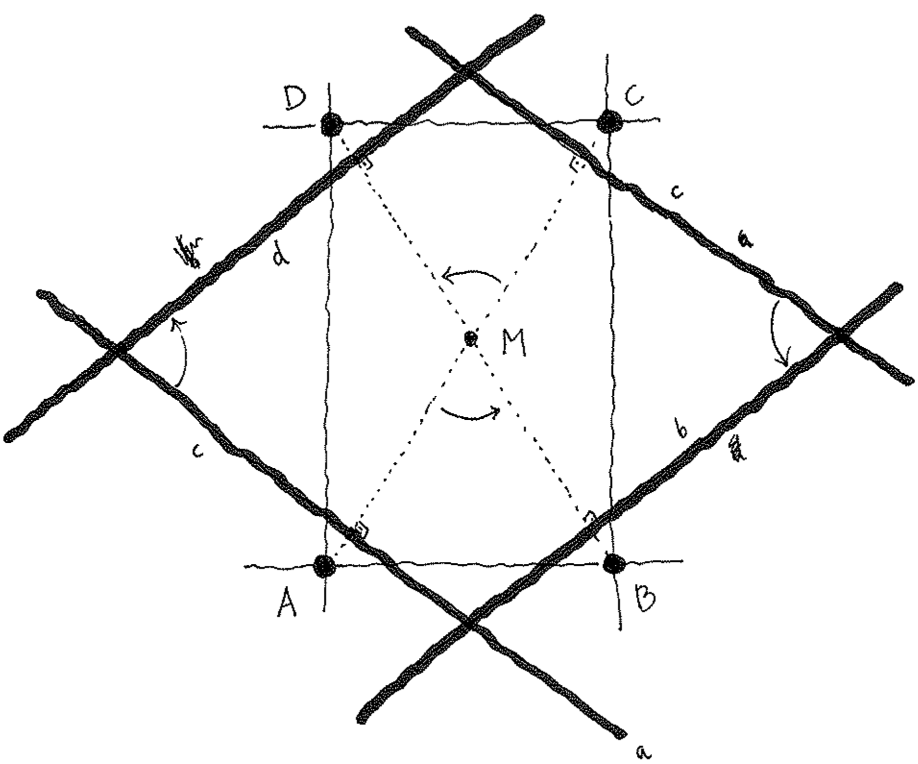
Reciprocation



Cem Tezer

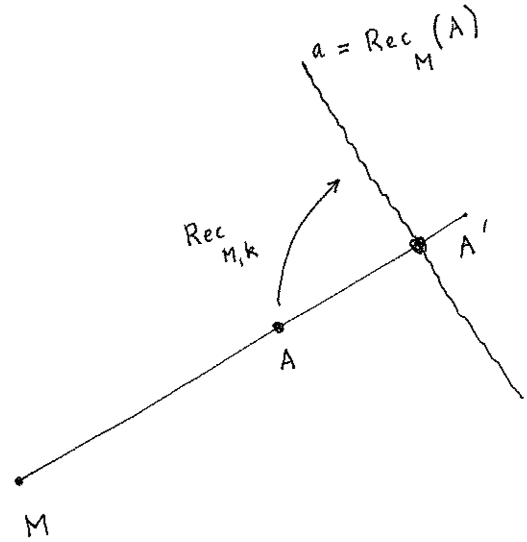
Ankara, 2009.

Reciprocation is a geometric transformation which is closely related to inversion and constitutes an excellent introduction to projective geometry.

The reciprocation [redacted] $\mathrm{Rec}_{M,k}$ of center $M \in \mathbb{R}^2$ and power $k \in \mathbb{R} - \{0\}$ is the map

$$\mathrm{Rec}_{M,k} : \underbrace{\mathbb{R}^2 - \{M\}}_{\text{points not coinciding with } M} \longrightarrow \{\text{lines not through } M\}$$

where $\mathrm{Rec}_{M,k}(A)$ is the line through $A' = \mathrm{Inv}_{M,k}(A)$ perpendicular to MA'. Notice that $\mathrm{Rec}_{M,k}$ is a bijective map.



