Lecture 8

Inversion

Given $M \in \mathbb{R}^2$, $\alpha \neq 0$, the <u>inversion</u> of center M, power α is the map

$$\mathrm{Inv}(M,\alpha) : \mathbb{R}^2 - \{M\} \longrightarrow \mathbb{R}^2 - \{M\}$$

under which X is mapped into X' iff M, X, X' are collinear and $MX . MX' = \alpha$.

<u>Immediate properties</u> :

1) $\varphi = \mathrm{Inv}(M,\alpha)$ is <u>involutive</u>, that is $\varphi \circ \varphi = \mathrm{Id}$.

* Conventionally one extends $\mathbb{R}^2$ by introducing an ideal point "∞" "at infinity" and stipulating that $\mathrm{Inv}(M,\alpha)$ interchanges M and ∞.

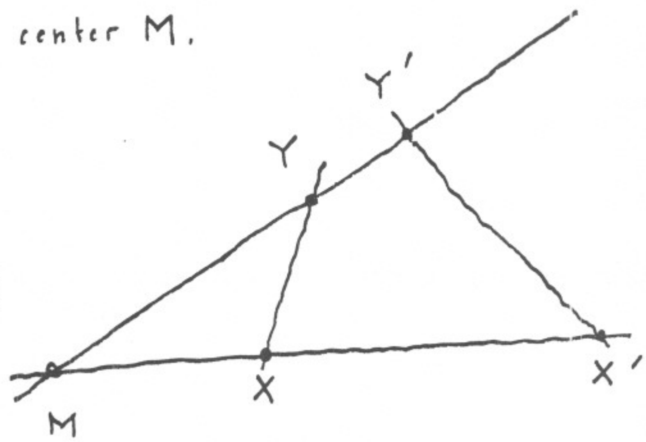
2) If $\alpha > 0$, the circle γ of center M and radius $\sqrt{\alpha}$ is left (pointwise) invariant by $\mathrm{Inv}(M,\alpha)$. Notice that the interior and exterior of γ are interchanged under the same transformation.

γ is hence called the "circle of inversion". $\mathrm{Inv}(M,\alpha)$ is also referred to as "inversion <u>in</u> γ" rather like "reflection in k".

Lemma: Given and inversion F_φ and any points X, Y the points $X, Y, \varphi(X), \varphi(Y)$ are concyclic.

Proof: F with center M.

Observe the similarity of the triangles MXY and ~~MX~~' MY'X' (!)



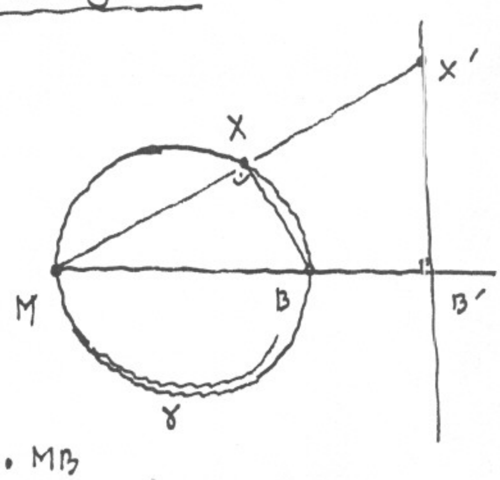
Theorem: ~~It~~ Let F be an inversion with center M.

1) F maps a line through M into itself. (Not pointwise...)

2) F maps a circle through M into a line not through M.

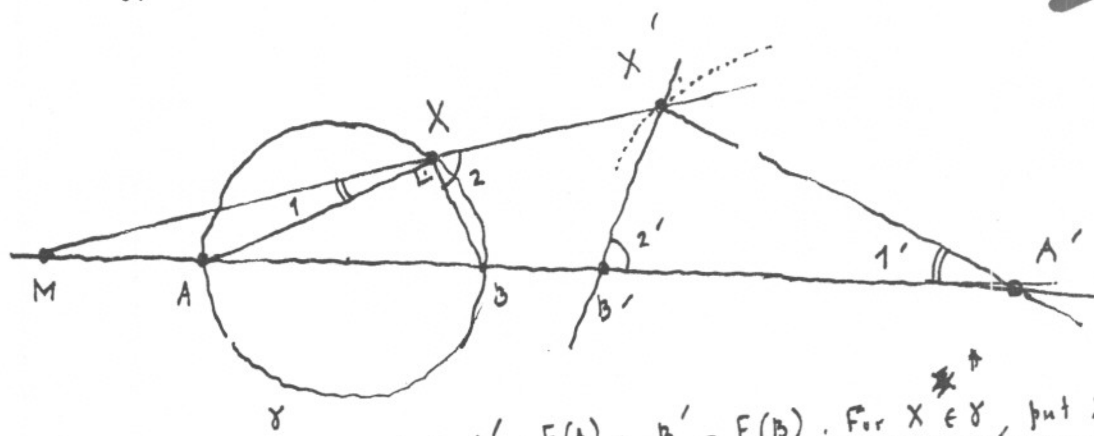
3) F maps a circle not through M into a circle not through M.

