Problems in Geometry (1)

1. Let ABC be a triangle with orthocenter H and let AH, BH, CH meet BC, CA, AB in $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, respectively.

(A) Prove that $HA \cdot H\widetilde{A} = HB \cdot H\widetilde{B} = HC \cdot H\widetilde{C}$. [1]

(B) If ABC is an acute angled triangle, prove that H is the incenter of $\widetilde{A}\widetilde{B}\widetilde{C}$. [2]

(C) What happens if A is an obtuse angle ?

2. Let ABC be a triangle with incenter I and excenters I_a, I_b, I_c.

(A) Prove that I is the orthocenter of the triangle $I_aI_bI_c$.

(B) Prove that the circumcircle of a triangle passes through the midpoint of a line segment joining any two of the excenters.

(C) Prove that the circumcircle of a triangle passes through the midpoint of a line segment joining the incenter with any one of the excenters.

3. In a triangle ABC with centroid G, let AG, BG, CG meet BC, CA, AB in A', B', C' respectively. Let the parallel to AB through C intersect $B'C'$ in K. Prove that

(A) AK is parallel to CC' and $|AK| = |CC'|$, $|A'K| = |BB'|$.

(B) Prove that the the line CA is the median of the triangle $AA'K$ through the point A.

(C) Prove that the length of the median of the triangle $AA'K$ through the point A is $3b/4$.

4. In a triangle ABC, let m_a, m_b, m_c denote the lengths of the medians through A, B, C respectively.

(A) Prove that $2m_a \le b + c$. [3]

(B) Prove that $m_a + m_b + m_c \le a + b + c$.

(C) Prove that

$$\frac{3}{4}\Big(a + b + c\Big) \le m_a + m_b + m_c .$$

Can any of these inequalities degenerate into an equality ? [4]

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[1]Note that the triangles $H\widetilde{A}C$ and $H\widetilde{C}A$ are similar.

[2]Note that $\widetilde{A}$, H, $\widetilde{B}$, C are concyclic and hence $< \widetilde{B}\widetilde{A}H =< \widetilde{B}CH = 90 - A$.

[3]Let A' be the midpoint of $[B, C]$. Let the point T be chosen on AA' such that A' is also the midpoint of $[A, T]$. Consider the paralelogram $ABTC$ and the triangle ATC.

[4]Remember the triangle $AA'K$ in Problem 3. Its sidelengths are m_a, m_b, m_c. What are the lengths of the medians of $AA'K$? .

5. Let ABC be a triangle with orthocenter H and centroid G. Let M_a, M_b, M_c be the centroids of HBC, HCA, HAB respectively.

(A) Prove that M_bM_c is parallel to BC. [5]

(B) Prove that G is the orthocenter of the triangle $M_aM_bM_c$.

6. In a triangle ABC, let k, l, m be the perpendicular bisectors of $[B, C]$, $[C, A]$, $[A, B]$. Given X_n for each $n \in \mathbb{Z}$ with $X_{3n} \in k$, $X_{3n+1} \in l$, $X_{3n+2} \in m$ such that $A \in X_{3n+1}X_{3n+2}$, $B \in X_{3n+2}X_{3n+3}$, $C \in X_{3n}X_{3n+1}$, prove that $X_n = X_{n+6}$ for all $n \in \mathbb{Z}$. [6]

7. Consider a triangle ABC with incenter I. Given X_n for each $n \in \mathbb{Z}$ with $X_{3n} \in BC$, $X_{3n+1} \in CA$, $X_{3n+2} \in AB$ such that $IC \perp X_{3n}X_{3n+1}$, $IA \perp X_{3n+1}X_{3n+2}$, $IB \perp X_{3n+2}X_{3n+3}$ for all $n \in \mathbb{Z}$, prove that $X_n = X_{n+6}$ for all $n \in \mathbb{Z}$.

8. Let ABC be a triangle with orthocenter H and let AH, BH, CH meet BC, CA, AB in $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, respectively. Let the feet of perpendiculars from $\widetilde{A}$ on CA and AB be X and X' respectively. Similarly let the feet of the perpendiculars from $\widetilde{B}$ on AB and BC be Y and Y', of $\widetilde{C}$ on BC and CA be Z and Z' respectively.

(A) Prove that YZ' is parallel to BC . What can you say about ZX' and XY' ?

(B) Prove that X, X', B, C are concyclic. What can you say about Y, Y', C, A and Z, Z', A, B ?

(C) Prove that $|XX'| = |YY'| = |ZZ'|$.

(D) Prove that X, Y, Z, X', Y', Z' are concyclic.[7]

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[5] Remember that in the triangle HBC the point M_b divides $[HA']$ in the ratio $-1 : 2$. Similar observations in ABC, HCA, HAB .

[6] Start with $X_0 \in k$ and construct $X_1 \in l$, $X_2 \in m$ and so on. Let $\alpha =< X_0CB$. Compute the values of $< X_1AC$, $< X_2BA$, $< X_3CB$, $< X_4AC$, $< X_5BA$. What do you notice ?

[7] The circle containing the points X, Y, Z, X', Y', Z' is referred to as the **Taylor Circle** and its discovery is attributed to H. M. Taylor (1842 - 1927) about whom I could find no information.