In the presence of a reciprocation, points are denoted by upper case characters $A, B, C, \ldots,$ [redacted] P and lower case characters $a, b, c, \ldots, p$ are reserved for the lines which are their images under the reciprocation in question.

Quite explicitly, in the context of a reciprocation φ

$$a = \varphi(A), \qquad B = \varphi^{-1}(b) \quad \underline{\text{always}} \ldots$$

Convention : $\quad A^{\neq M} \xleftrightarrow[\text{Rec}_{M,k}]{} a \not\ni M$

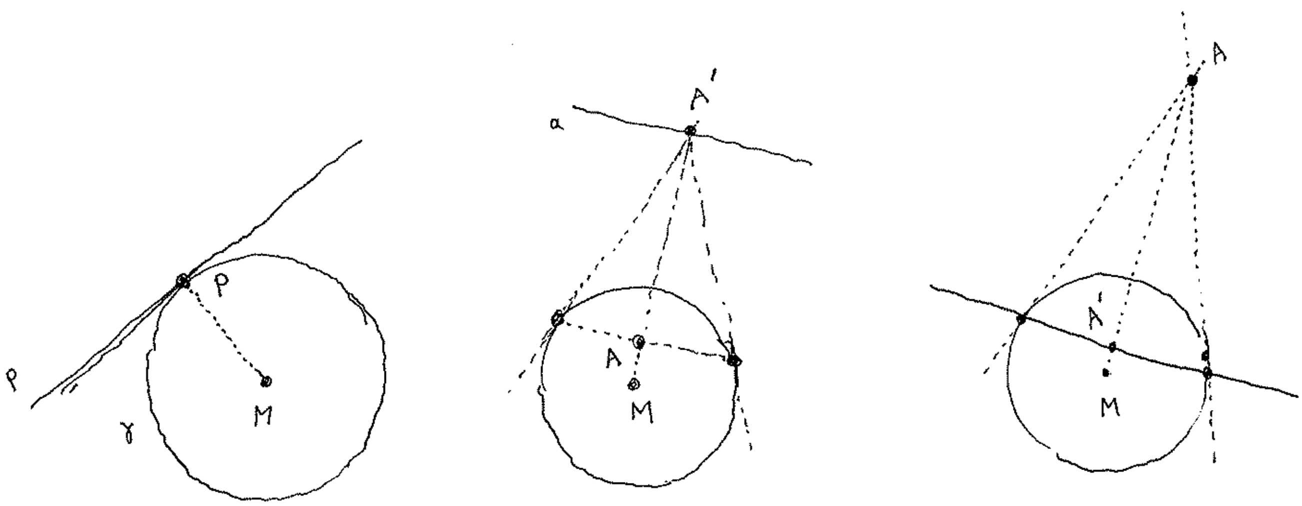
The traditional expressions are :

"A is the <u>pole</u> of a"

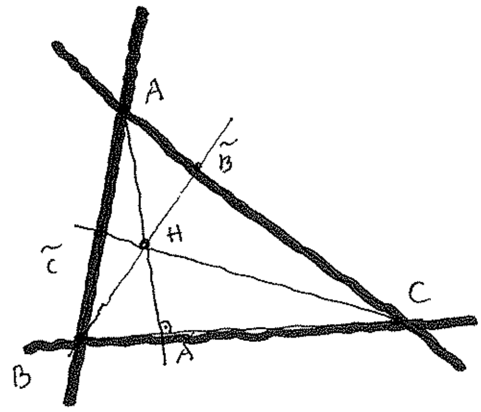
"a is the <u>polar</u> of A" .

If $k > 0$, then the circle of radius $\sqrt{k}$ is of fundamental importance in connection with $\text{Rec}_{M,k}$. Often one refers to $\text{Rec}_{M,k}$ as the "reciprocation in the circle γ" where γ is the circle of center M, radius $\sqrt{k}$.

