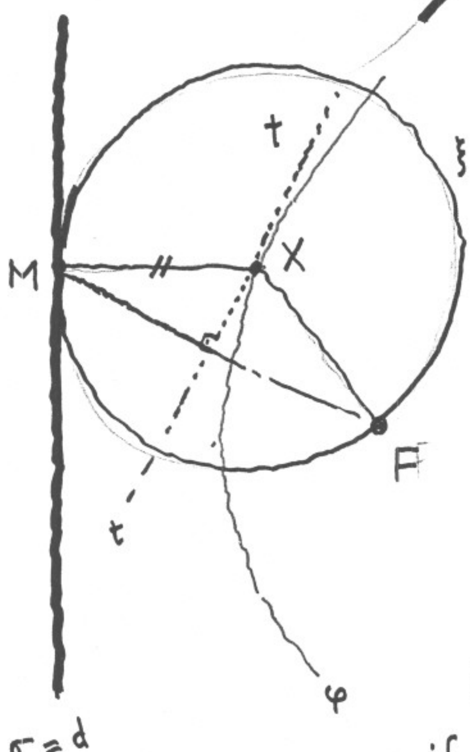
Remark : The second characterisation misses the circle! (The circle arises when $\pi \parallel \pi'$ in which case d cannot be defined.

Characterisation III

This is in essence only characterisation II in disguise...



Terminology: "Directrix Circle"



Given a circle σ with center F and radius $2a$ and F' lying inside σ, the centers of circles (ξ) through F' and tangent to σ describe an ellipse.

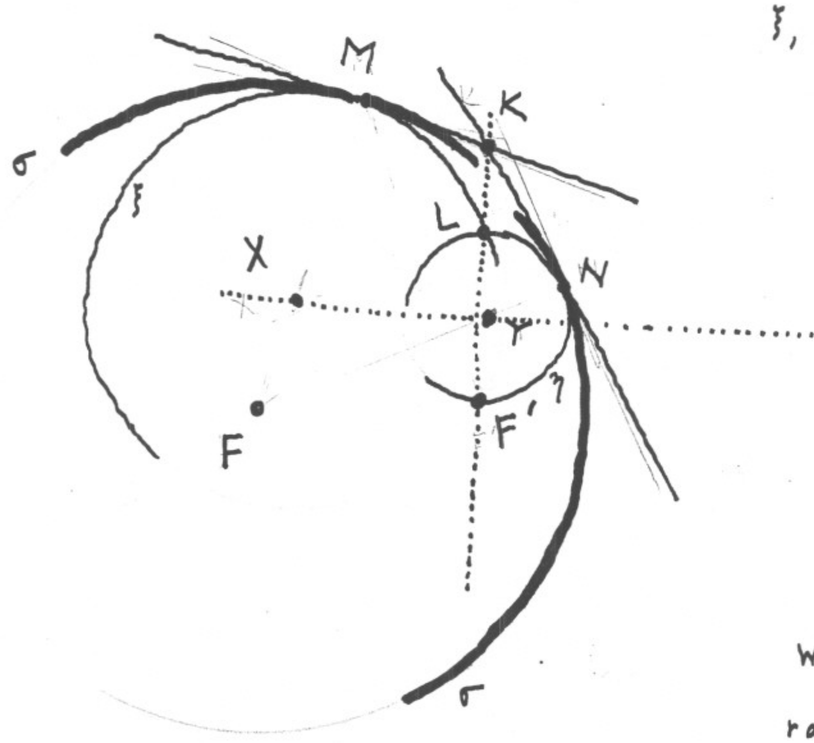
Circles ξ – which are at first sight ~~the~~ a little elusive – can be easily constructed starting from the point of tangency M: ~~the~~ The perpendicular bisector t of $[MF']$ meets MF in X, the center!

(One branch! of) a hyperbola arises if F' lies outside σ.

A parabola is obtained simply by assigning the role of σ to the directrix d.

The line t has a deep significance: It is the tangent to the curve φ at the point X.

Too see this: Consider $X, Y \in \varphi$. Let circles ξ, η of respective centers X, Y through F' touch σ at M, N.



Let $\xi \cap \eta = \{F', L\}$

If the tangents to σ at M, N meet in K which is clearly the radical center of σ, ξ, η then

LF' passes through K. Consequently $XY \perp KF' = LF'$.
(Equivalently, keeping M fixed move N towards M!)

Keeping X fixed, move Y on φ towards X.

As $Y \longrightarrow X$, M, K, N, L merge into one point and

(All the while $XY \perp KF'$ retains its validity!)

$XY \rightsquigarrow t =$ the perpendicular bisector of $[F', M]$.

