Problems in Geometry (8)

1. (A) What is the image of the line $2x+y=2$ under TR_{u} where $\mathsf{u}=[\,-1,3\,]$?

(B) What is the image of the line $x=1$ under $\mathrm{Rot}_{(0,0),\pi/4}$?

(C) What is the image of the line $x+y=1$ under Ref_k , where k is the line $y=2x$?[1]

(D) Let P, Q be distinct points. Prove that half-turns in P and Q together give rise to a translation. Which ?

(E) What is the image of the circle $x^2+y^2-2x-8=0$ under $\mathrm{HT}_{(5,1)}$?

(F) What is the image of the circle $x^2+y^2-2y-3=0$ under $\mathrm{Hom}_{(2,1),3}$?

(G) What is the image of the line $x+2y-5=0$ under $\mathrm{Hom}_{(1,1),3}$?

2. Consider a triangle ABC and equilateral traiangles BUC, CVA, AWB which have the same orientation as ABC. Let M_a, M_b, M_c be the respective centres of the equilateral triangles BUC, CVA, AWB.

(A) Prove that $\mathrm{Rot}_{M_a,2\pi/3}\circ\mathrm{Rot}_{M_b,2\pi/3}=\mathrm{Rot}_{M_c,4\pi/3}$.

(B) Prove that $M_aM_bM_c$ is an equilateral triangle. [2]

3. Consider a convex positively oriented quadrangle $ABCD$ with positively oriented equilateral triangles BAP, CBQ, DCR, ADS, constructed on its sides. Let

$$\alpha=\mathrm{Rot}_{P,\pi/3}\circ\mathrm{Rot}_{Q,\pi/3}\circ\mathrm{Rot}_{R,\pi/3}$$

$$\beta=\mathrm{Rot}_{Q,\pi/3}\circ\mathrm{Rot}_{R,\pi/3})\circ\mathrm{Rot}_{S,\pi/3}$$

(A) Prove that α and β are halfturns. Find their fixed points.

(B) Let A' be the image of A under $\beta\circ\alpha$. What can you say about the quadrangle $AA'BD$?

(C) Consider the point J such that JQR be a positively oriented isosceles triangle with $\angle QJR=2\pi/3$. If K is the midpoint of $[A,D]$ evaluate the angles $\angle PKJ$, $\angle KJP$, $\angle JPK$.

4. Consider a convex positively oriented pentagon $ABCDE$ with positively oriented squares $BAPQ$, $CBRS$, $DCTU$, $EDVW$, $AEXY$ constructed on its sides with respective centers M_a, M_b, M_c, M_d, M_e.

[1] Obviously the angle that the first line makes with the horizontal is $\arctan 2$. Compute the angle which the image of k makes with the horizontal.

[2] A romantically resilient rather than historically well-founded tradition ascribes this result to Napoléon Bonaparte !

(A) What is the image of A under the isometry $\Psi = \mathrm{Rot}_{M_d,-\pi/2} \circ \mathrm{Rot}_{M_c,-\pi/2} \circ \mathrm{Rot}_{M_b,-\pi/2} \circ \mathrm{Rot}_{M_a,-\pi/2}$?

(B) What is the image of Y under Ψ ?

5. Consider a triangle ABC and squares $BPQC$, $CRSA$, $AUVB$ which have the same orientation as ABC. Let M_a M_b M_c be the respective centres of the squares $BPQC$, $CRSA$, $AUVB$.

(A) Prove that $|AM_a| = |M_bM_c|$.

(B) Prove that AM_a is perpendicular to M_bM_c . [3]

(C) Prove that AM_a,BM_b, CM_c are concurrent.

7. Given distinct points X, Y, Z let $\mathrm{Hom}_{X,\lambda}(B) = C$, $\mathrm{Hom}_{Y,\mu}(C) = A$, $\mathrm{Hom}_{Z,\nu}(A) = B$ where $\lambda, \mu, \nu \notin \{0, 1\}$.

(A)Write down and prove a necessary and sufficient condition in terms of λ, μ, ν for X, Y, Z to be collinear. [4]

(B) State and prove the analogous condition for AX, BY, CZ to be concurrent.

8. Let ABC be a triangle with orthocenter H, centroid G, and circumcircle O.

(A) Prove that the respective circumcenters O_a, O_b, O_c of the triangles HBC, HCA, HAB are the reflections of O in BC, CA, AB respectively.

(B) Prove that a half-turn sends $ABCH$ into $O_aO_bO_cO$. Which ?

9. Let the circles Γ, Γ' of respective centers O, O' and points A, A' be given. Prove that it is in general possible to construct points $B \in \Gamma, A' \in \Gamma'$ such that AA' is parallel to BB' and $|AA'| = |BB'|$. [5]

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[3] Consider rotations $\mathrm{Rot}_{M_a,\pi/2}$ and $\mathrm{Rot}_{M_c,\pi/2}$ and notice that they combine to give rise to a halfturn about the midpoint K of $[A, C]$. Argue that KM_aM_c is a right isosceles triangle and argue further with $\mathrm{Rot}_{K,\pi/2}$.

[4] Theorem of Menelaus !

[5] Consider the effect of TR_{u} where $\mathsf{u} = \overrightarrow{AA'}$ on Γ .