Proof:



2) Consider circle γ through M. Let $[M, B]$ be a diameter, let $B' = F(B)$. For any $X \in \gamma$, $F(X)$ is on
M*
the perpendicular to MB erected at B'.

3)



Let γ be a circle not through M, [AB] be a diameter thereof not through M. Let $A' = F(A)$, $B' = F(B)$. For $X \in \gamma$, put $X' = F(X)$. Note that $1 = 1'$ as A, A', X, X' are concyclic. $2 = 2'$ as B, B', X, X' are concyclic. Hence

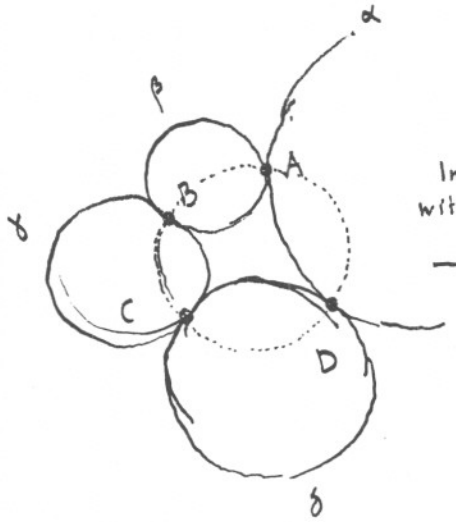
$\measuredangle B'XB = 1' + 2' = \pi/2$.

Remarks: 1) Circles tangent at M are transformed into parallel lines.

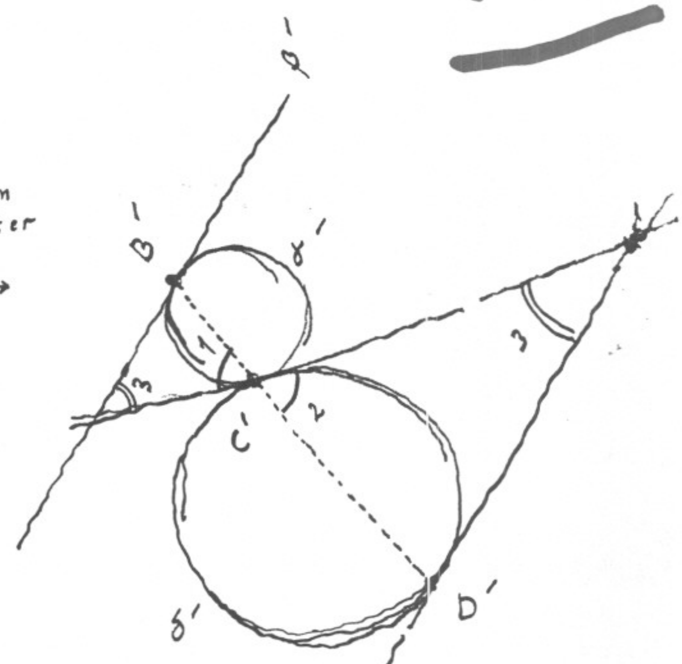
2) Let $\alpha > 0$. If $\mathrm{Inv}(M, \alpha)$ transforms the circle γ into the circle γ', then M is the external center of homothety of γ and γ'.

Application ① In attacking a problem by means of an inversion one tries to achieve simplification by transforming circles into lines.

