Lecture 7

Isometries and Homotheties

§7. Geometric Transformations I

(A) Isometries :

Def : An isometry is a bijection of the plane which preserves distances.

The following three instances [redacted] should be familiar to everybody :

Example 1 : Translation by a vector $\vec{u}$

$$\boxed{Tr_{\vec{u}}(x)} = x + \vec{u} \quad \text{for each } x \in \mathbb{R}^2.$$

Example 2 : Rotation about P, through angle θ:

$$\boxed{Rot(P,\theta)(X)} = P + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (X - P)$$

Important special case : $Rot(P,\pi)$: Half-turn about P.

Example 3 : Reflection in line k

$$\boxed{Ref_k(X)} = X'$$

Either $X \in k$ and $X' = X$

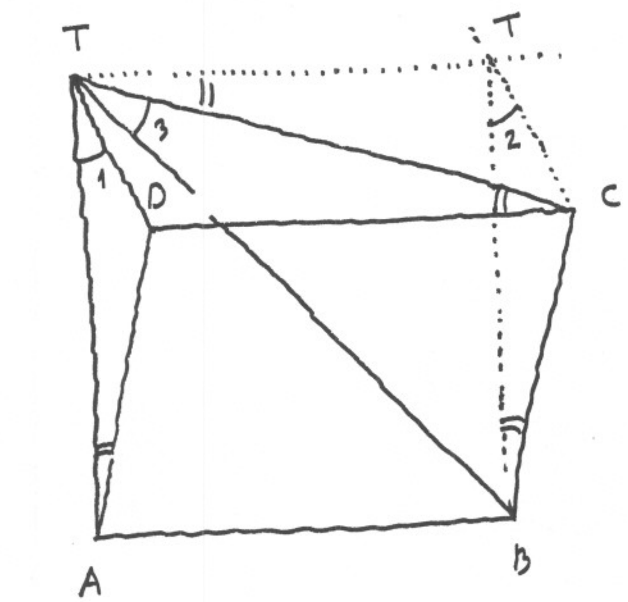
or $X \notin k$ and $X' \neq X$

$XX' \perp k$, midpoint of $[X, X']$ on k.

Although introduced as special cases, the above will be seen to contain the essential meaning of isometries...

Application 1 : (Translation...)

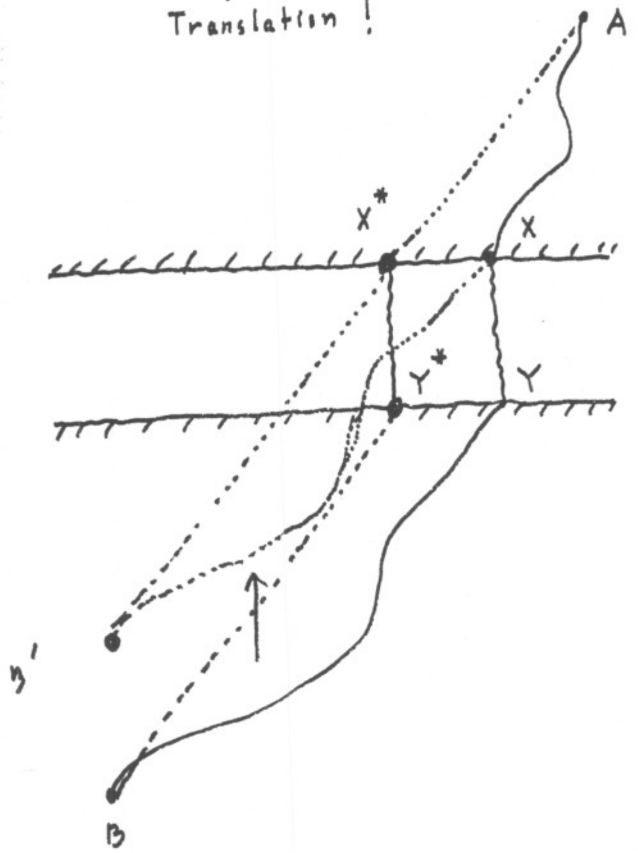
A problem: Given a parallelogram $ABCD$ and a point T such that $\measuredangle TAD = \measuredangle DCT$; prove that $\measuredangle ATD = \measuredangle BTC$.



Notes: Let $T' = Tr_{\overrightarrow{AB}}(T)$ and note that T, T', C, B concyclic. Hence $3 = 2 = 1$.

Application (2) The bridge-across-river problem...

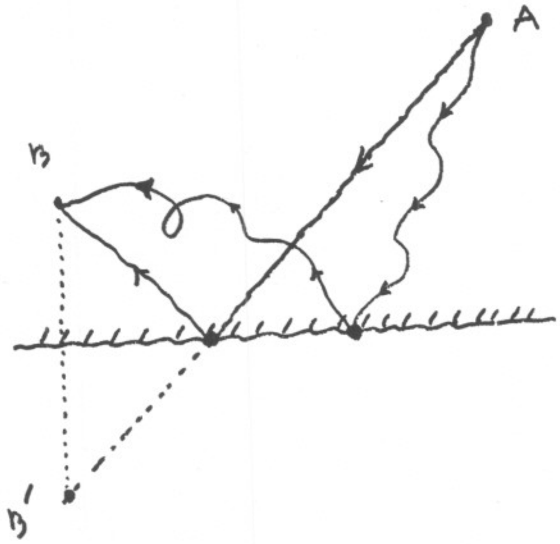
Translation !



Find the [redacted] position of the bridge which makes the distance $|AX|+|XY|+|YB|$ minimum.

