Note Title 13.02.2020

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ODTÜ Matematik Bölümü

Math 709 - General Topology

- Differentiable Manifolds
- Intersection Theory
- Vector Bundles
- Characteristic Classes and some applications.

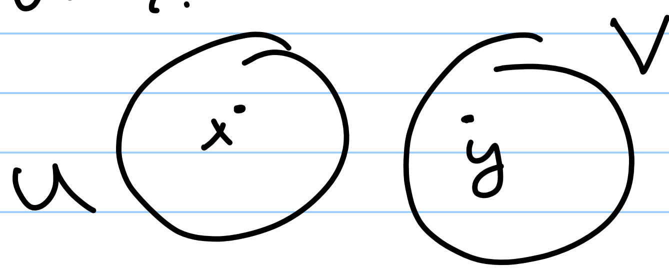
Book: Türevlenebilir manifoldlara giriş

ODTÜDEN

Differentiable Manifolds

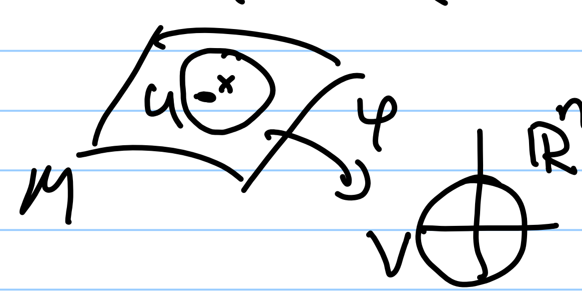
Definition: A topological manifold is a Hausdorff and second countable topological space, which is locally Euclidean.

Hausdorff $x, y \in X$, $x \neq y \Rightarrow \exists\, U, V \subseteq X$ open subsets such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.



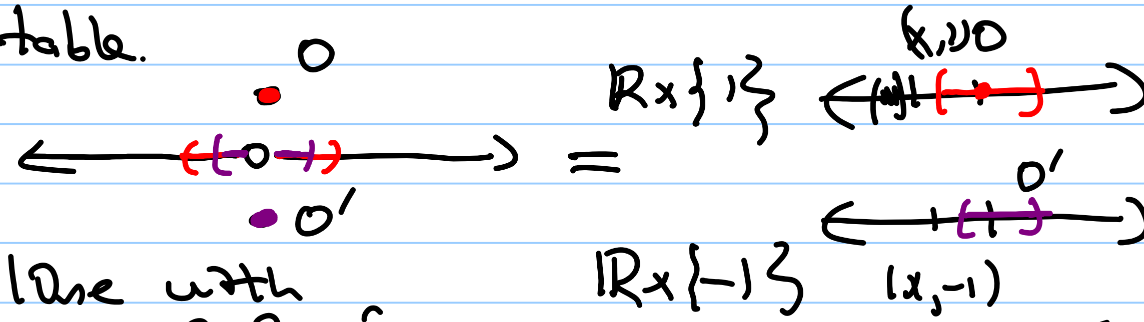
Second Countable: X has a countable basis $\mathcal{B}$.

Locally Euclidean: $x \in X$, $\varphi : U \to V$ homeomorphism s.t. $x \in U \subseteq X$ and $V \subseteq \mathbb{R}^n$ open subset.



Remark: In order to embedd a manifold into some Euclidean space we must require that it is Hausdorff and second countable.

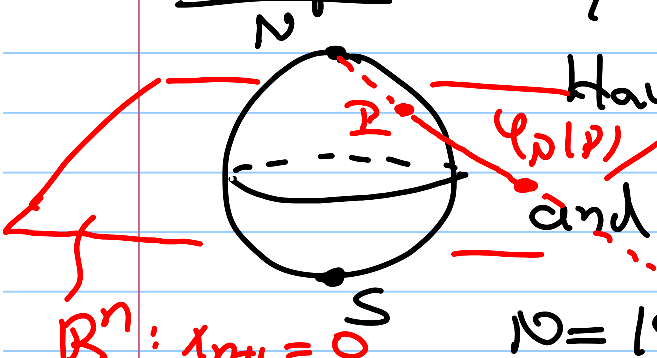
Ex:



The line with double origin is locally Euclidean but not Hausdorff.