Typical is the following:



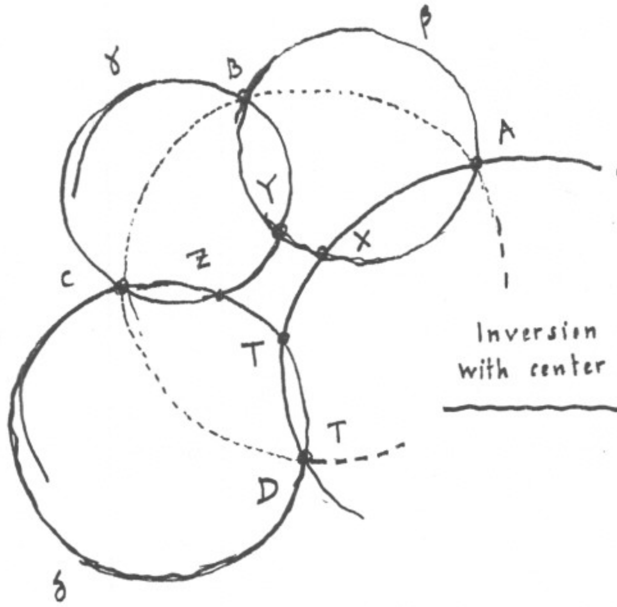
Inversion with center A



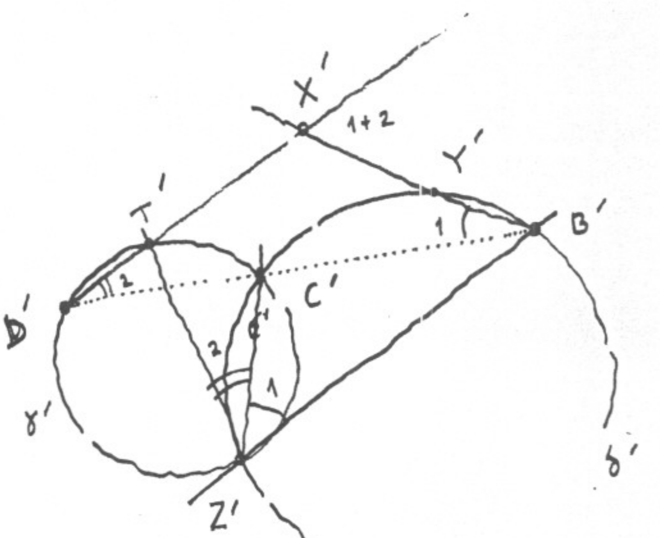
"Prove that A, B, C, D are concyclic or collinear"

"Prove that B', C', D' are collinear"

Application (1')



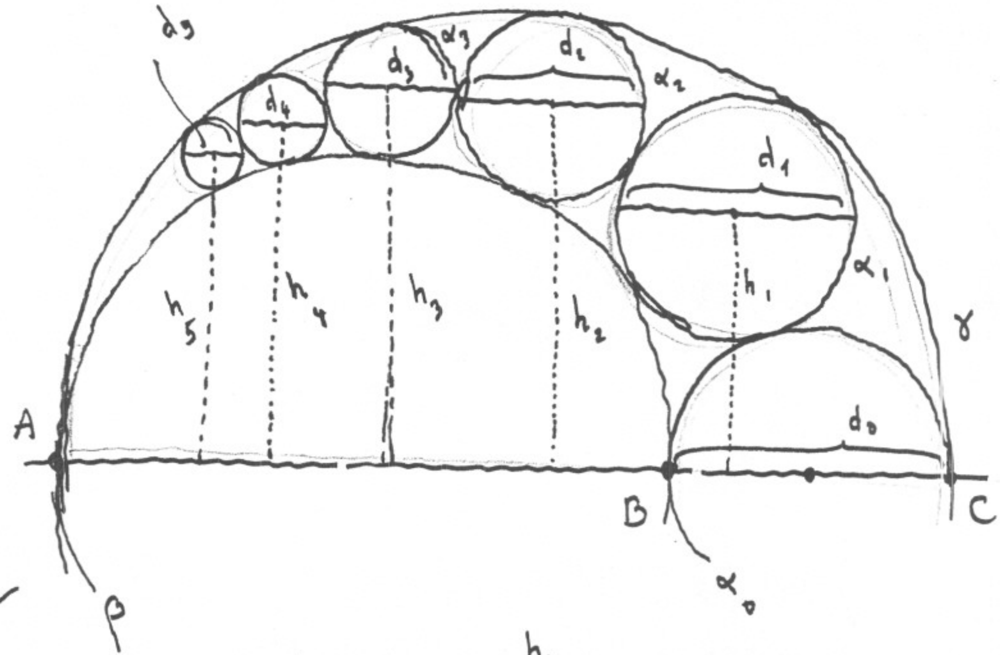
Inversion with center A



"A, B, C, D are concyclic. Prove that X, Y, Z, T are concyclic"