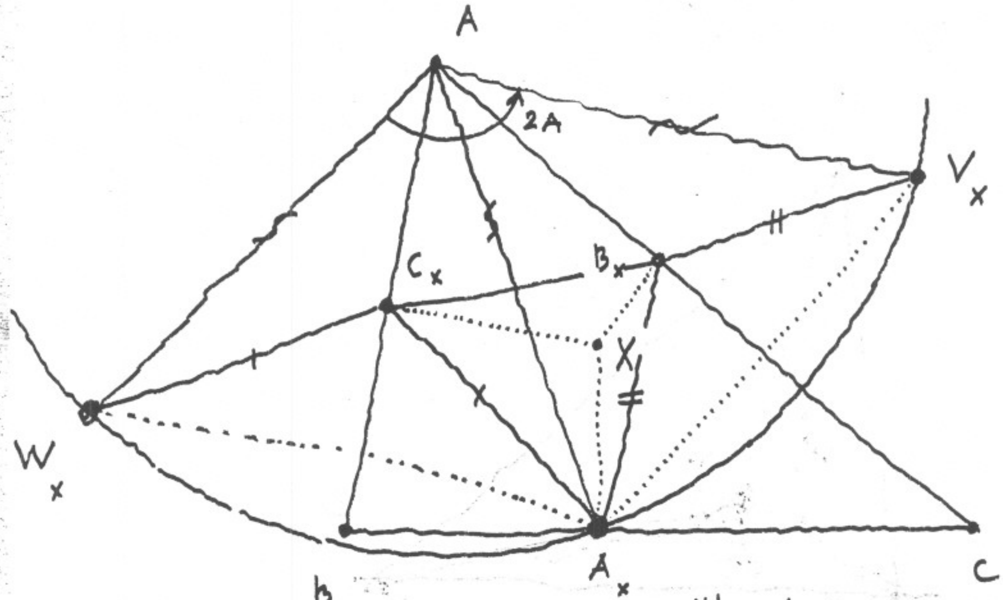
Reflection only !

Application (3) A historic problem



Application ④ → Reflection

The Fagnano Problem



The solution is $X = A$ if $A \geqslant 90°$

with $A, B, C \leqslant 90°$

Given a triangle ABC, find the position of X which makes $|A_xB_x| + |B_xC_x| + |C_xA_x|$ minimum, where A_x, B_x, C_x are the feet of the perpendiculars from X on BC, CA, AB.

Existence : Real analysis.

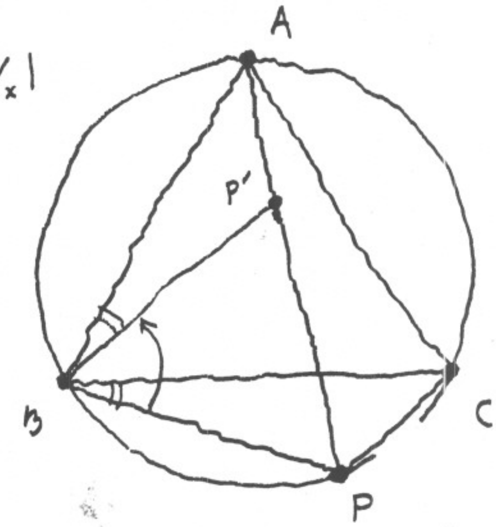
Note :

$$|B_xC_x| + |C_xA_x| + |A_xB_x| \geqslant |W_xC_x| + |C_xB_x| + |B_xV_x|$$

$$\geqslant |W_xV_x| = 2|AA_x| \sin 2A$$

which attains its least value if AA_x^* is an altitude. $\Longrightarrow X = H$

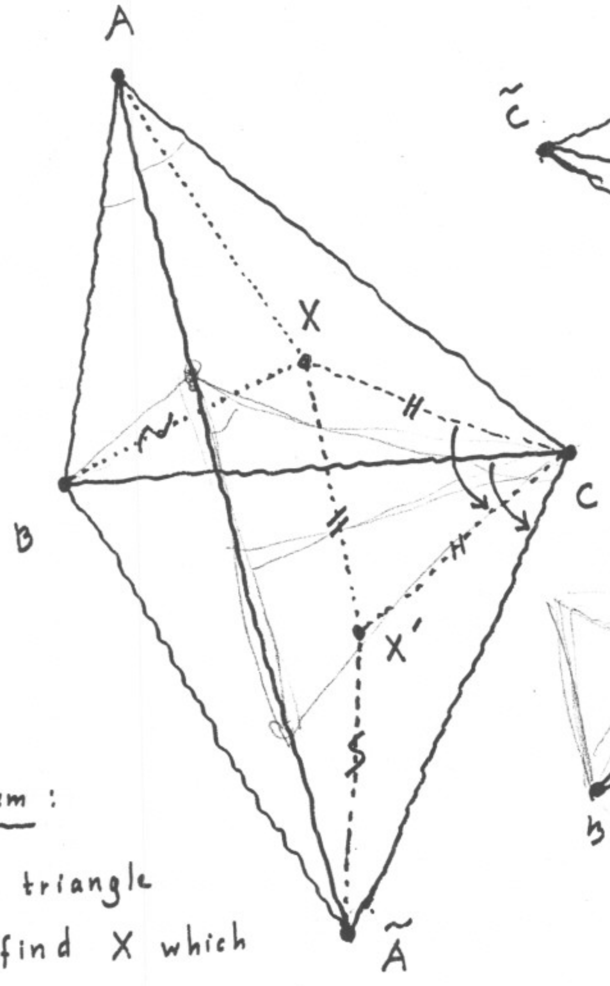
Application ⑤ – Rotation !

