# Problems in Geometry (3)

**1.** Given a triangle $ABC$, let $A'$ be the midpoint of $[B, C]$ and consider $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$ such that $BY$ and $CZ$ meet on $AA'$. Prove that $YZ$ is parallel to $BC$.[1]

**2.** Given a triangle $ABC$, consider $P \in BC - \{B, C\}$, $Q \in CA - \{C, A\}$, $R \in AB - \{A, B\}$ such that $AP$, $BQ$, $CR$ are concurrent. Let $QR$, $RP$, $PQ$ meet $BC$, $CA$, $AB$ in $X$, $Y$, $Z$ respectively. Prove that

(A) $X$, $Y$, $Z$ are collinear.

(B) $AP$, $BY$, $CZ$ are concurrent or parallel.

**3.** Prove that in a triangle $ABC$ the altitude through $A$, the median through $B$ and the internal angle bisector through $C$ are concurrent iff $\sin A = \cos B \tan C$ .

**4.** Given a quadrilateral $ABCD$, consider $X \in AB - \{A, B\}$, $Y \in BC - \{B, C\}$, $Z \in CD - \{C, D\}$, $T \in DA - \{D, A\}$.

(A) Prove that if $X$, $Y$, $Z$, $T$ are collinear then

$$\frac{XA}{XB} \cdot \frac{YB}{YC} \cdot \frac{ZC}{ZD} \cdot \frac{TD}{TA} = 1 \quad .$$

(B) Is the converse true ?

**5.** Given a triangle $ABC$, let $A'$, $B'$, $C'$ be midpoints of $[B, C]$, $[C, A]$, $[A, B]$ respectively. Consider $L \in BC - \{B, C\}$, $M \in CA - \{C, A\}$, $N \in AB - \{A, B\}$ such that $AL$, $BM$, $CN$ are concurrent. If $P$, $Q$, $R$ are midpoints of $AL$ $BM$, $CN$, respectively prove that $PA'$, $QB'$, $RC'$ are concurrent. [2]

**6.** In a triangle $ABC$ , let $J_a \in BC$, $J_b \in CA$, $J_c \in AB$ be chosen such that $AJ_a$, $BJ_b$, $CJ_c$ be the respective internal angle bisectors at $A$, $B$, $C$ . Let the perpendicular bisectors of $[A, J_a]$, $[B, J_b]$, $[C, J_c]$ intersect $BC$, $CA$, $AB$ in $X, Y, Z$ respectively. Prove that $X, Y, Z$ are collinear.

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[1] It is sufficient to show that $\frac{YC}{YA} = \frac{ZB}{ZA}$ !

[2] Observe that $P$, $Q$, $R$ lie on the sides of the "medial triangle" $A'B'C'$ .

**7.** Consider $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$. Let $X' \in BC - \{B, C\}$, $Y' \in CA - \{C, A\}$, $Z' \in AB - \{A, B\}$. be reflections of $X, Y, Z$ in the respective midpoints of $[B, C]$ $[C, A]$, $[A, B]$.

(A) Prove that $AX, BY, CZ$ are concurrent or parallel iff $AX', BY', CZ'$ are concurrent or parallel. [3]

(B) Prove that $X'$, $Y'$, $Z'$ are collinear iff $X, Y, Z$ are collinear. [4]

(C) Prove that the areas of the triangles $XYZ$ and $X'Y'Z'$ are equal.

**8.** (A) Let the incircle of the triangle $ABC$ touch $BC$, $CA$, $AB$ in $S$, $T$, $U$ respectively. Prove that $AS$, $BT$, $CU$ concur. [5]

(B) Let the respective excircles opposite $A$, and $B$ and $C$ touch $BC$, $CA$, $AB$ in $S_a$, $T_a$, $U_a$ and $S_b$, $T_b$, $U_b$ and $S_c$, $T_c$, $U_c$ respectively. Prove that $AS_a$, $BT_b$, $CU_c$ concur. [6]

(C) Prove that $N$ is the isotomic conjugate of $\Gamma$.

**9.** Consider a triangle $ABC$ and squares $BPQC$, $CRSA$, $AUVB$ which have the same orientation as $ABC$. Let $M_a$ $M_b$ $M_c$ be the respective centres of the squares $BPQC$, $CRSA$, $AUVB$. Prove that $AM_a$, $BM_b$, $CM_c$ are concurrent.

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[3] If $AX, BY, CZ$ concur in $M$ (an ordinary point or a point "at infinity") and $AX', BY', CZ'$ concur in $M^\circ$ (again an ordinary point or a point "at infinity"), the points $M$ and $M^\circ$ are said to be *isotomic conjugates* of one another with respect to the triangle $ABC$.

[4] If $X, Y, Z$ lie on the line $m$ and $X', Y', Z'$ lie on the line $m^\circ$ the lines $m$ and $m^\circ$ are said to be *isotomic conjugates* of one another with respect to the triangle $ABC$.

[5] The point in which $AS$, $BT$, $CU$ concur is a remarkable point of $ABC$, called the *Gergonne point* of $ABC$. It is usually denoted by $\Gamma$.

[6] The point in which $AS_a$, $BT_b$, $CU_c$ concur is a remarkable point of $ABC$, called the *Nagel point* of $ABC$. It is usually denoted by $N$.