"B', C', D' are collinear. Prove that X', Y', Z', T' are ~~collinear~~ concyclic or collinear"

Application ②

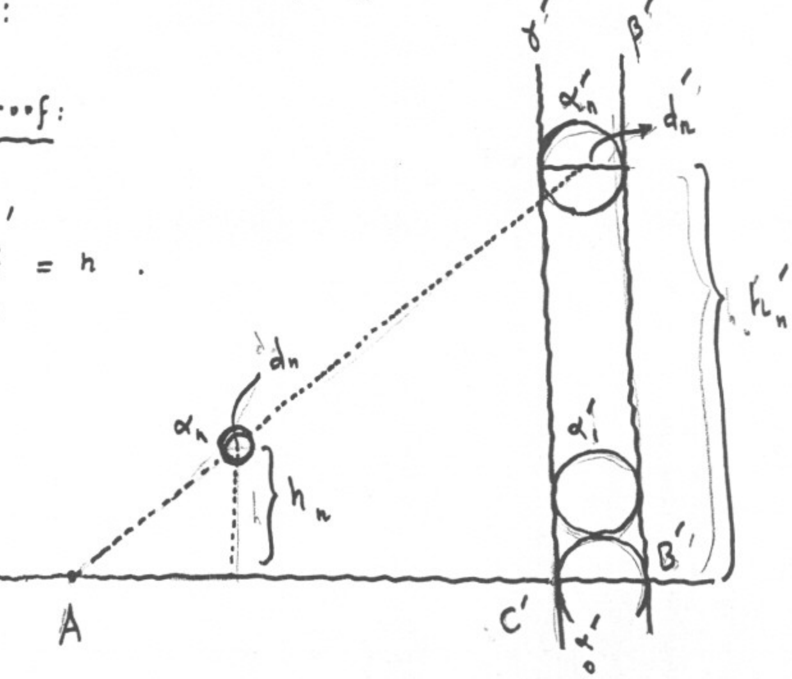


inversion with center A:

Theorem: $\dfrac{h_n}{d_n} = n$ (Pappus, + 4. century)

Proof:

$$\frac{h_n}{d_n} = \frac{h_n'}{d_n'} = n \ .$$



The next result illustrates the manner in which inversion can be employed to obtain new theorems from old theorems.

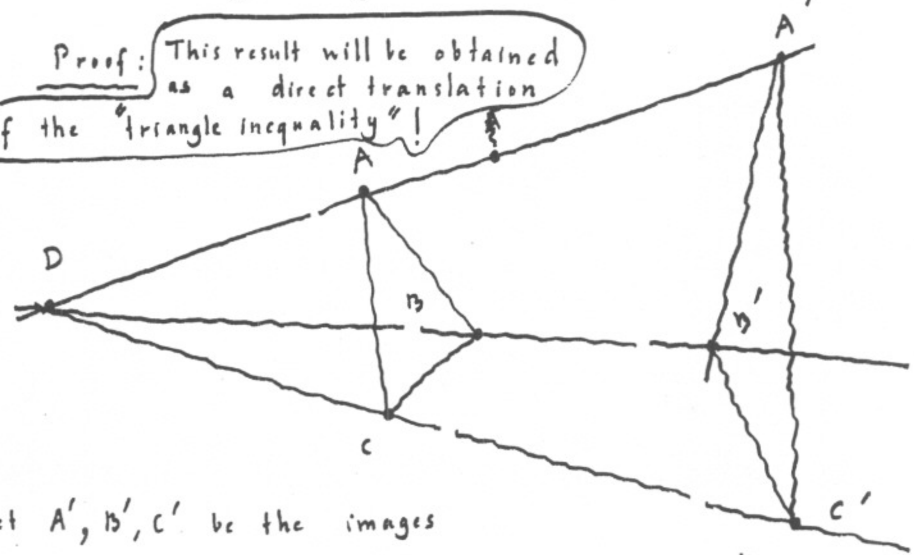
Application (3): ("The Inequality of Ptolemy" "Batlamyus" (+2))

Theorem: For any distinct points A, B, C, D

$$|AC| \cdot |BD| \leq |AB||CD| + |BC||DA|$$

where equality holds iff A, B, C, D are concyclic or collinear and ordered cyclically...