$(x,1)\sim(x,-1)$ if $x\neq 0$

Example $S^n=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1}\mid x_1^2+\dots+x_{n+1}^2=1\}$



Hausdorff, Second Countable and locally Euclidean.

$N=(0,\dots,0,1)$, $S=(0,\dots,0,-1)$.

$U_N=S^n\setminus\{N\}$, $U_S=S^n\setminus\{S\}$.

$\varphi_N: U_N\to\mathbb{R}^n$,

$\varphi_N(x_1,\dots,x_{n+1})=\left(\frac{x_1}{1-x_{n+1}},\dots,\frac{x_n}{1-x_{n+1}}\right)$

$\varphi_S : U_S \longrightarrow \mathbb{R}^n$

$\varphi_S(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}} \right)$

φ_N and φ_S are both homeomorphism with inverses

$\varphi_N^{-1} : \mathbb{R}^n \longrightarrow U_N,$

$\varphi_N^{-1}(y_1, \dots, y_n) = \left(\frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_n}{1+\|y\|^2}, \frac{\|y\|^2 - 1}{1+\|y\|^2} \right)$

$\|y\|^2 = y_1^2 + \dots + y_n^2$, and

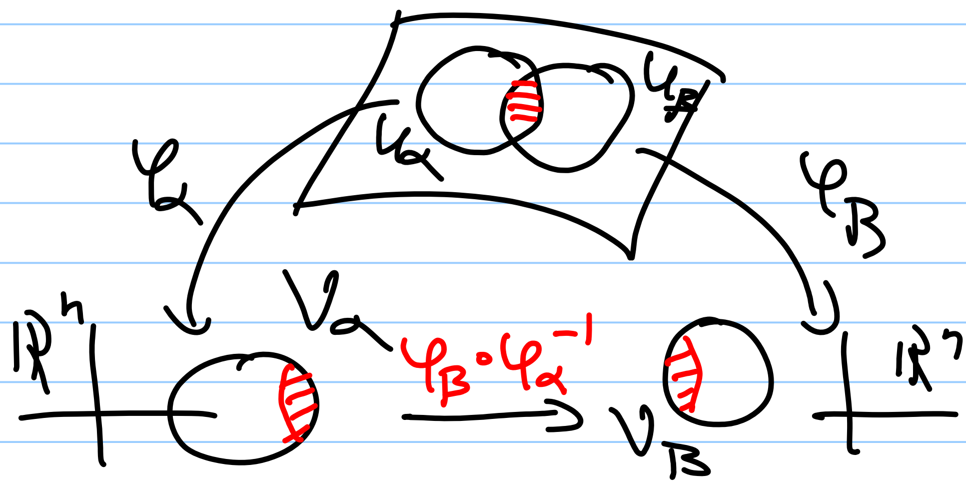
$\varphi_S^{-1} : \mathbb{R}^n \longrightarrow U_S,$

$\varphi_S^{-1}(y_1, \dots, y_n) = \left(\frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_n}{1+\|y\|^2}, \frac{1-\|y\|^2}{1+\|y\|^2} \right)$

$\Rightarrow S^n$ is locally Euclidean.

<u>Definition:</u> Let M be a topological manifold with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \Lambda}$, when $U_\alpha \subseteq M$ open, $V_\alpha \subseteq \mathbb{R}^n$ open, $\varphi_\alpha : U_\alpha \to V_\alpha$ homeomorphism, for each $\alpha \in \Lambda$ and $M = \bigcup_{\alpha \in \Lambda} U_\alpha$. If all compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$, wherever they are defined, are

smooth maps of open subsets of Euclidean spaces then we say that the atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in\Lambda}$ defines a smooth manifold structure on M.



