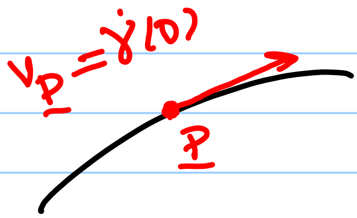
Note Title 24.02.2020

$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $v_p \in T_p\mathbb{R}^n$

$D\Phi_p(v_p) = ?$

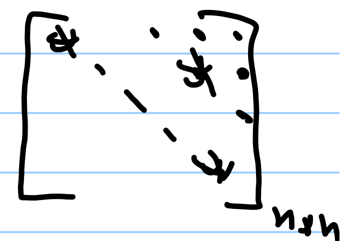
$D\Phi_p(v_p) = \frac{d}{dt}\Phi(\gamma(t))\Big|_{t=0}$, $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}$

$\gamma(0) = p$, $\dot{\gamma}(0) = v_p$



Ex $\Phi : M(n) \longrightarrow S(n)$ $\quad 1+2+\dots+n = \frac{n(n+1)}{2}$

$M(n) = n\times n$ - Real matrices
$= \mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$



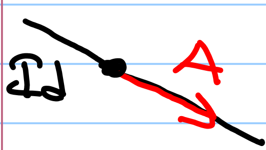
$S(n) = n\times n$ - Symmetric matrices
$= \{[a_{ij}] \in M(n) \mid a_{ij} = a_{ji}\}$
$= \mathbb{R}^{n(n+1)/2}$

$Id \in M(n)$, $D\Phi_{Id}(A) = ?$

$T_{Id}M(n) = M(n)$, $T_{Id}S(n) = S(n)$

$\Phi : M(n) \longrightarrow S(n)$, $\Phi(Q) = Q^TQ$

$D\Phi_{Id}(A) = \frac{d}{dt}\Phi(\gamma(t))\Big|_{t=0}$



$\gamma(t) = Id + tA$

$$D\Phi_{Id}(A) = \frac{d}{dt}\left(\gamma(t)^T \gamma(t)\right)\Big|_{t=0}$$
$$= \frac{d}{dt}\left[(Id + tA)^T (Id + tA)\right]\Big|_{t=0}$$
$$= \frac{d}{dt}\left(Id + tA^T + tA + t^2 A^T A\right)\Big|_{t=0}$$
$$= \left(0 + A^T + A + 2t A^T A\right)\Big|_{t=0}$$
$$= A^T + A.$$

So $D\Phi_{Id}(A) = A^T + A$.

<u>Claim:</u> $D\Phi_{Id} : T_{Id} M(n) \longrightarrow T_{Id} S(n)$ is onto.

<u>Proof:</u> If $B \in T_{Id} S(n) = S(n)$, then $B = B^T$,
$B = \frac{B}{2} + \frac{B^T}{2} = D\Phi_{Id}(A)$, when $A = \frac{B}{2}$.

<u>Tangent Space of Manifolds</u>

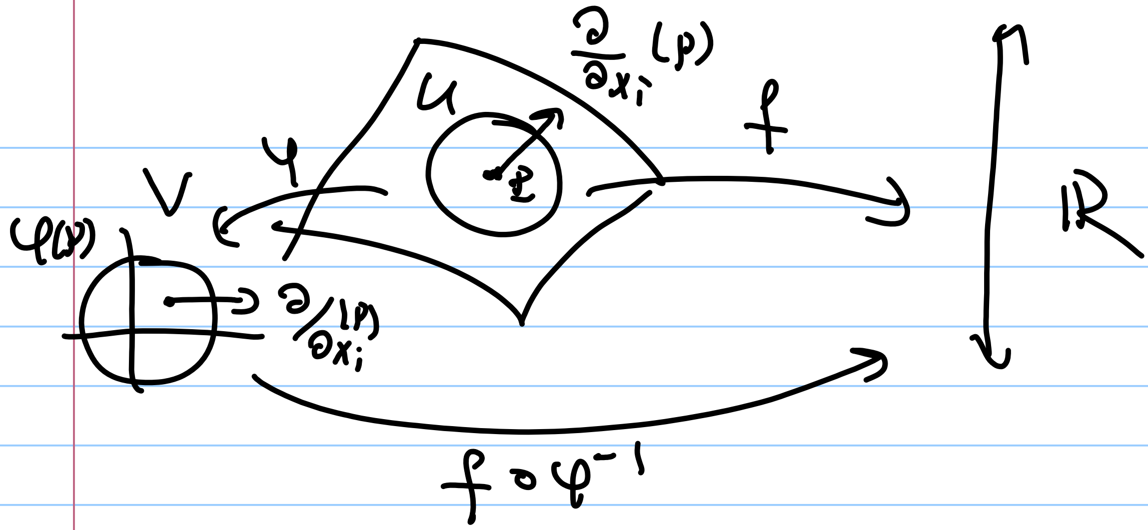
M smooth manifold, $p \in M$, $\varphi : U \to V$