Example (1)

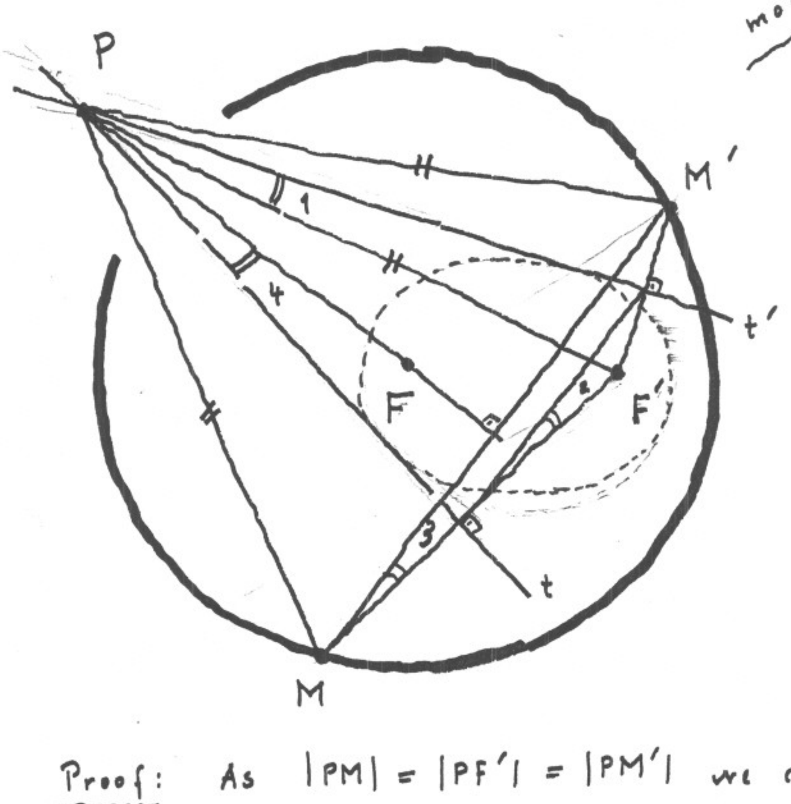
Alone on the basis of the above observation we can give a beautiful solution of Pr 4 "Selected Problems VI"

Theorem : (The "First Theorem of Poncelet")

In a conic section with foci F, F', if tangents t, t' intersect in P, then

$$\measuredangle (t, PF) = \measuredangle (PF', t')$$

mod π



Proof: As $|PM| = |PF'| = |PM'|$ we conclude $PF \perp MM'$.

Hence $\measuredangle (t, PF) = 4 = 3 = 2 = 1 = \measuredangle (PF', t')$.

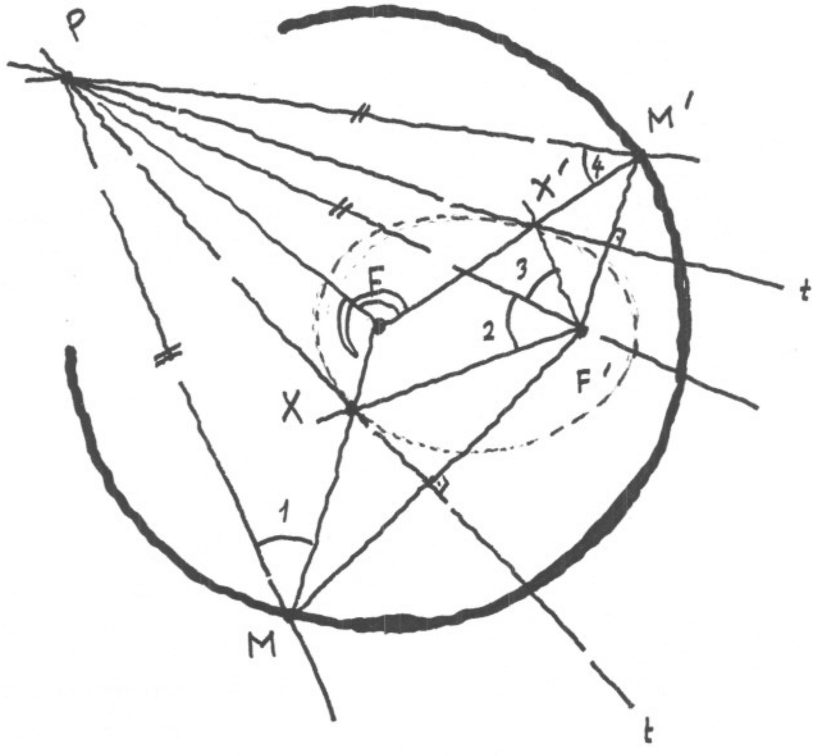
Example (2) Pr 6 "Selected Problems VI"

Theorem: (The "Second Theorem of Poncelet")

Given points X, X' on a conic section of foci F', F if the tangents at X, X' intersect in P, then

$$\measuredangle(XF', PF') = \measuredangle(PF', X'F')$$

(and of course
$$\measuredangle(XF, PF) = \measuredangle(PF, FX').$$



Proof: Again note that PF is the perpendicular bisector of $[M, M']$.

Example (3) Pr. 7 "Selected Problems VI"