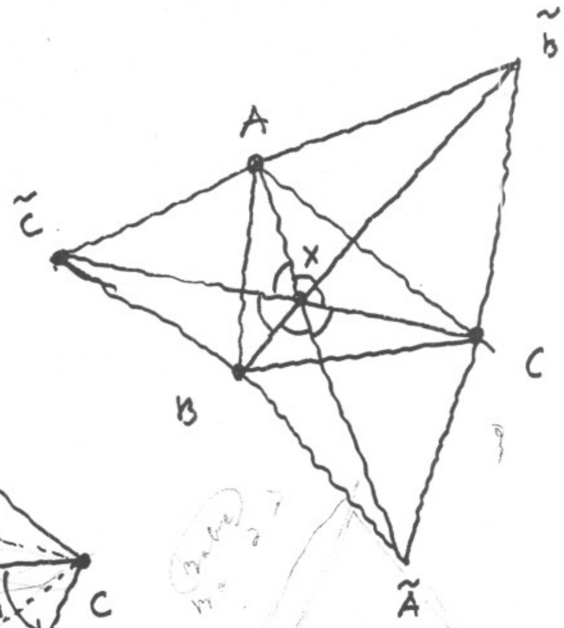


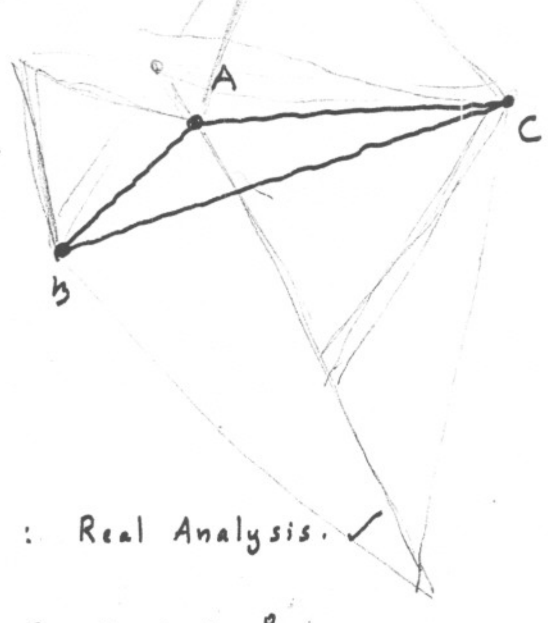
Prove $|PB| + |PC| = |PA|$:

Note : $\mathrm{Rot}(B, \pi/3)$: $\left.\begin{array}{l} C \longrightarrow A \\ B \longrightarrow B \\ P \longrightarrow P' \end{array}\right\}$ ✓

Application (6) — Rotation ! (~~The~~ A "Fermat Problem")







Problem :

Given a triangle ABC, find X which minimizes $|AX| + |BX| + |CX|$.

Existence : Real Analysis. ✓

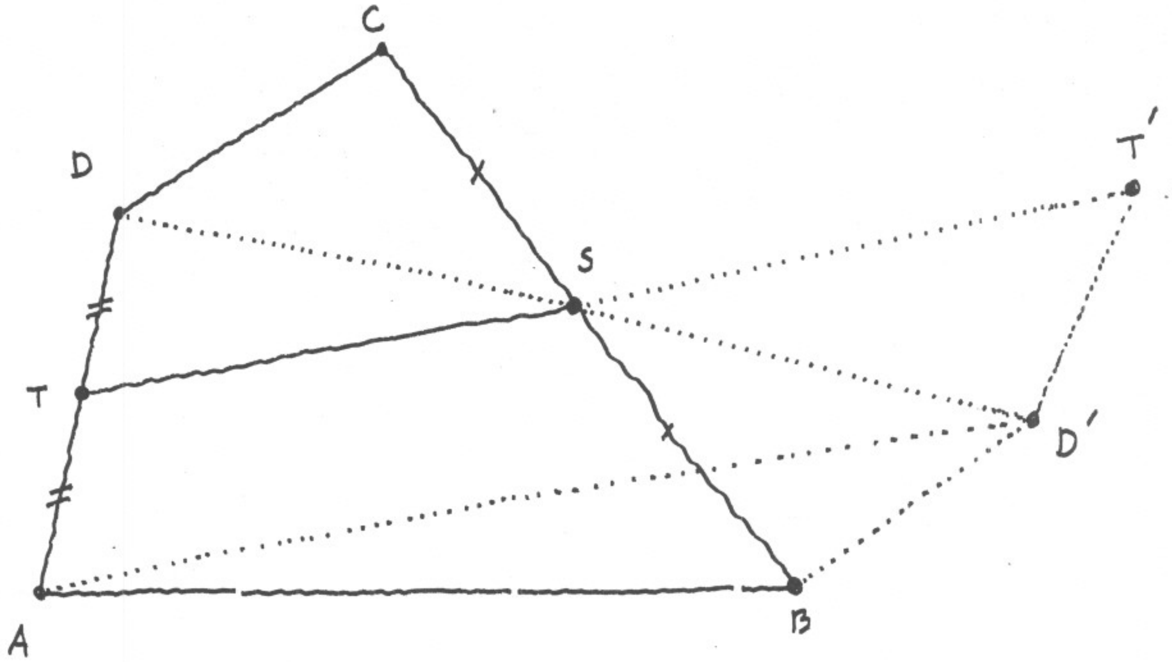
The case with $A, B, C \leq 120^\circ$.

For any point X (Inside?) ABC $\quad Rot(C, \frac{\pi}{3}) : \begin{array}{l} C \to C \\ X \to X' \\ B \to \tilde{A} \end{array}$

$$|XA| + |XB| + |XC| = |XA| + |XX'| + |X'\tilde{A}| \geq A\tilde{A} .$$

X must lie on $A\tilde{A}$ if it exists !

Application (7) — Half-turn.



Problem: S, T respective midpoints of [A, D], [B, C] respectively. Prove that

$$2\,|ST| \leq |AB| + |CD|$$

equality holding iff ABCD is a trapezoid (Better: $AB \parallel CD$)

Solution: Consider half-turn with center S.

Observe :

$$\left.\begin{array}{l} T \longrightarrow T' \\ D \longrightarrow D' \\ C \longrightarrow B \end{array}\right\}$$

Consequently $AD'T'T$ is a parallelogram.