$\varphi_\beta \circ \varphi_\alpha^{-1} \in C^\infty$.

Back to the Example:

$\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$

$$(y_1, \dots, y_n) \overset{\varphi_S^{-1}}{\longmapsto} \Big(\underbrace{\frac{2y_1}{1+\|y\|^2}}_{x_1}, \dots, \underbrace{\frac{2y_n}{1+\|y\|^2}}_{x_n}, \underbrace{\frac{1-\|y\|^2}{1+\|y\|^2}}_{x_{n+1}} \Big)$$

$$\downarrow \varphi_N$$

$$\left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right)$$

$$\frac{x_1}{1-x_{n+1}} = \frac{2y_1/1+\|y\|^2}{1-\left(\frac{1-\|y\|^2}{1+\|y\|^2}\right)} = \frac{2y_1}{2\|y\|^2} = \frac{y_1}{\|y\|^2}$$

So, $(\varphi_N \circ \varphi_S^{-1})(y_1, \dots, y_n) = \frac{1}{\|y\|^2}(y_1, \dots, y_n)$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$. Thus S^n is a smooth manifold of dimension n.

__Ex__: $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$

$\varphi_N : S^1 \setminus \{N\} \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \frac{x}{1-y}$

$\varphi_S : S^1 \setminus \{S\} \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \frac{x}{1+y}$

$\varphi_N^{-1} : \mathbb{R} \longrightarrow S^1 \setminus \{N\}$, $t \longmapsto \left(\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}\right)$

$\varphi_S^{-1} : \mathbb{R} \longrightarrow S^1 \setminus \{S\}$, $t \longmapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$.

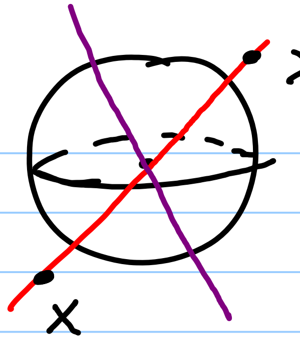
Moreover,

$$\varphi_N \circ \varphi_S^{-1} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$$

$$t \longmapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \longmapsto \frac{\frac{2t}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}} = \frac{2t}{2t^2} = \frac{1}{t}$$

__Example__ $\mathbb{RP}^n$: the real projective space of dimension n.

$\mathbb{RP}^n$ = the space of lines in $\mathbb{R}^{n+1}$ through the origin



$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} \Big/ \begin{array}{l} x \sim \lambda x \\ x \in \mathbb{R}^{n+1} \setminus \{0\} \\ \lambda \in \mathbb{R} \setminus \{0\}. \end{array}$$

$\mathbb{RP}^n$ is second countable since $\mathbb{R}^{n+1}$ is second countable.

<u>Exercise:</u> $\mathbb{RP}^n$ is Hausdorff.

$[x_0 : x_1 : \quad : x_n] = \{ \lambda (x_0, x_1, \dots, x_n) \mid \lambda \neq 0 \}$, the equivalence class containing the point $(x_0, x_1, \dots, x_n)$.

$U_i = \{ [x_0 : x_1 : \quad : x_n] \mid x_i \neq 0 \}$, $i = 0, \dots, n$.

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$,

$[x_0 : \quad : x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \widehat{\frac{x_i}{x_i}}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right)$

$\varphi_i^{-1} : \mathbb{R}^n \longrightarrow U_i$, $(y_1, \dots, y_n) \longmapsto [y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n]$.

φ_i is a homeomorphism for each i.

Hence, $\mathbb{RP}^n$ is a topological manifold of dimension n.

$\varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$

$(y_1, \dots, y_n) \xmapsto{\varphi_i^{-1}} [y_1 : \quad : \underset{|}{y_j} : \quad : y_{i-1} : 1 : y_i : \dots : y_n)$

$$\varphi_j \downarrow$$

$$\left(\frac{y_1}{y_j}, \dots, \widehat{\frac{y_j}{y_j}}, \dots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_{i+1}}{y_j}, \dots, \frac{y_n}{y_j}\right)$$

are clearly smooth and thus the atlas $\{\varphi_i\}_{i=0}^{n}$ defines a smooth structure on $\mathbb{RP}^n$.