With respect to a reciprocation in the circle γ each point $P \in \gamma$ is sent to the line p tangent to γ at P. The following are rather self evident configurations involving a point A and its polar a under the reciprocation in the circle γ



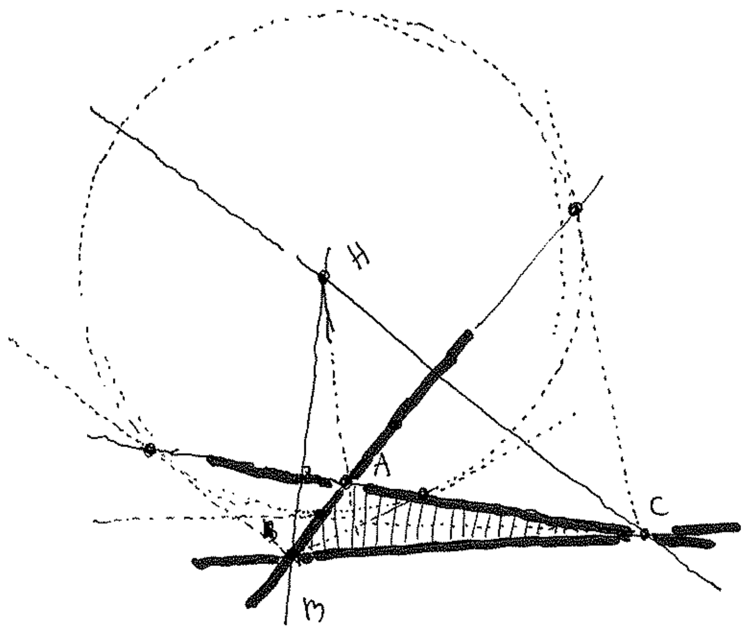
Even when $k<0$, it is a good idea to think of a reciprocation of power k as a "reciprocation in a circle", albeit an imaginary one.



<u>Example:</u> Let $\tilde{A}, \tilde{B}, \tilde{C}$ be the feet of the perpendiculars from A, B, C on BC, CA, AB.

Let $k = HA \cdot H\tilde{A} = HB \cdot H\tilde{B} = HC \cdot H\tilde{C}$.

Notice that the reciprocation $\mathrm{Rec}_{H,k}$ sends A, B, C into $a = BC$, $b = CA$, $c = AB$.



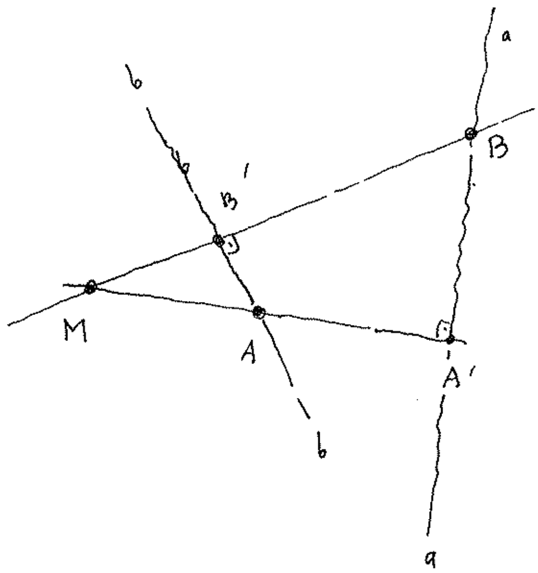
If ABC is obtuse, then $k>0$ and $\mathrm{Rec}_{H,k}$ may be consider as the reciprocation in the center of center H and radius $\sqrt{k}$. This circle is called the <u>polar circle</u> of the triangle ABC. (of [redacted] course, the polar circle is imaginary in an acute angled triangle...)

Simplest properties :

1. A reciprocation preserves the incidence relations in the following sense:

$$\boxed{A \in b \quad \text{iff} \quad B \in a}$$

to be quite explicit, let φ be the reciprocation.
We have $A \in b$ iff $\varphi^{-1}(b) \in \varphi(A)$



Proof: Consider $B \in a$. Observe that A, B, A', B' are concyclic hence $AB' \perp MB$ $\therefore$ $A \in b$.