Equality obtains if B lies on $[A, D']$ etc. ...

in which case $AB = BD' \parallel CD$.

Basic Properties of Isometries :

1 Let ISO (non-standard notation!) denote the set of isometries of $\mathbb{R}^2$.
ISO constitues a group under functional composition : $Id \in ISO$, $\varphi \in ISO \rightsquigarrow \varphi^{-1} \in ISO$, $\varphi, \psi \in ISO \rightsquigarrow \psi \circ \varphi \in ISO$.

2 Isometries preserve collinearity and betweenness...
(A simple application of the triangle inequality !)

3 Isometries preserve angles up to a change of sign.

4 (↗ 2) An isometry leaving distinct points P, Q fixed, leaves all points on the line PQ fixed.

5 Important compositions :

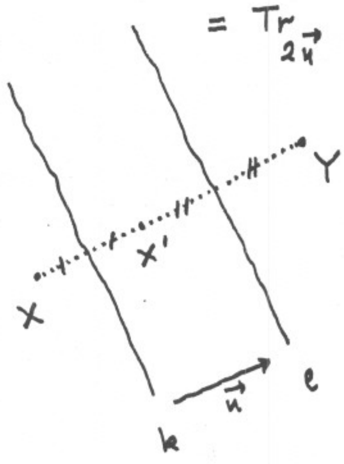
(a) Two reflections : $Ref_\ell \circ Ref_k$

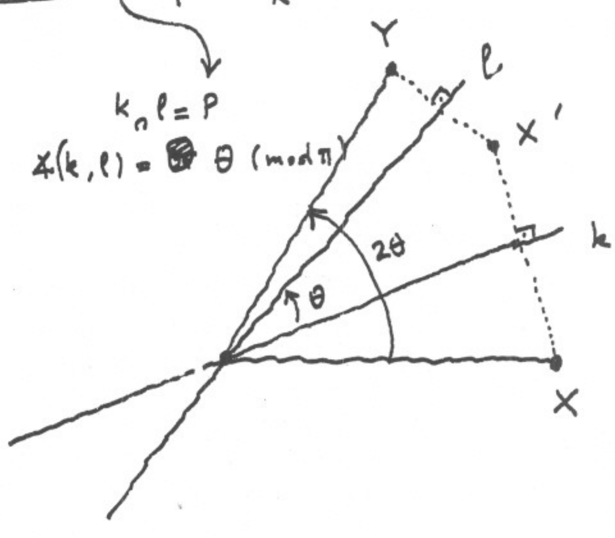
$k = \ell \longmapsto \underline{\underline{Id}}$. "involutive"

$k \parallel \ell$

$= Tr_{2\vec{u}}$

$k \cap \ell = P$
$\measuredangle(k, \ell) = \theta \pmod{\pi}$





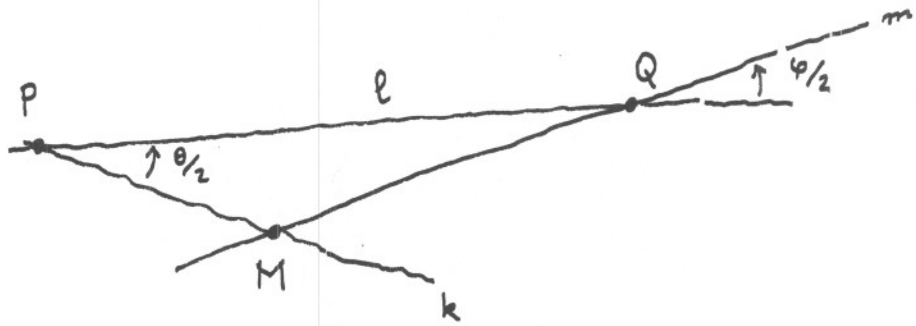
(b) Two rotations: Let P, Q be distinct points.

Given $\theta, \varphi \pmod{2\pi}$ we wish to know

$$\mathrm{Rot}(Q, \varphi) \circ \mathrm{Rot}(P, \theta)$$

Remember that $\mathrm{Rot}(P,\theta) = \mathrm{Ref}_\ell \circ \mathrm{Ref}_k$ for any lines k, ℓ intersecting in P with $\measuredangle(k,\ell) = \theta/2 \pmod{\pi}$.

Let $\ell = PQ$. Choose k through P with $\measuredangle(k,\ell) = \theta/2 \pmod{\pi}$
m through Q with $\measuredangle(\ell, m) = \varphi/2 \pmod{\pi}$



Notice that $\measuredangle(k,m) = \dfrac{\theta+\varphi}{2}$ and

$$\mathrm{Rot}(Q,\varphi) \circ \mathrm{Rot}(P,\theta) = \mathrm{Ref}_m \circ \mathrm{Ref}_\ell \circ \mathrm{Ref}_\ell \circ \mathrm{Ref}_k$$
$$= \mathrm{Ref}_m \circ \mathrm{Ref}_k$$