$p \in U \subseteq M$ open, $V \subseteq \mathbb{R}^n$ open

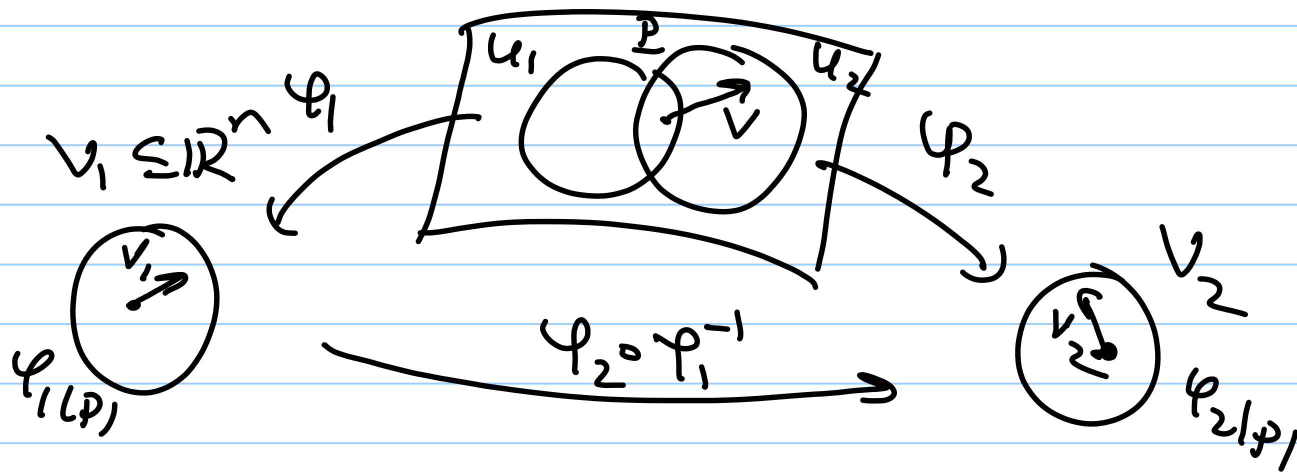
$$T_p M = T_p U \xrightarrow[\cong]{D\varphi_p} T_{\varphi(p)} V = T_{\varphi(p)} \mathbb{R}^n$$

$\varphi = (x_1, x_2, \ldots, x_n)$ $\quad T_{\varphi(p)} \mathbb{R}^n = \text{span}\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^n$

$f : M \to \mathbb{R}$, $\frac{\partial}{\partial x_i}(f)\Big|_p = \frac{\partial}{\partial x_i}\left(f \circ \varphi^{-1}\right)\Big|_{\varphi(p)}$



$$\frac{\partial}{\partial x_i}(p)(f) \doteq \frac{\partial}{\partial x_i}(f\circ\varphi^{-1})$$