Proof: This result will be obtained as a direct translation of the "triangle inequality"!



Let A', B', C' be the images of A, B, C under an inversion of center D and power k (immaterial what...)

As $ABD \cong B'A'D$ we have

$$\frac{|A'B'|}{|AB|} = \frac{|DA'|}{|DB|} = \frac{|DA'||DA|}{|DA||DB|} = \frac{|k|}{|DA||DB|}$$

Hence

$$|A'B'| = \frac{|AB||k|}{|DA||DB|}, \quad |B'C'| = \frac{|BC||k|}{|DB||DC|}, \quad |A'C'| = \frac{|AC||k|}{|DA||DC|}$$

Since

$$|A'B'| + |B'C'| \geqslant |A'C'|$$

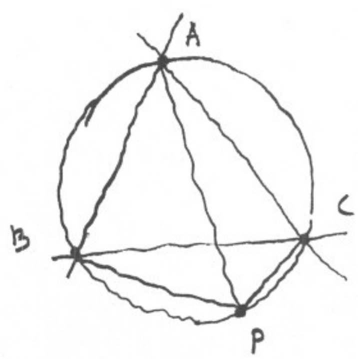
we obtain

$$|AB||CD| + |BC||DA| \geqslant |AC||BD|$$

where equality occurs iff B' lies on $[A', C']$.

Application (3′):

Revisiting Application (5) of Lecture 7.



ABC equilateral. By the Ptolemy

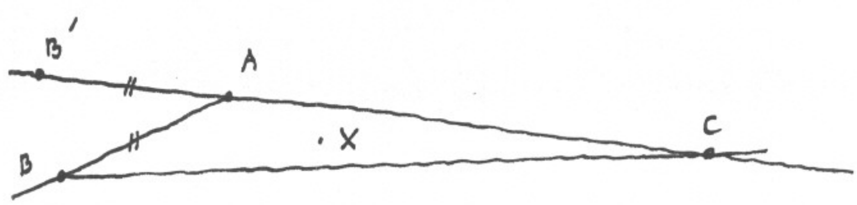
$$|PB| \cdot |AC| + |PC| \cdot |AB| = |PA| \cdot |BC|$$

hence

$$|PB| + |PC| = |PA|$$ order!

Application (3″):

The Toricelli-Fermat problem with $A \geqslant 120°$:



Choose B' on AC such that A is between B' and C and $|AB'| = |AB|$. Consequently $\measuredangle B'AB \leqslant 60°$ and $|BB'| \leqslant |AB|$. By the "Ptolemy", for any X we have

$$|XA| + |XB| = \frac{|XA||AB'| + |XB||AB'|}{|AB'|} \geqslant \frac{|XA||BB'| + |XB||AB'|}{|AB'|}$$

$$\overset{(*)}{\geqslant} \quad \frac{|XB'|\,|AB|}{|AB|} = |XB'|$$

hence

$$|XA|+|XB|+|XC| \geqslant |XB'|+|XC| \overset{(**)}{\geqslant} |B'C| = |AB|+|AC|$$
$$= \underset{\overset{\shortparallel}{0}}{|AA|} + |AB| + |AC| \quad .$$

Note that equality holds iff (*) and (**) reduce to equalities simultaneously, in which case X is on the circumcircle of ABB' not on the same $\underbrace{\text{arc as } B'}_{\text{bounded by } A,\, B'}$ and $X \in [B'C]$

hence $X = A$.

Inversion can be employed to full advantage only after clarifing its effect on angles :

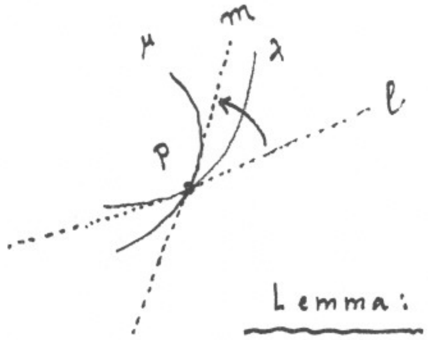
Theorem : A inversion leaves angles unchanged in absolute value and changes their signs.

