Problems in Geometry (6)

1. Characterise the set of points whereof the power with respect to a fixed circle is a constant.

2. Compute the power of the point $P(x_0, y_0)$ with respect to the circle $x^2+y^2+2ax+2by+c=0$. Write down the equation of the radical axis of the circles $x^2+y^2+2a_1x+2b_1y+c_1=0$ and $x^2+y^2+2a_2x+2b_2y+c_2=0$.

3. (A) Consider fixed non-concentric circles Γ_1, Γ_2. Prove that the set of points whereof the powers with respect to Γ_1 and Γ_2 differ by a constant is a line parallel to the radical axis of Γ_1 and Γ_2.

(B) Prove that the set of points whereof the ratio of powers with respect to two given circles Γ_1, Γ_2 is a constant is a circle the centre of which lies on the line joining centres of Γ_1, Γ_2.

4. (A) Given two circles which are orthogonal to one another, is it possible for the centre of one to lie on the other ?

(B) Let each one of the circles C_1, C_2 intersect the circles Γ_1, Γ_2 orthogonally. Prove that the radical axis of C_1, C_2 is the line joining the centres of Γ_1, Γ_2 .

5. Consider circles Γ and Δ which are tangent to the line k at the points $C, D \in k$ respectively. Prove that the radical axis of Γ and Δ bisects the line segment $[C, D]$.

6. Given a circle γ and coaxial [1] circles Γ_1, Γ_2, Γ_3, prove that the radical axes of γ and Γ_1, γ and Γ_2, γ and Γ_3 are concurrent or parallel.

7. Given triangle ABC, let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be the feet of the perpendiculars from A, B, C on BC, CA, AB respectively. Let BC and $\tilde{B}\tilde{C}$, CA and $\tilde{C}\tilde{A}$, AB and $\tilde{A}\tilde{B}$ meet in X, Y, Z respectively.

(A) Prove that X, Y, Z are collinear by employing the theorems of Menelaus and Ceva.

(B) Prove the same result by demonstrating that X, Y, Z lie on the radical axis of the circumcircle and the 9-point-circle of ABC. [2]

(C) Prove that the line containing X, Y, Z is perpendicular to the Euler line of ABC.

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[1] By this expression it is meant that the three pairs of circles Γ_1 and Γ_2, Γ_2 and Γ_3, Γ_3 and Γ_1 have the same radical axis.

[2] Consider the circle with diameter $[BC]$.

8. Let (I) touch BC, CA, AB in S, T, U respectively. Let (I_a) and (I_b) and (I_c) touch BC, CA, AB in S_a, T_a, U_a and S_b, T_b, U_b and S_c, T_c, U_c respectively. Let Y, Z be the respective midpoints of the line segments $[T_b, T_c]$, $[U_b, U_c]$.

(A) Suppose that (I_b) and (I_c) are not congruent. (Y and Z are distinct !) Employ the thorem of Menelaus to prove that YZ bisects the line segment $[B, C]$.

(B) Compute the powers of Y, Z with respect to the circles (I_b) and (I_c).

(C) Suppose that (I_b) and (I_c) are not congruent. Prove without using the thorem of Menelaus that YZ bisects the line segments $[B, C]$ and $[S_b, S_c]$.

(D) Let A', B', C', be the midpoints of $[B, C]$, $[C, A]$ $[A, B]$. Prove that the incenter of $A'B'C'$ is the radical center of (I_a), (I_b), (I_c).

$\diamond$ The incenter of $A'B'C'$ is called the *Spieker point* of ABC, denoted by Σ. $\diamond$

(E) Prove that Σ is the center of gravity of the set $[B, C] \cup [C, A] \cup [A, B]$.[3]

9. (A) Given a triangle UVW and points $P \in UW$, $Q \in UV$, let β, γ be circles with respective diameters $[V, P]$, $[W, Q]$. Prove that the orthocenter of UVW lies on the radical axis of β and γ.

(B) Consider a quadrangle $ABCD$ with $AC \cap BD = \{F\}$. Let K, L be the orthocenters of AFD, BFC, let S, T be the midpoints of $[A, B]$, $[C, D]$ respectively. Prove that $KL \perp ST$.[4]

(C) Let Y, Z be the respective centroids of CFD, AFB. prove that $KL \perp YZ$.

[3] Concentrate each edge of the triangle into a point lying at its midpoint. Let that point have a mass proportional to the length of the edge.

[4] Consider the circles of diameters $[A, B]$,$[C, D]$.