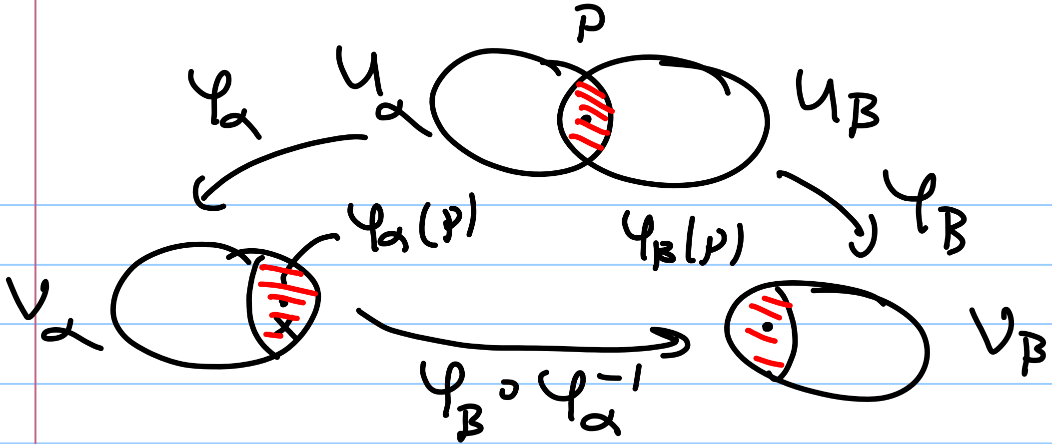
$D\varphi_1(v_1) = v = D\varphi_2(v_2)$ if and only if

$D(\varphi_2\circ\varphi_1^{-1})_{\varphi_1(p)}(v_1) = v_2$.

<u>Tangent Bundle</u> $M = \bigcup_\alpha U_\alpha$, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in A}$

is an atlas for M.

$$M = \bigcup_\alpha U_\alpha = \dot{\bigcup_\alpha} V_\alpha \Big/ \begin{array}{l}(\varphi_\beta\circ\varphi_\alpha^{-1})(x)\sim x \\ \forall x\in \varphi_\alpha(U_\alpha)\cap\varphi_\beta(U_\beta)\end{array}$$

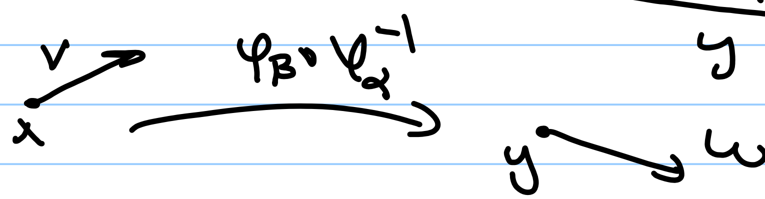


$$M = \bigcup_\alpha U_\alpha = \dot{\bigcup_\alpha} V_\alpha \Big/ x \sim (\varphi_\beta \circ \varphi_\alpha^{-1})(x)$$

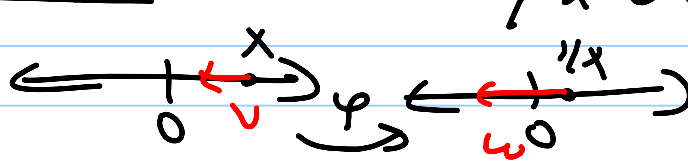
$$\underline{T_* M}_{\text{2n-dim'l smooth manifold}} = \bigcup_\alpha T_* U_\alpha = \dot{\bigcup_\alpha} \overbrace{T_* V_\alpha}^{V_\alpha \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} \text{ open}} \Big/ (x,v) \sim (y,w)$$

$$T_* V_\alpha = V_\alpha \times \mathbb{R}^n \longrightarrow T_* V_\alpha = V_\beta \times \mathbb{R}^n$$

$$(x,v) \longmapsto \Big(\underbrace{(\varphi_\beta \circ \varphi_\alpha^{-1})(x)}_{y},\ \underbrace{D(\varphi_\beta \circ \varphi_\alpha^{-1})_{(x)}(v)}_{w}\Big)$$