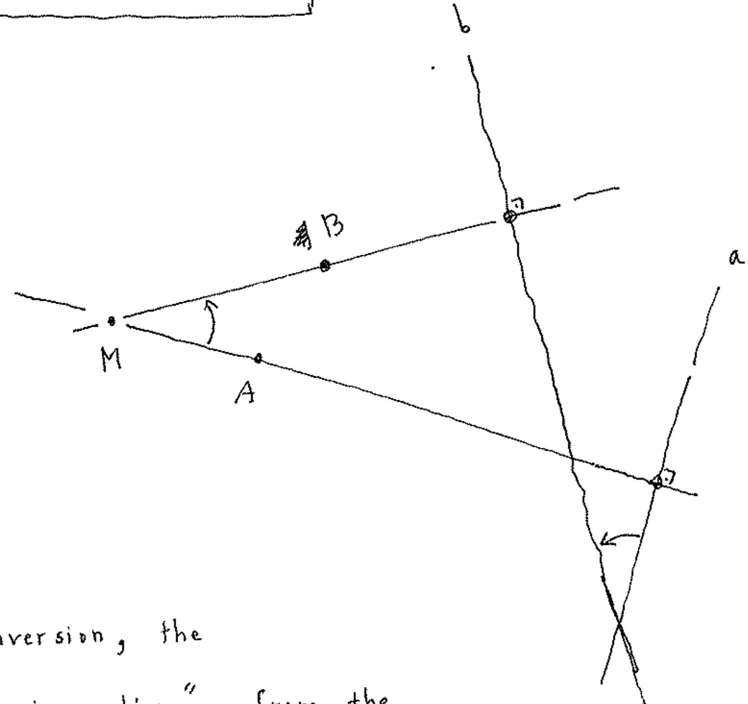
It is impossible to exaggerate the importance of this very simple fact: Under reciprocation (and its inverse!) collinear points turn into concurrent lines, concurrent lines turn into collinear points. As a consequence, duals of theorems involving solely incidence relations are valid. (For instance the Desargues theorem, the Pappus theorem!)

(For a reciprocation of center M ...)

2. Yet another powerful property of breathtaking simplicity:

$$\boxed{\measuredangle(MA, MB) = \measuredangle(a, b)}$$

Proof: Hardly necessary:



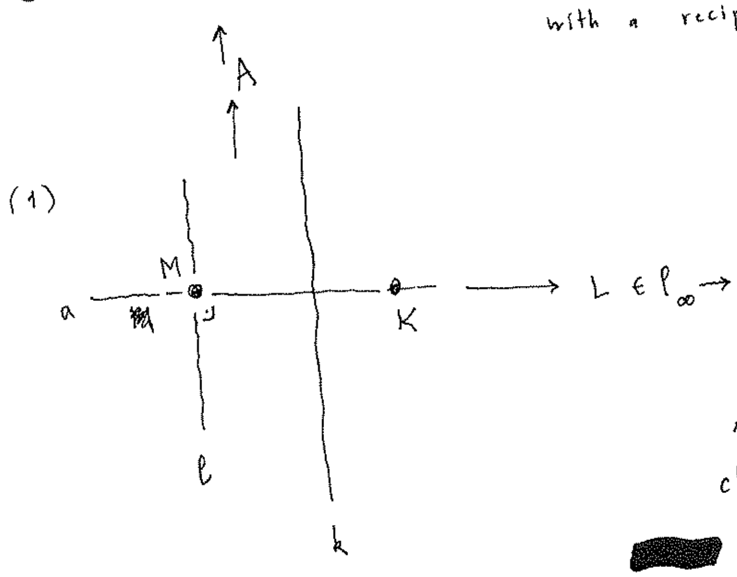
The Completion:

Just as in the case of inversion, the exclusion of the "centre of reciprocation" from the domain of definition of this transformation is an obvious defect. A "completion" is called for: Given $\varphi = Rec_{M,k}$ We introduce an "ideal" line ℓ_∞ (the "line at infinity") as the polar of M. This necessitates the following: each line through M is the polar of a(n ideal) point on ℓ_∞.

This is a perfect picture: The line at infinity ℓ_∞ is perfectly tangible as it consists of poles of lines through M.

The geometric artifice obtaine from $\mathbb{R}^2$ (as identified with the Euclidean plane) by augmenting it with a "line at infinity" [redacted] is called the <u>real</u> <u>projective</u> <u>plane</u>, $\boxed{\mathbb{RP}^2}$

Inherent in the very construction of $\mathbb{RP}^2$ ([redacted] that arises as a token of the presence of a reciprocation!) [redacted] are certain interesting incidence relations: (As above we are working with a reciprocation of center M!)