More precisely : Let F be an inversion with center M. Consider curves γ, δ intersecting in $P \neq M$. We have

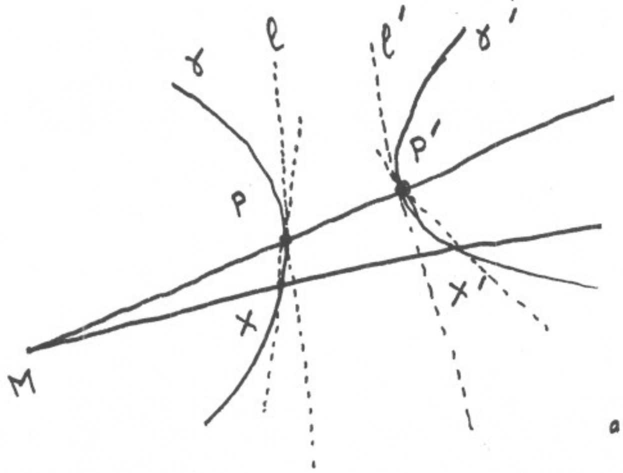
$$\measuredangle_{F(P)}(F\circ\gamma, F\circ\delta) = -\measuredangle_P(\gamma,\delta)$$

(In this statment we may take γ, δ as "directed" curves in which case $\measuredangle_P(\gamma,\delta) \in \mathbb{R}/_{2\pi\mathbb{Z}}$ or as "undirected" curwes in which case $\measuredangle_P(\gamma,\delta) \in \mathbb{R}/_{\pi\mathbb{Z}}$.)

Final clarification: Given (smooth) curves λ, μ intersecting in P we understand $\sphericalangle_P(\lambda,\mu)$ to be $\sphericalangle(\ell, m)$ where ℓ, m are tangent lines to λ, μ respectively at P.



Lemma: $$\sphericalangle_P(MP,\gamma) = -\sphericalangle_{\underbrace{F(P)}_{= P'}}(MP', \underbrace{\gamma'}_{= F\circ\gamma})$$



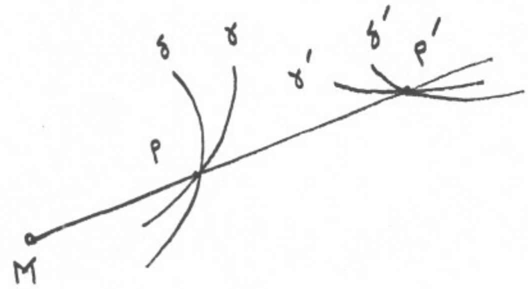
Proof: Consider X "near" P. Let ℓ, ℓ' be tangents to γ, γ' at P, P' respectively. Let $X' = F(X)$. Since X, X', P, P' are concyclic

$$\sphericalangle(MP, PX) = -\sphericalangle(MP', P'X')$$

$$\downarrow \quad \text{as } X \to P,\ X' \to P' \quad \downarrow$$

$$\sphericalangle(MP, \ell) = -\sphericalangle_{P'}(MP', \ell')$$

$$\sphericalangle_P(\gamma, \ell) = -\sphericalangle_{P'}(\gamma', \ell').$$



Back to the proof of the theorem:

$$\sphericalangle_P(\gamma,\delta) = \sphericalangle_P(\gamma, MP) + \sphericalangle_P(MP,\delta)$$

$$\cancel{= -\sphericalangle_{P'}(MP',\gamma') + \sphericalangle_{P'}(\delta', MP')}$$

$$= -\sphericalangle_{P'}(\gamma', MP') - \sphericalangle_{P'}(MP',\delta')$$

$$= -\sphericalangle_{P'}(\gamma',\delta').$$

Application (4): "The Feuerbach Theorem"

K.W. Feuerbach (1800-1834)