a translation if $\theta + \varphi = 0 \pmod{2\pi}$

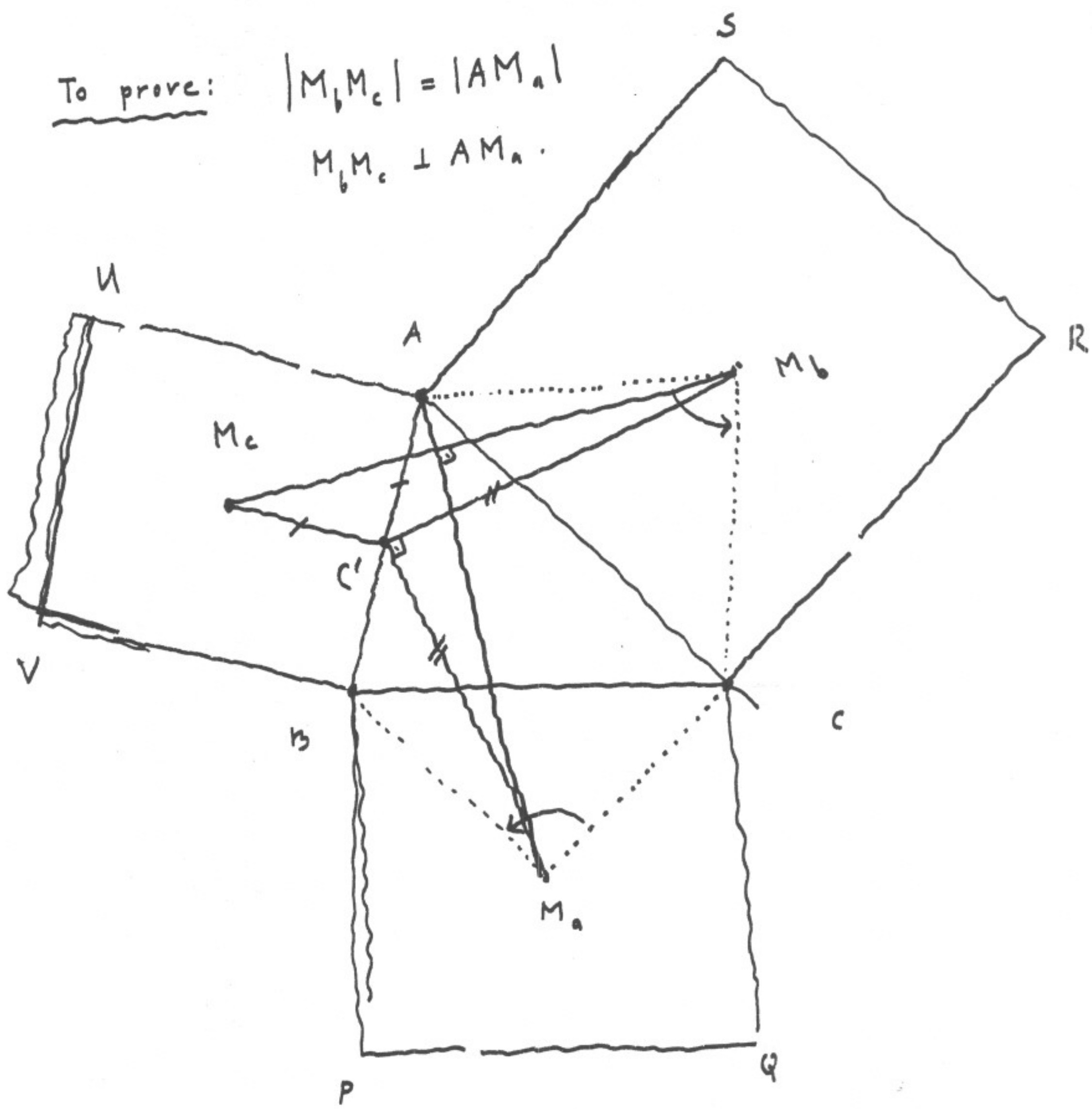
$= \mathrm{Rot}(M, \theta+\varphi)$ where $k \cap m = \{M\}$ if $\theta + \varphi \neq 0 \pmod{2\pi}$

Application : The facts we have noticed about the way in which two rotations are composed can be employed to great advantage. The following is a classic :

Problem 3, Selected problems VII :

To prove: $|M_b M_c| = |A M_a|$

$M_b M_c \perp A M_a$.



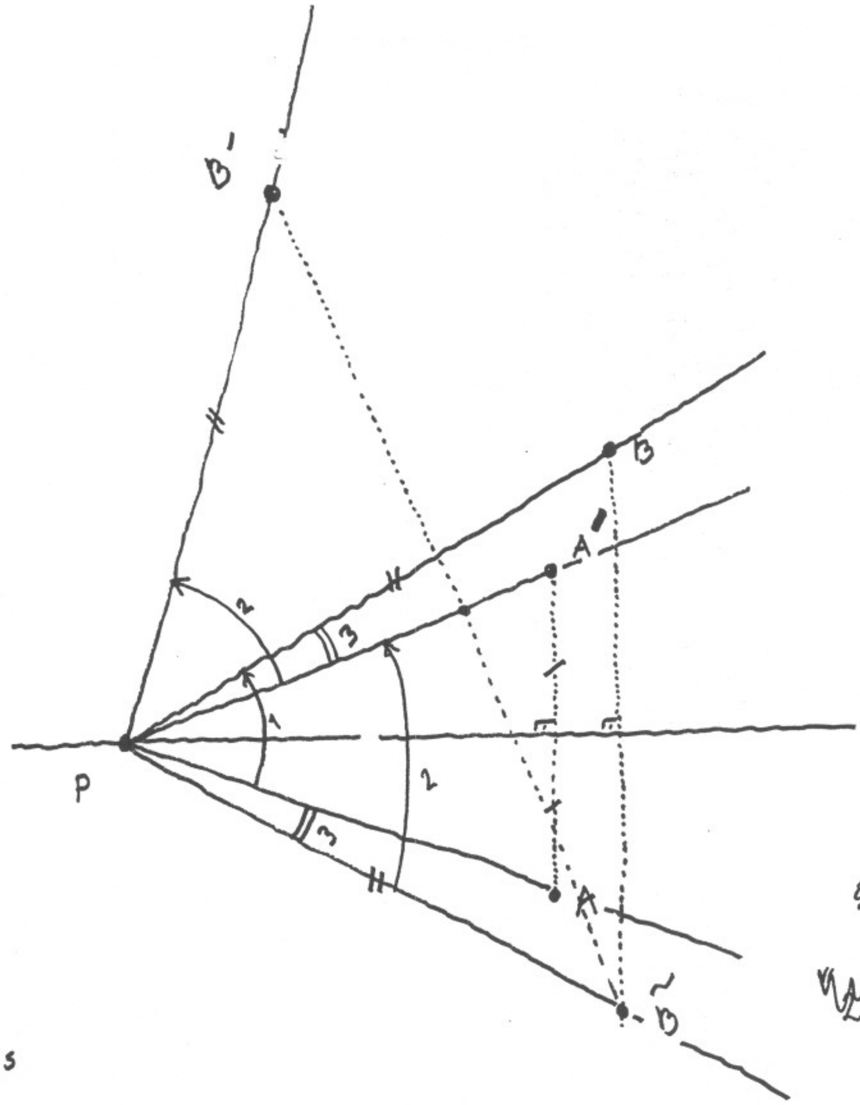
Note that $Rot(M_a, \pi/2) \circ Rot(M_b, \pi/2)$ is a half-turn sending A into B. ∴ Its center is the midpoint C' of $[A,B]$. This shows that $M_a C' M_b$ is an isosceles right triangle, and $Rot(C', \pi/2)$: $M_a \longrightarrow M_b$, $A \longrightarrow M_c$ ✓.