Let ℓ be a line through M. For any line $k \parallel \ell$

$MK \perp$ [redacted] k, ℓ

As $k \to \ell$ without changing its alignment [redacted] $K \to L$ the pole of ℓ.

Thus $L \in \ell_\infty$, the pole of M is the "point at infinity" which lies in a direction $\perp$ to ℓ!

So we understand L as a the point in which MK (which is $\perp$ to ℓ!) intersects ℓ_∞ .....!

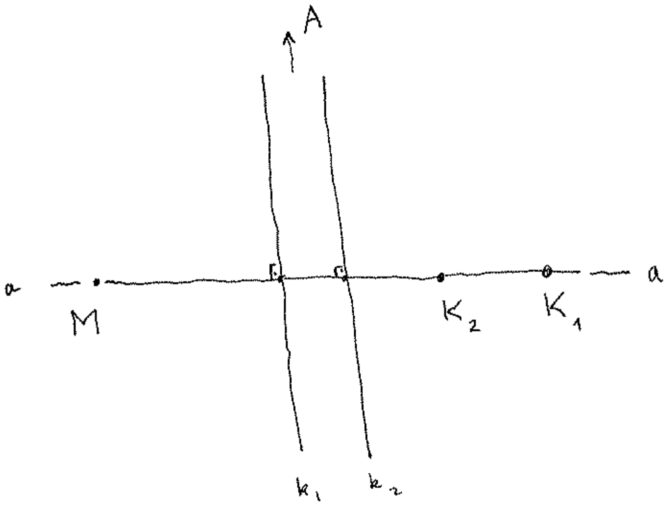
(2) [redacted] In (1) ~~A~~ letting [redacted] $a = MK$ it is seen that $A \in \underset{\ell_\infty}{\underset{\shortparallel}{m}} \cap k$. This goes to show that AM is the polar of $a \cdot \ell_\infty = L$ $\therefore$ AM is ℓ. Therefore $A = \ell \cap \ell_\infty$. Note that the parallel lines k, ℓ must ~~inters~~ intersect in A which is "at infinity".

in particular $A \in k$

in particular $A \in \ell$

(3) Any two parallel lines intersect in a "point at infinity"

The case (2) takes care of the situation where. If k_1, k_2 are parallel and do not incide in M, then $a = K_1 K_2$ is a line through M and hence $A \in \rho_\infty$ = polar of M lies on k_1 and k_2!