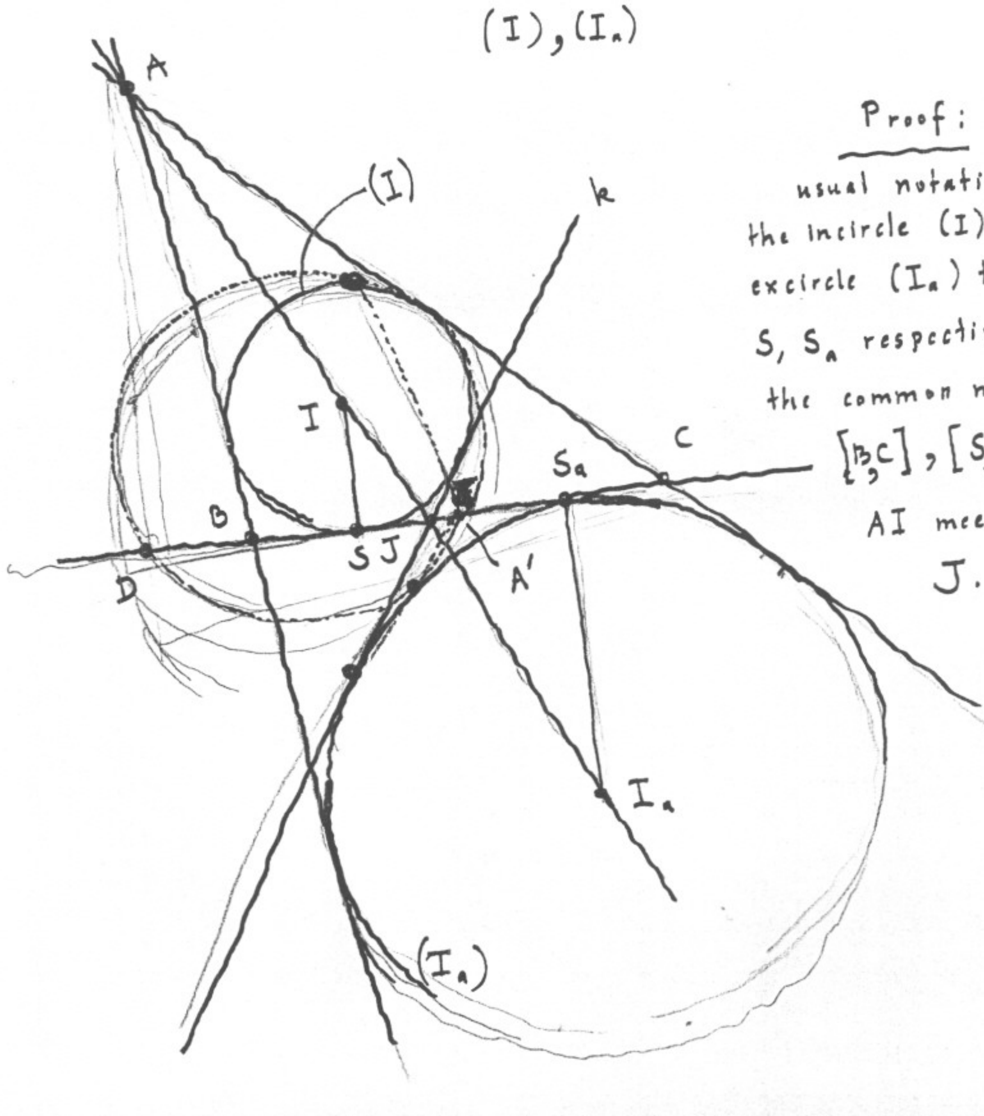
Remark: An inversion leaves ~~invariant~~ a circle orthogonal to the circle of invariant. (Not pointwise!)

J. Coolidge (1873-1954) call this result "the most beautiful theorem in elementary geometry that has been discovered since the time of Euclid."

Theorem: In a triangle, the 9-point circle is tangent to all tritangent circles.

(I), (I_a)

i.e. the incircle and the excircles...



Proof: Employ the usual notation: Let the incircle (I) and the excircle (I_a) touch BC in S, S_a respectively, A' be the common midpoint of $[B,C]$, $[S, S_a]$, let AI meet BC in J.

Check that: $|SA'| = |S_a A'| = \frac{a}{2} - (s-b) = \frac{|b-c|}{2}$

$$|A'J| = \frac{a}{2} - \frac{ac}{b+c} = \frac{a|b-c|}{2(b+c)}$$

Let D be the foot of the perpendicular from A on BC and K be the midpoint of [AH].

$$|A'D| = \left|\frac{a}{2} - c\cos B\right| = \frac{a}{2} - \frac{a^2+c^2-b^2}{2a} = \frac{|b^2-c^2|}{2a}$$

If F is the inversion of center A' and power $|SA'|^2$; we see that F leaves (I), (I_a) fixed ~~that~~ yet F sends D into J. Thus the image of the nine-point circle under F must be a line k through J and perpendicular to $A'K \parallel AO$. But AO is the reflection of AH in AI. Consequently k is the reflection of BC in AI hence tangent to (I) and (I_a). So is the 9-point circle!