6 Classification:

(i) If an isometry leaves three non-collinear points fixed, then it is the identity map.
(This observation is equivalent to the following:
If two isometries agree at three non-collinear points, then they are identical.)

(ii) Suppose an isometry φ leaves two distinct points A, B fixed. We have already observed that all points on AB are fixed by φ. Consider $C \notin AB$. There are two possibilities: (ii 1) $\varphi(C) = C$. In this case, obviously $\varphi = \mathrm{Id}$. (ii 2) $\varphi(C) \neq C$: Let C' be the foot of the perpendicular from C onto AB. We have $\varphi(C') = C'$. Thus $\varphi(CC') = CC'$ and $\varphi(C) \in CC'$. Since $\varphi(C) \neq C$, φ must be Ref_k.

(iii) Suppose an isometry φ has unique fixed point P. Consider distinct points $A, B \neq P$ let $\varphi(A) = A'$, $\varphi(B) = B'$. Let k be the perpendicular bisector of $[A, A']$, put $\tilde{B} = \mathrm{Ref}_k(B)$.



$\measuredangle APB = \measuredangle A'PB'$

"2" = "1"

$\cancel{\text{"2"} = \text{"1"} + \text{"3"}}$

As

$$\measuredangle \tilde{B}PA' = \measuredangle \tilde{B}PA + \measuredangle APB - \measuredangle A'PB$$
$$= \measuredangle APB = \measuredangle A'PB'$$

we conclude from $|P\tilde{B}| = |PB| = |PB'|$ that

$$\mathrm{Ref}_{PA'}(\tilde{B}) = B'$$

Since $\mathrm{Ref}_{PA'} \circ \mathrm{Ref}_k : \begin{array}{l} P \longrightarrow P \\ A \longrightarrow A' \\ B \longrightarrow B' \end{array}$

we conclude that $\varphi = \mathrm{Ref}_{PA'} \circ \mathrm{Ref}_k = \mathrm{Rot}(P, \Theta)$

where $\Theta = 2\measuredangle(k, PA') = \measuredangle APA'$.

(iv) Finally, suppose that an isometry φ leaves no point fixed. For any $P \in \mathbb{R}^2$ let $P' = \varphi(P)$ and put $\vec{u} = \overrightarrow{PP'}$. The isometry $Tr_{-\vec{u}} \circ \varphi$ has at one fixed point and can be treated as in the above items.

In any case we observe that <u>any isometry can be expressed as the concatenation of at most three reflections</u>.

(B) Homothety:

"center of homothety" "ratio of similitude"

Definition: Given $P \in \mathbb{R}^2$ and $\alpha \in \mathbb{R} - \{0\}$, the homothety on $\mathbb{R}^2$ with "center" P and "ratio of similitude" α is a map

$$\mathrm{Hom}(P, \alpha) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

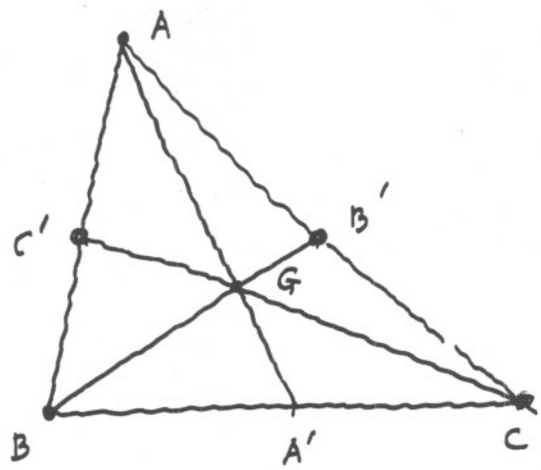
defined by

$$\mathrm{Hom}(P, \alpha)(X) = P + \alpha (X - P).$$

Special Case : Hom $(P, -1)$ = Rot (P, π)

It can be easily checked that a homothety maps lines into ~~[illegible]~~ parallel lines, preserves angles.

Application (1) : The "medial homothety" in a triangle :



Hom $(G, -\frac{1}{2})$

$A \longrightarrow A'$

$B \longrightarrow B'$

$C \longrightarrow C'$

The punch-line :

H is mapped into the orthocentre of $A'B'C'$ which is O :

Thus: Hom $(G, -\frac{1}{2})$ (H) = O.

Important consequence : O, G, H are collinear "the Euler Line" and $GO : GH = -1 : 2$.

Action of a homothety on a circle :

Let $\Gamma(O, R)$ stand for the circle of radius R, center O.

