Problems in Geometry (2)

1. Given a rectangle ABCD prove that $|PA|^2+|PC|^2 = |PB|^2+|PD|^2$ for any point P.

2. Given an acute angled positively oriented triangle ABC, let O and H be the circumcenter and the orthocenter of ABC, respectively. Let AH intersect BC in $\tilde{A}$.

(A) Prove that $|AH| = 2R\cos A$.[1]

(B) Prove that $|\tilde{A}H| = 2R\cos B\cos C$.[2]

(C) Prove that OH is perpendicular to AH iff $\tan B\tan C = 3$.[3]

3. Let P be a point on the circumcircle of a triangle ABC such that AP is the internal bisector of the angle A . Prove that the following conditions are equivalent [4] [5]

(i) $|PA| = 2|PB|$, (ii) $2\sin\left(\frac{A}{2}\right) = \cos\left(\frac{B-C}{2}\right)$, (iii) $2a = b + c$.

4. In a triangle ABC prove that the following assertions are equivalent:

(i) $A = 90^\circ$, (ii) $r + r_b + r_c = r_a$, (iii) $r_b r_c = r r_a$. [6]

5. Generalise the previous result by proving that in a triangle ABC the following conditions are equivalent:

(i) $\cos A = \dfrac{\alpha-1}{\alpha+1}$, (ii) $r_b + r_c = \alpha(r_a - r)$, (iii) $r_b r_c = \alpha r r_a$.

where $\alpha \neq -1$ is a constant. Compute the angle A in a triangle ABC in which

$$3r + r_b + r_c = 3r_a \ .$$

6. Given a counterclockwise oriented right triangle ABC with $< A = \pi/2$, consider the counterclockwise oriented squares $CPQA$, $ARSB$, $BTUC$. Let $m = |PU|$, $n = |ST|$ and prove that $m^2 - n^2 = 3(b^2 - c^2)$.[7]

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[1] Note that $|AH| = 2|OA'|$. Consider the triangle BOA' .

[2] Consider the triangle BOA' .

[3] Note that $OH \perp AH$ iff $|\tilde{A}H| = |OA'|$.

[4] Apply the sine rule in the triangle ABP .

[5] Note that $B = \left(\frac{B+C}{2}\right) + \left(\frac{B-C}{2}\right)$, $C = \left(\frac{B+C}{2}\right) - \left(\frac{B-C}{2}\right)$, $A = \frac{\pi}{2} - \left(\frac{B+C}{2}\right)$.

[6] Try to obtain $a^2 = b^2 + c^2$ and employ The theorem of Pythagoras.

[7] Apply the cosine rule in the triangles PCU and TBS .

7. In a triangle ABC prove that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$.

8. In a triangle ABC prove that $\Delta = \frac{4}{3}\sqrt{\mu(\mu - m_a)(\mu - m_b)(\mu - m_c)}$ where $2\mu = m_a + m_b + m_c$. [8]

9. Let $ABCD$ be a convex quadrilateral with side lenths $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$ and diagonal of length $m = |AC|$. Let s be the semiperimeter of $ABCD$ defined by $2s = a + b + c + d$. Let Δ_{XYZ} stand for the area of a triangle XYZ.

(A) Prove that $16\Delta_{ABC}^2 = 4a^2b^2 - (a^2 + b^2 - m^2)^2$.[9]

(B) Prove that $16\Delta_{ACD}^2 = 4c^2d^2 - (c^2 + d^2 - m^2)^2$.

(C) Prove that $32\Delta_{ABC}\Delta_{ACD} = 2(a^2 + b^2 - m^2)(c^2 + d^2 - m^2) - 8abcd\cos(A + C)$.[10]

(D) Let Δ stand for the area of the quadrilateral. Prove that

$$\Delta^2 = (s - a)(s - b)(s - c)(s - d) - abcd\cos^2\left(\frac{A + C}{2}\right) .$$ [11], [12]

(E) In ancient times people were led by rough experiments to believe that the area of the quadrilateral $ABCD$ was given by the formula

$$\Delta = \frac{1}{4}(a + c)(b + d) \quad .$$

Prove that this formula is correct only for rectangles and gives rise to too large a value for the area otherwise !

10. Let $ABCD$ be a convex quadrilateral of area Δ with side lenths $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$ and diagonals of length $m = |AC|$, $n = |BD|$.

(A) Prove that $16\Delta^2 = 4m^2n^2\sin^2\theta$.

(B) Prove that $16\Delta^2 = 4m^2n^2 - (a^2 - b^2 + c^2 - d^2)^2$. [13]

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[8] Consider the triangle $AA'K$ presented in the Problems (1) Pr. 5 . Compare its area with the area of ABC.

[9] Start with $2\Delta_{ABC} = ab\sin A$, square both sides and employ the cosine rule in ABC .

[10] Start with $4\Delta_{ABC}\Delta_{ACD} = abcd\sin A\sin C$, replace $\sin A\sin C$ by $-\cos A + C + \cos A\cos C$ and employ the cosine rule in ABC and ACD .

[11] Start with $\Delta^2 = \Delta_{ABC}^2 + \Delta_{ACD}^2 + 2\Delta_{ABC}\Delta_{ACD}$ and compute.

[12] This formula is usually ascribed to the legendary Indian geometer Brahmagupta (~11th century). Notice how it reduces to the beautiful formula $\Delta = \sqrt{(s - a)(s - b)(s - c)(s - d)}$ for inscriptible quadrangles.

[13] Let $x = |AS|$, $y = |BS|$, $z = |CS|$, $t = |DS|$ and note $16\Delta^2 = 4m^2n^2\sin^2\theta = 4m^2n^2\left(1 - \cos^2\theta\right)$ and $m = x + z$, $n = y + t$, $2xy\cos\theta = x^2 + y^2 - a^2$, $2xt\cos(\pi - \theta) = x^2 + t^2 - d^2$ etc where $\theta = \angle BAD$. This formula usually ascribed to a mathematician named Bretschneider about whom, I regret to say that I know nothing.