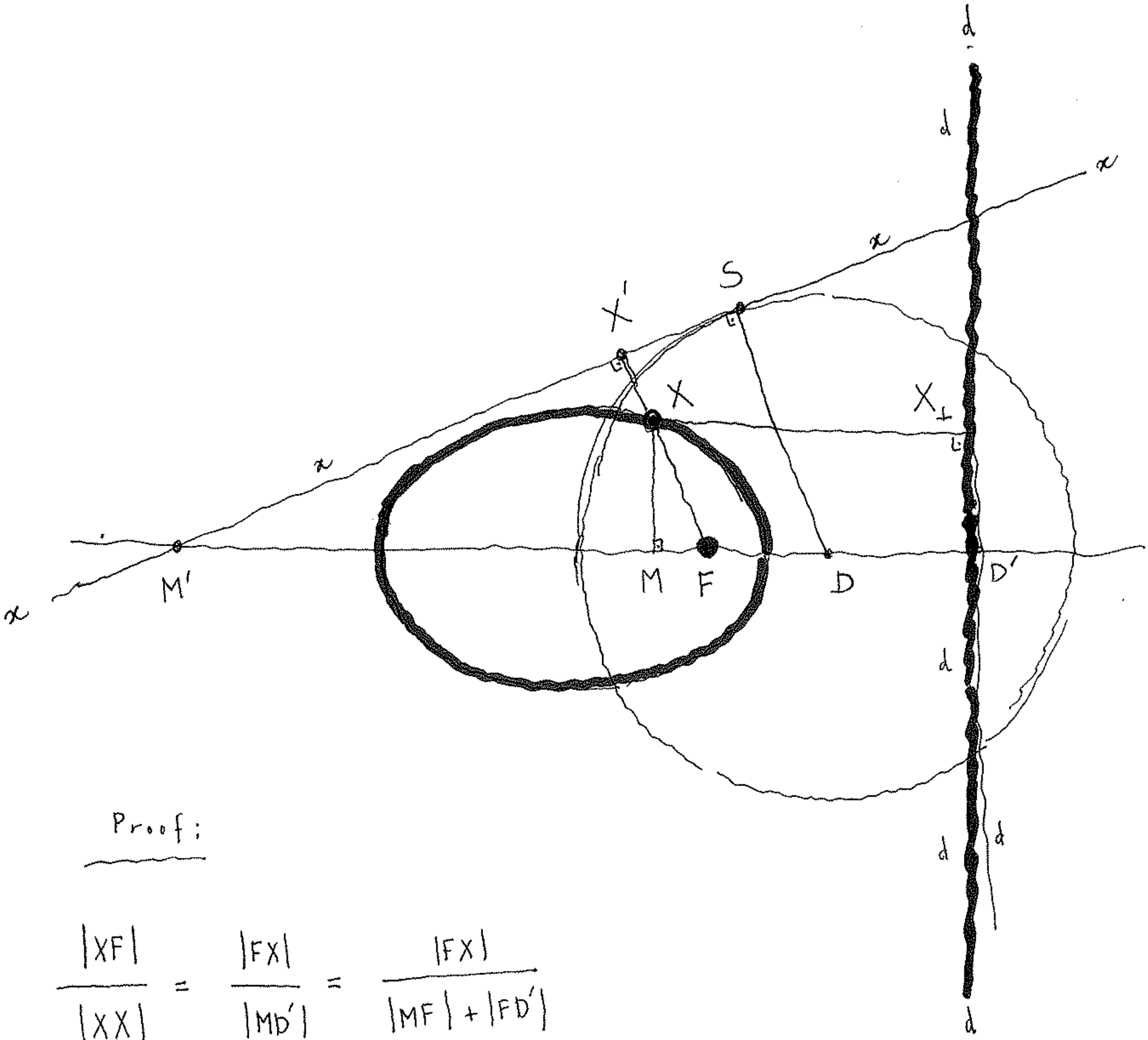
Remark: In $\mathbb{RP}^2$ any two distinct lines intersect in a unique point. In $\mathbb{RP}^2$ through any two distinct points there passes a unique line.

Theorem: Consider a circle γ of center D and radius r and a point F. Let $\varphi = Rec_{F,k}$ for some $k \in \mathbb{R} - \{0\}$.

[1] If $D = F$, then the poles of tangents to γ constitute a circle of center D and radius $\frac{|k|}{r}$

[2] If $D \neq F$, then the poles of the tangents to γ constitute a conic section of focus F, directrix d and eccentricity $\varepsilon = \frac{|FD|}{r}$.

Remark: The rather circumlocutious theorem above tells us in essence that circles turn into conic sections (and of course, vice versa) under reciprocations. The resulting conic section is a circle if $F = D$ (i.e. $\varepsilon = 0$!), an $\left\{\begin{array}{l}\text{ellipse}\\ \text{hyperbola}\\ \text{parabola}\end{array}\right\}$ if F is $\left\{\begin{array}{l}\text{inside}\\ \text{outside}\\ \text{on}\end{array}\right\}$ the circle in question...



Proof:

$$\frac{|XF|}{|XX_\perp|} = \frac{|FX|}{|MD'|} = \frac{|FX|}{|MF|+|FD'|}$$

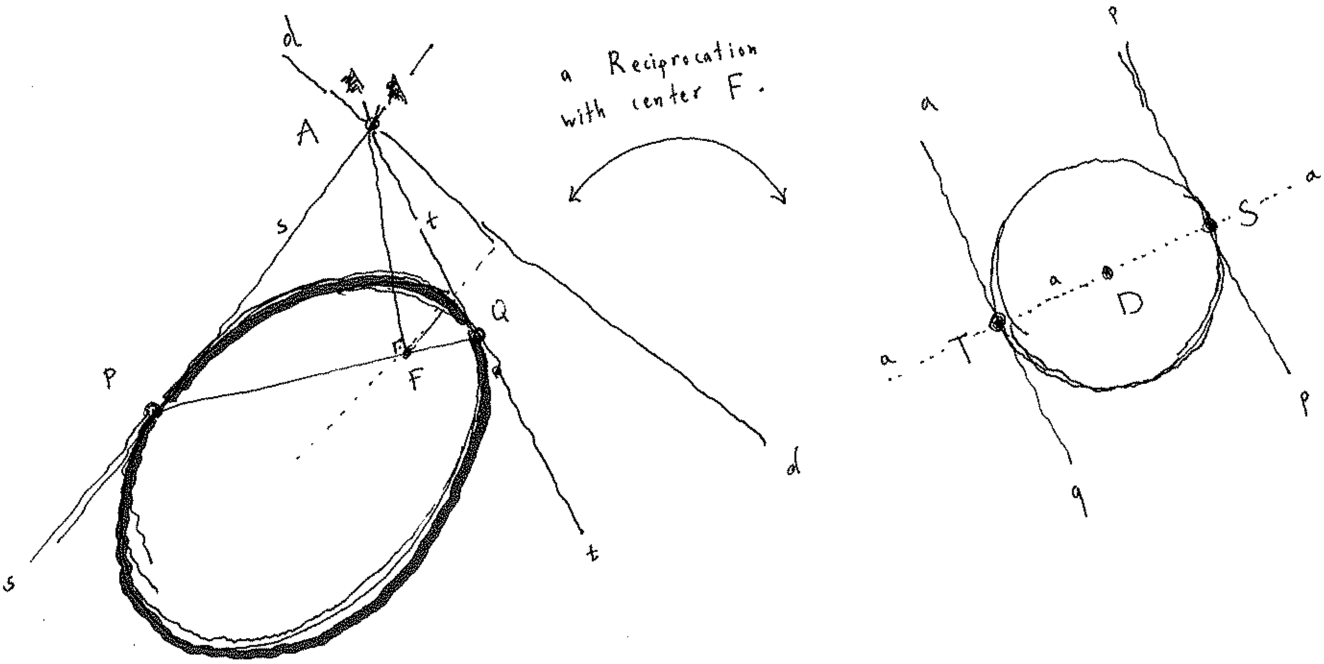
$$= \frac{\frac{|k|}{|FX'|}}{\frac{|k|}{|M'F|}+\frac{|k|}{|FD|}}$$

$FM'X' \cong$ [redacted] $DM'S$

$$= \frac{|FM'|\,|FD|}{|FX'|\left(|M'F|+|FD|\right)} = \frac{|FM'|}{|FX'|}\cdot\frac{|FD|}{|M'D|} = \frac{|DM'|}{|DS|}\cdot\frac{|FD|}{|M'D|} = \frac{|FD|}{r}$$