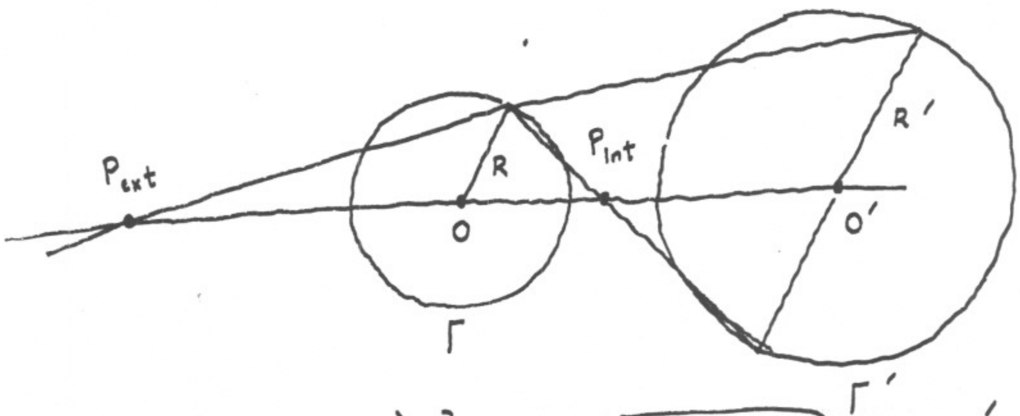
It can be readily checked that

$$\mathrm{Hom}(P, \lambda)\,(\Gamma(O, R)) = \Gamma(O', |\lambda| R)$$

where $O' = \mathrm{Hom}(P, \lambda)(O)$.

There is a striking converse:

Generically two circles are homothetic in two ways:



If $\left.\begin{array}{l}\Gamma = \Gamma(O, R) \\ \Gamma' = \Gamma(O', R')\end{array}\right\}$ are $\boxed{\begin{array}{l}\text{non-concentric} \to O \neq O' \\ \text{non-congruent} \to R \neq R'\end{array}}$

then

$$\Gamma \xrightarrow[\mathrm{Hom}(P_{int},\, -R'/R)]{\mathrm{Hom}(P_{ext},\, R'/R)} \Gamma'$$

where

$$P_{ext}O : P_{ext}O' = R : R'$$

and

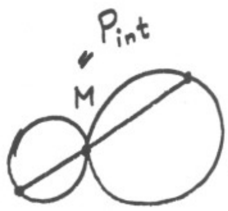
$$P_{int}O : P_{int}O' = -R : R'.$$

Non-generic situations :

$O = O'$: In this case we may take $P_{int} = P_{ext} = O = O'$

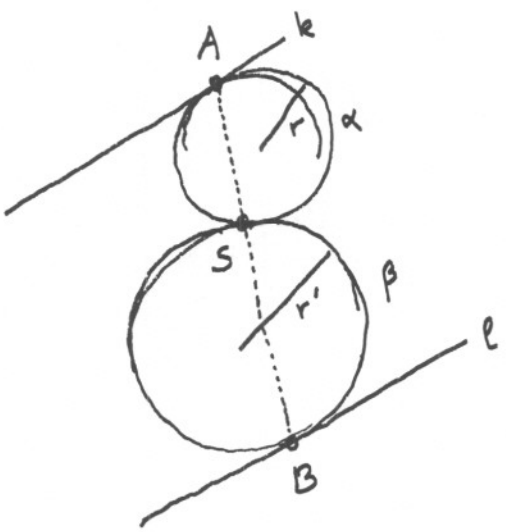
$O \neq O$ but $R = R'$: P_{int} is the midpoint of $[O, O']$ but "P_{ext} lies at infinity".

Interesting situations :



This, by the way, is a good way of defining "externally tangent" and "internally tangent" pairs of circles...

Application (2) : A problem : $k \parallel \ell$ etc. Prove that A, B, S are collinear.



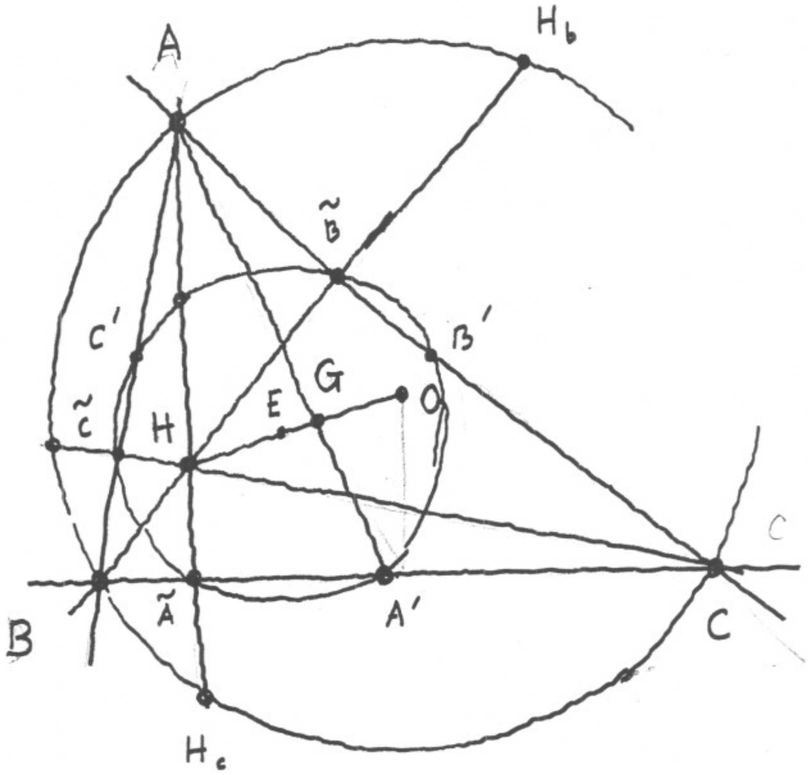
Solution : Let $\lambda = ? \pm r \quad r'$