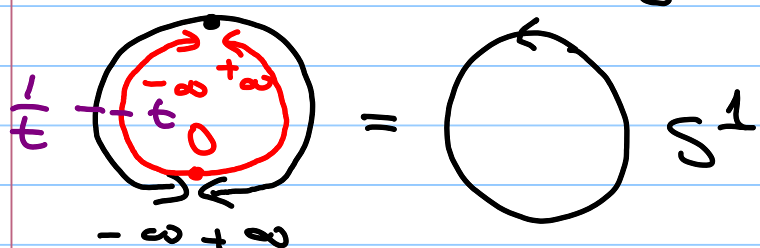
<u>Ex</u> $\mathbb{RP}^1 = U_0 \cup U_1$, $\quad U_0 = \{[x_0 : x_1] \mid x_0 \neq 0\}$

$\qquad U_1 = \{[x_0 : x_1] \mid x_1 \neq 0\}$

$$U_0 \xrightarrow{t} \mathbb{R}, \quad [x_0 : x_1] \longmapsto x_1/x_0 = t$$

$$U_1 \longrightarrow \mathbb{R}, \quad [x_0 : x_1] \longmapsto x_0/x_1 = 1/t$$

$$\mathbb{RP}^1 = \underset{0}{\mathbb{R}} \,\dot\cup\, \mathbb{R} / t \sim \frac{1}{t}, \quad t \neq 0.$$



<u>Complex Projective Space $\mathbb{CP}^n$:</u>

$$\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} \Big/ (z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$$

$$\lambda \in \mathbb{C}, \lambda \neq 0.$$

$\mathbb{C} = \mathbb{R}^2$, $\mathbb{C}^{n+1} = \mathbb{R}^{2n+1}$

$\mathbb{CP}^n$ Hausdorff and second countable.

$U_i = \{[z_0 : z_1 : \cdots : z_n] \mid z_i \neq 0\} \subseteq \mathbb{CP}^n$ open

$\varphi_i : U_i \to \mathbb{C}^n = \mathbb{R}^{2n}$,

$\varphi_i([z_0 : \cdots : z_n]) = \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right)$

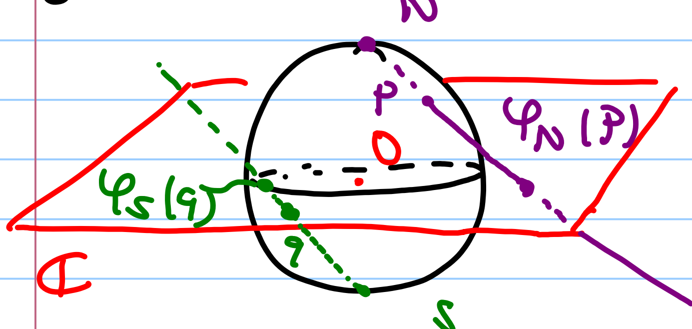
$\varphi_i^{-1} : \mathbb{C}^n \to U_i$

$\varphi_i^{-1}(w_1, \ldots, w_n) = [w_1 : \quad : w_{i-1} : 1 : w_i : \cdots : w_n]$

$\varphi_j \circ \varphi_i^{-1}$ is a C^∞ function of open subsets of $\mathbb{C}^n = \mathbb{R}^{2n}$, and thus $\mathbb{CP}^n$ is a smooth 2n-dimensional manifold.

Ex: $\mathbb{CP}^1 = \mathbb{C} \dot\cup \mathbb{C} \Big/ z \sim \frac{1}{z},\ z \neq 0$

This description shows that $\mathbb{CP}^1$ is just the Riemann sphere.