Example: $S^1 = \mathbb{R} \,\dot\cup\, \mathbb{R} \big/ x \sim \frac{1}{x}\ ,\ x \neq 0$



$$T_* S^1 = T_* \mathbb{R} \,\dot\cup\, T_* \mathbb{R} \Big/ (x,v) \sim \Big(\frac{1}{x}, \overbrace{\frac{-v}{x^2}}^{w}\Big)$$

$$D\varphi_x(v) = -\frac{1}{x^2}(v) = -\frac{v}{x^2}$$

Vector Fields $\quad T_* M \xrightarrow{\ \pi\ } M$ smooth map

$$(x,v) \longmapsto x$$

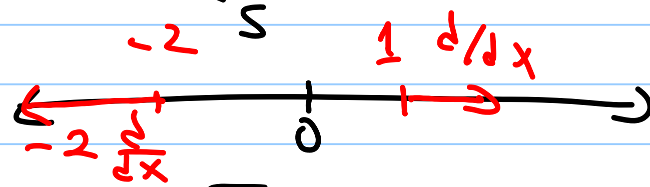
A vector field on M is a section

$s : M \longrightarrow T_x M$. Hence, $\pi \circ s = id_M$

$$T_xM \underset{s}{\overset{\pi}{\rightleftarrows}} M \qquad \begin{aligned} s(x) &= (x, v(x)) \\ \pi(x, v(x)) &= x \end{aligned}$$

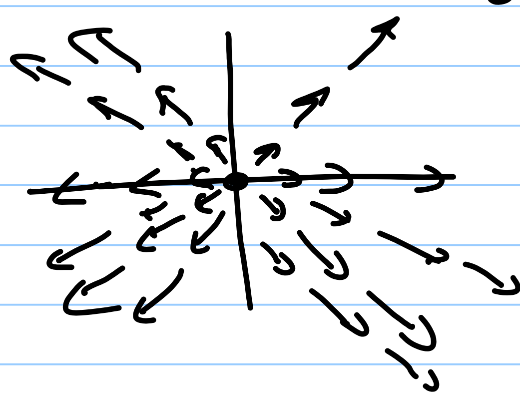
<u>Example</u> $M = \mathbb{R}$, $T_x\mathbb{R} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{R}$

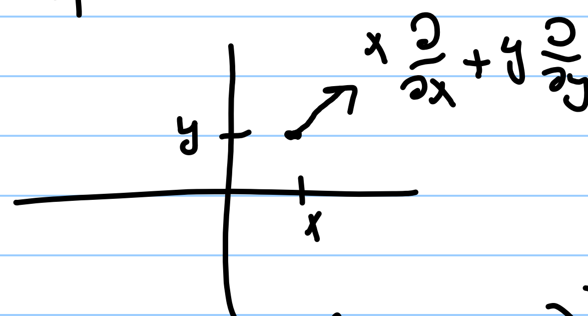
$s(x) = x \frac{d}{dx}$



<u>Example</u> $M = \mathbb{R}^2$, $T_xM \underset{s_1}{\overset{\pi}{\rightleftarrows}} M$

$s_1(x,y) = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$

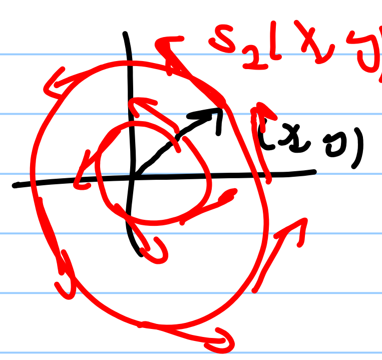




Radial vector field on $\mathbb{R}^2$

$s_2 : M = \mathbb{R}^2 \longrightarrow T_xM = T_x\mathbb{R}^2$

$s_2(x,y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$



$\underline{\text{Ex}}$ $M = S^2$, $T_x M = T_x S^2$

$(x,y,z) \rightsquigarrow (-y, x, 0)$

$\downarrow$

$-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$

$$S(x,y,z) = \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}\right)\Big|_{(x,y,z)}$$