Note : $Hom(S, \lambda)$: $k \longrightarrow \ell$, $\alpha \longrightarrow \beta$

Hence $A \longrightarrow B$

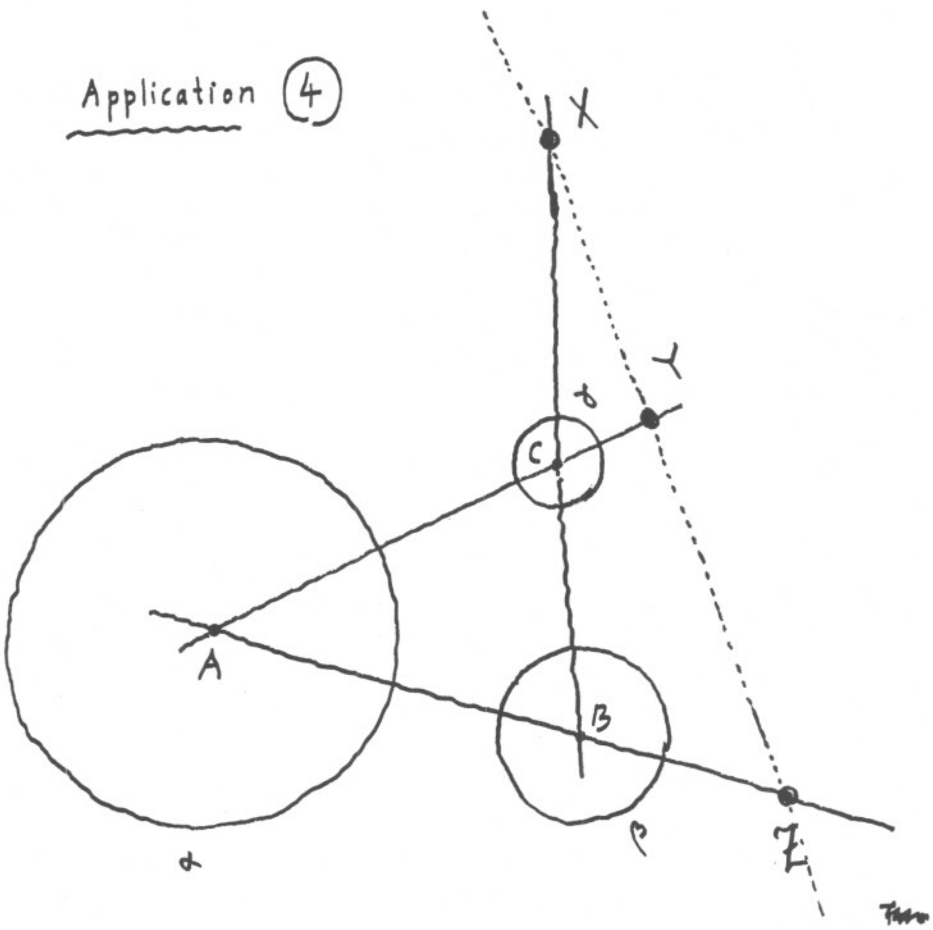
and therefore ✓.

Application ③ The 9-point circle revisited !



Clearly the "medial homothety" sends (O) the circum-circle of ABC into the circumcircle of $A'B'C'$. Where is the center of the second homothety which does the same thing? Of course, the ratio of similitude must be $\frac{1}{2}$ this time! A simple inspection shows that the required homothety is Hom$(H, \frac{1}{2})$. This fact provides us with a wealth of information: Such as: Midpoints of $[H,A]$, $[H,B]$, $[H,C]$, $[H,H_a]$, $[H,H_b]$, $[H,H_c]$ lie on the circumcircle of $A'B'C'$ ⟶ The name "9-points-circle" The illustration!

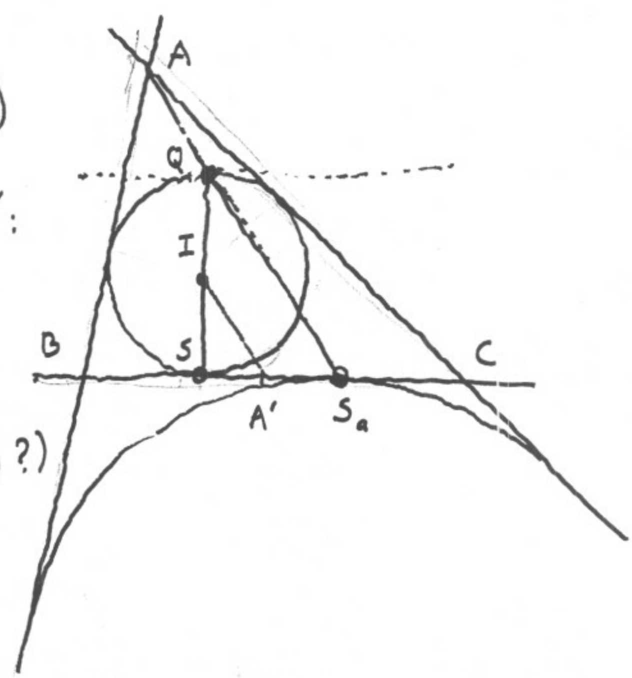
Application 4



A classical and rather too well-known result: External homothety circles of pairs from among three circles "in general position" are collinear! A simple application of the Menelaus.

Application 5

First apply the homothety: $\mathrm{Hom}(A, r_a/r)$



Medial homothety strikes again!

1) I is the Nagel point of $A'B'C'$;
2) G, I, N are collinear and

$$GI : GN = -1 : 2.$$

Observe: If $[S, Q]$ is a diameter of (I) then A, Q, S_a are collinear. (Why?) Thus $A'I \parallel AS_a$; this shows (!) that medial homothety sends N into I!