$\varphi_N \circ \varphi_S^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$
$z \longmapsto z^{-1}$

Example: $\mathbb{HP}^n$: Quaternionic projective n-space

$\mathbb{H} = \mathbb{R}^4 \quad (x_1, x_2, x_3, x_4) \longleftrightarrow x_1 + i x_2 + j x_3 + k x_4$

$i^2 = j^2 = k^2 = -1, \quad i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j$

$j \cdot i = -k, \quad k \cdot j = -i, \quad i \cdot k = -j$

$\mathbb{HP}^n = \mathbb{H}^{n+1} \setminus \{0\} / v \sim \lambda v, \quad \lambda \in \mathbb{H} \setminus \{0\}.$

$\mathbb{HP}^1 = \mathbb{H} \dot\cup \mathbb{H} / p \sim \frac{1}{p}, \quad p \neq 0.$

Remark: $\mathbb{RP}^1 = S^1, \quad \mathbb{CP}^1 = S^2, \quad \mathbb{HP}^1 = S^4$

$\mathbb{RP}^1 \downarrow w_i$ Stiefel Whitney, $\mathbb{CP}^1 \downarrow c_i$ Chern Classes, $\mathbb{HP}^1 \downarrow$ Pontryagin classes

Tangent Space and Tangent Bundle

$p \in \mathbb{R}^n$, $T_p \mathbb{R}^n$ = The set of all derivations on the ring of smooth functions defined near $p \in \mathbb{R}^n$.

$v_p(f) \in \mathbb{R}, \quad f : U \to \mathbb{R}$ smooth, $p \in U \subseteq \mathbb{R}^n$

- v_p linear : $v_p(af + bg) = a\, v_p(f) + b\, v_p(g)$, $a, b \in \mathbb{R}$

• $v_p(fg) = v_p(f)\,g(p) + f(p)\,v_p(g)$

<u>Ex</u> $\mathbb{R}^n$, $x_1, \dots, x_n$ coordinate on $\mathbb{R}^n$

$\left(\frac{\partial}{\partial x_i}\Big|_p\right)(f) = \frac{\partial f}{\partial x_i}(p)$ is a derivation.

<u>Proposition</u> The set of all derivations $T_p\mathbb{R}^n$ is a vector space of dimension n and $\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}_{i=1}^{n}$ is a basis for $T_p\mathbb{R}^n$.

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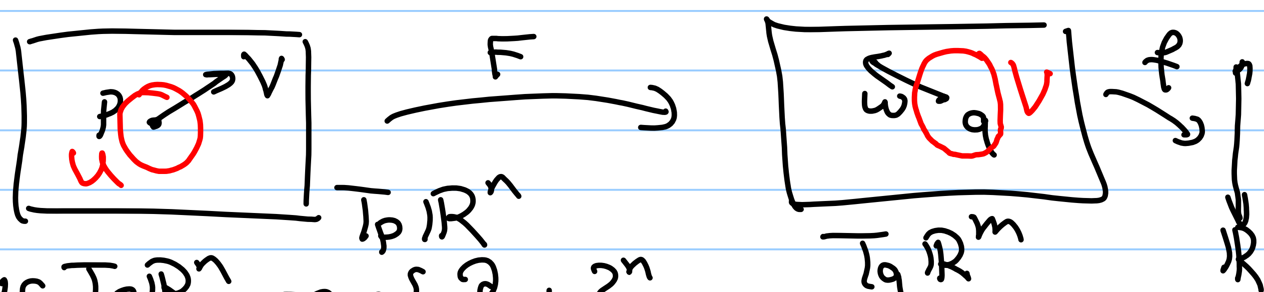
Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. Then for any $p \in \mathbb{R}^n$ we have a linear map

$$DF(p) : T_p\mathbb{R}^n \longrightarrow T_q\mathbb{R}^m, \quad q = F(p)$$



$v \in T_p\mathbb{R}^n = \text{span}\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}_{i=1}^{n}$ $\quad w = DF(p)(v)$.