$r = |DS|$

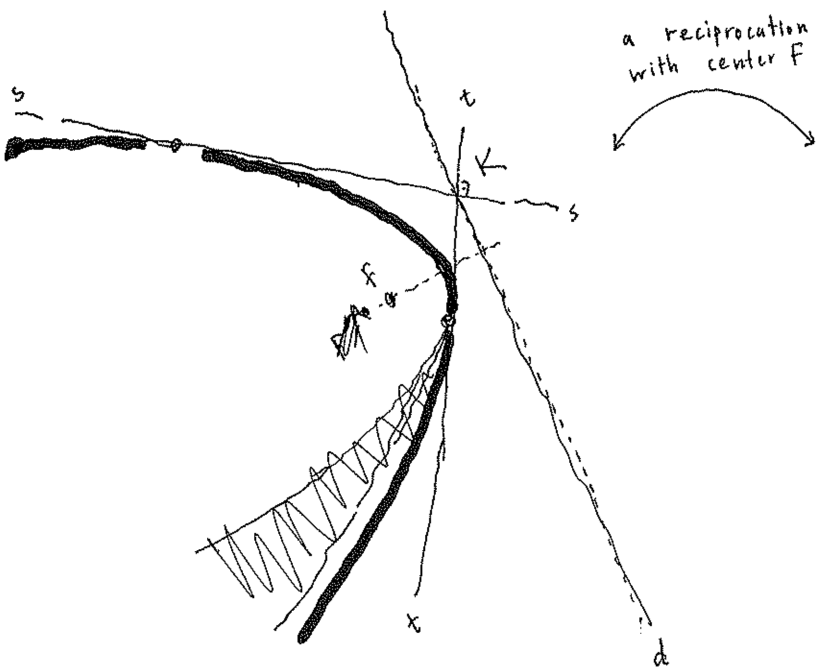
Application 1

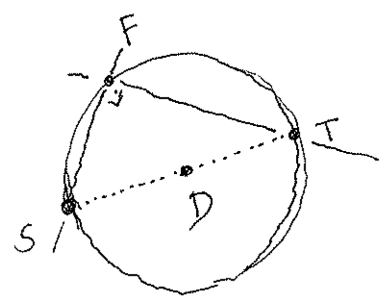


Since $p \parallel q$ we conclude that $F \in PQ$.

Clearly $a = ST \perp p$ hence $FA \perp FP$.

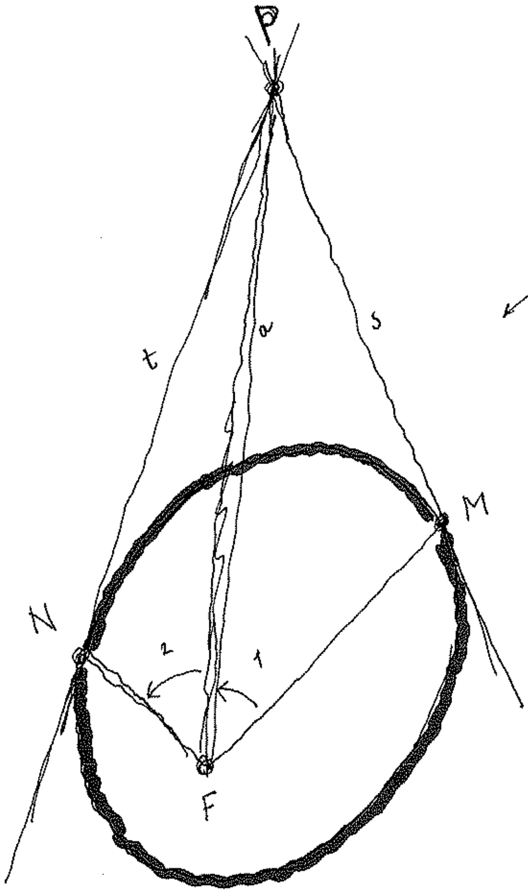
Application 2



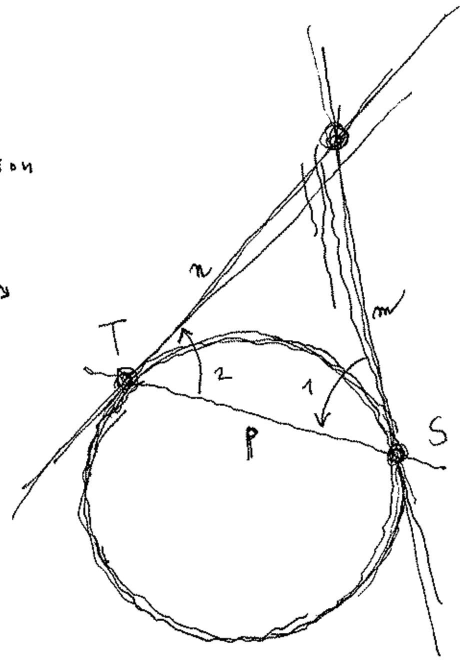


$s \perp t \text{ iff } \underbrace{D \in ST}_{FS \perp FT \text{ iff}} \text{ iff } K \in d.$

Application (3)



A reciprocation
of center F



As $\measuredangle(m, p) = \measuredangle(p, n)$

we obtain $\measuredangle(FM, FP) = \measuredangle(FP, FN)$

"Poncelet II".