Problems in Geometry (7)

1. Let φ be an ellipse with foci F, F'. If the tangent lines at $A, B \in \varphi$ intersect in P , prove that $PF \perp FA$ iff $F \in AB$. [1]

2. Let φ be an ellipse with foci F, F'. Let t, t' be the tangents to φ which intersect in P . If H, H' are respectively the feet of the perpendiculars from F on t, t', prove that PF' is perpendicular to HH'. [2]

3. Let φ be an ellipse of foci F, F'. Let $\varphi \cap FF' = \{A, A'\}$. Let ℓ, ℓ' be the tangents to φ. at A, A'. For any tangent t to φ at $M \in \varphi$ let $t \cap \ell = \{P\}, t \cap \ell' = \{P'\}$.

(A) Prove that $< (FP, FP') =< (F'P, F'P') = \pi/2$ [3] .

(B) Prove that $FP, F'P'$ intersect on the normal to φ at M. [4]

4. Let φ be an ellipse of foci F, F'. Consider $X \in \varphi$. Let Q, Q' be the points in which the normal of φ at X intersects the perpendiculars to XF, XF' erected at F, F' respectively. Prove that, the perpendicular bisector of $[FF']$ bisects $[Q, Q']$.

5. Consider an ellipse φ with foci F, F'. Let a line through F meet φ in X, X'. If the normals to φ at X, X' intersect in N, prove that the parallel to FF' through N bisects $[XX']$. [5]

6. (A) Let φ be an ellipse. Prove that the set of points from which the tangents to φ are perpendicular to one another, is a circle. [6] [7]

(B) Let φ, ψ be ellipses with common foci. Prove that the locus of the point of intersection of tangents to φ, ψ which are perpendicular to each other, is a circle.

---

[1] Employ the second theorem of Poncelet.

[2] P, F, H, H' are concyclic. Employ the first theorem of Poncelet.

[3] Apply the second theorem of Poncelet to the tangent lines t, l first and then to the tangent lines t, l' .

[4] The normal at M is the internal bisector of FMF'. $FP, F'P'$ are external bisectors of what ?

[5] Use the characterisation III to construct the normals.

[6] The Monge circle.

[7] Use the characterisation III and remember that for any rectangle $ABCD$

$$|PA|^2 + |PC|^2 = |PB|^2 + |PD|^2$$

holds for any point P .

7. Notice that Characterisation II is the same as Characterisation III for a parabola.

(A) Describe tangents to a parabola by means of Characterisation II. [8]

(B) State and prove the first theorem of Poncelet for a parabola.

(C) State and prove the second theorem of Poncelet for a parabola.

(D) On a parabola of focus F, consider points X, Y such that $F \in XY$. Let the tangents to the parabola at X, Y intersect in P . Prove that the following assertions are equivalent :

$$\begin{array}{rl} \text{(i)} & F \in XY \\ \text{(ii)} & FP \perp FX \\ \text{(iii)} & PX \perp PY \end{array}$$

8. (A) Consider a line d and a point $F \notin d$. Let circles ξ, η through F touch d at X, Y respectively and $\xi \cap \eta = \{F, M\}$. Prove that FM bisects $[XY]$.

(B) Prove that in a parabola the reflection of the focus in a tangent lies on the directrix.

(C) Consider a parabola φ, points $A, B \in \varphi$. Prove that the tangents to φ at A, B intersect on the directrix of φ iff AB goes through the focus of φ.

[8]Precisely, you are expected to describe tangent lines in terms of the focus and the directrix.