$w \in T_q\mathbb{R}^m$, $f : V \to \mathbb{R}$, $q \in V \subseteq \mathbb{R}^m$ open

$U = F^{-1}(V)$. So $f \circ F : U \to \mathbb{R}$ smooth function.

In this case $DF(p)(v)$ is defined to be the derivation given by

$$DF(p)(v)(f) = v(f \circ F), \quad v = \frac{\partial}{\partial x_i}\Big|_p$$

$$v(f \circ F) = \frac{\partial}{\partial x_i}(f \circ F)(p) \qquad F = (f_1, \dots, f_m)$$

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

$$= \frac{\partial}{\partial x_i}\left(f(f_1, f_2, \dots, f_m)\right)(p) \qquad q = F(p)$$

$$= \frac{\partial f}{\partial y_1}(q)\frac{\partial f_1}{\partial x_i}(p) + \frac{\partial f}{\partial y_2}(q)\frac{\partial f_2}{\partial x_i}(p) + \cdots + \frac{\partial f}{\partial y_m}(q)\frac{\partial f_m}{\partial x_i}(p)$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_i} & \frac{\partial f_2}{\partial x_i} & \dots & \frac{\partial f_m}{\partial x_i} \end{bmatrix}(p) \begin{bmatrix} \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial y_2} \\ \vdots \\ \frac{\partial f}{\partial y_m} \end{bmatrix}(q)$$

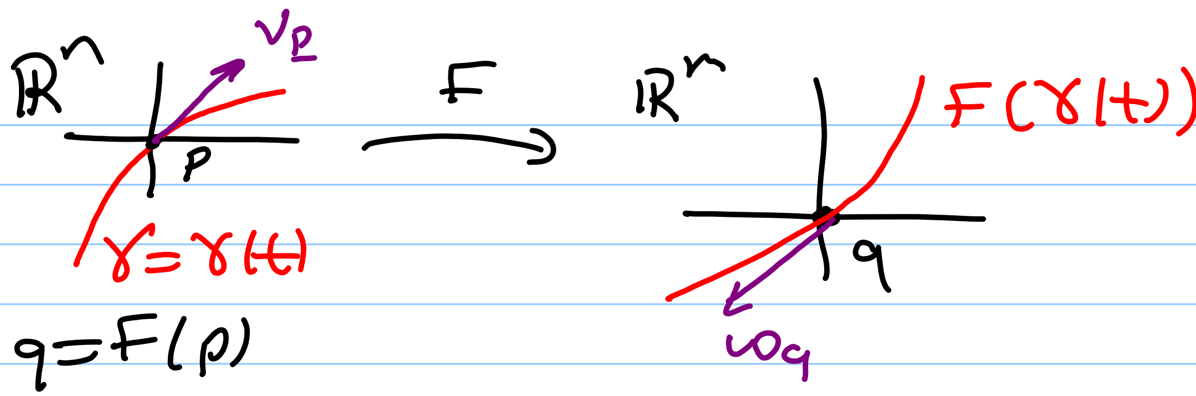
$$T_p\mathbb{R}^n = \mathrm{span}\left\{\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p\right\} \leftarrow \beta$$

$$T_q\mathbb{R}^m = \mathrm{span}\left\{\frac{\partial}{\partial y_1}\Big|_q, \dots, \frac{\partial}{\partial y_m}\Big|_q\right\} \leftarrow \beta'$$

$$DF(p) : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$$

$$[DF(p)]_{\beta}^{\beta'} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}(p) = \frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n)} = \frac{\partial F}{\partial(x_1, \dots, x_n)}(p)$$

Jacobian of F at $p \in \mathbb{R}^n$.



$q = F(p)$

$$w_q = \frac{d}{dt}\left(F(\gamma(t))\right)\Big|_{t=0}, \quad \frac{d\gamma}{dt}(0) = v_p$$

$$= DF(p)(v_p).$$